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The proof of an equivalence of D'alembert's and Cauchy's tests in the theory of numerical series

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Доказательство эквивалентности признаков Даламбера и Коши в теории числовых рядов

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Доказ еквівалентності ознак Даламбера і Коші в теорії числових рядів

In the paper it is shown that D'alembert's test and Cauchy's radical test are not independent one from another. It is proposed a transition scheme from one test to another and vice versa. The equivalence of the tests is proved for series with the monotonically decreasing terms. This fact is used to formulate a new test of the convergence for series with positive terms. The test is equivalent to Dalembert and Cauchy's radical tests, but it has some advantages. It can be applied to any series within Cauchy-D'alembert's theory.
Keywords: series, convergence, comparison tests, D'alembert's test, Cauchy's test, limit.

В статье показано, что признак Даламбера и радикальный признак Коши не являются независимыми друг от друга. Предложена схема перехода от одного признака к другому. Эквивалентность признаков доказана для рядов с монотонно убывающими членами. Этот факт использован для формулировки нового признака сходимости рядов с положительными членами. Новый признак эквивалентен признакам Даламбера и Коши, но имеет некоторые преимущества – он может применяться к любым рядам в рамках теории Коши-Даламбера.
Ключевые слова: ряд, сходимость, признак сравнения, признак Даламбера, признак Коши, предел.

У статті показано, що ознака Даламбера і радикальна ознака Коші не є незалежними одна від другої. Запропоновано схему переходу від однієї ознаки до другої. Еквівалентність ознак доведена для рядів з монотонно регресними членами. Цей факт використан для формулювання нової ознаки збіжності рядів з додатними членами. Нова ознака еквівалентна ознакам Даламбера і Коші, але має деяку перевагу вона може застосовуватися до будь-яких рядів в рамках теорії Коші-Даламбера.
Ключові слова: ряд, збіжність, ознака порівняння, ознака Даламбера, ознака Коші, границя.

Introduction

Tests of the convergence of series with positive terms such as Cauchy's and D'alembert's tests are usually applied to quickly converging series. In the limit form for any series $\sum_{n=1}^{\infty} u_n$, $u_n > 0$ these tests can be written down in the forms of inequalities [1]:

$$\lim_{n\to\infty} \sqrt[n]{u_n} \le 1,\ u_n \ge 0\ , \tag{1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \le 1\ ,\ u_n > 0\,. \tag{2}$$

The first inequality is called Cauchy's radical test, the second one is D'alembert's test. For both tests the sign of equality means uncertainty in a question of convergence of the series. In that case additional research is required. They require of application of more " delicate " tests.

The proof of both tests (1) and (2) are based on the comparison test with respect to the geometrical series $\sum_{n=0}^{\infty} q^n, q<1$. From this we can make a conclusion about equivalence of D'alembert's and Cauchy's tests. In the literature the direct connection between these tests is usually not considered, but only through the geometrical series (Figure.1).

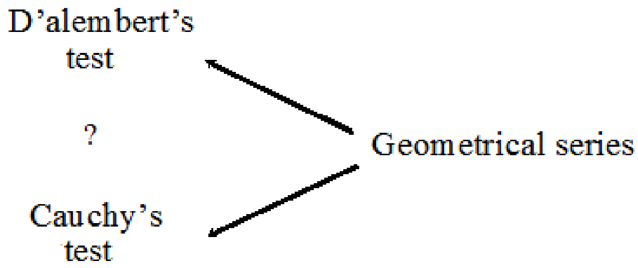

Figure. 1 – The scheme of a connection between D'alembert's and Cauchy's tests

In this work we will show a direct connection between D'alembert's and Cauchy's tests, moreover, how from one of the test follows another test and vice versa.

In the course of the proof of the tests there is appeared one more form of the test, which is equivalent both to D'alembert's and Cauchy's tests. This test is no more than the reformulation of D'alembert's and Cauchy's tests, but it has some advantages in comparison with D'alembert's and Cauchy's tests.

D'alembert's test as a corollary from Cauchy's radical test and vice versa

Let us address to Cauchy's radical test (1), and we will execute some transformations. We apply logarithm to both parts of the inequality (1) $\ln\left(\lim_{n\to\infty}\sqrt[n]{u_n}\right)\le \ln 1$. It is clear, if $\lim_{n\to\infty}\sqrt[n]{u_n}=0$ the logarithm $\ln\left(\lim_{n\to\infty}\sqrt[n]{u_n}\right)$ does not exist. In that case the range of definition must be extended to $\ln 0=-\infty$ and the inequality remains fair. This moment of the proof can be avoided if to consider not the limit form of the test, but to consider the inequality $\sqrt[n]{u_n}<1$ beginning from some number N_o.

A property of continuity of the function $y=\ln x$ allows inserting the logarithm sign after of the limit sign

$$\lim_{n\to\infty}\ln\sqrt[n]{u_n}\le 0 \Rightarrow \lim_{n\to\infty}\frac{\ln u_n}{n}\le 0 .$$

According to the necessary test we have $u_n \to 0$. Then, $\ln u_n \to -\infty$ at $n\to\infty$. The limit has uncertainty $\left\{\frac{\infty}{\infty}\right\}$ and the conditions of L'Hospital's rule are satisfied if the terms of the sequence $\{u_n\}$ are monotonically decreasing. Therefore

$$\lim_{n\to\infty}\frac{(\ln u_n)'}{n'}=\lim_{n\to\infty}\frac{(u_n)'}{u_n}\le 0 .$$

Let's represent the derivative $(u_n)'$ at $n \to \infty$ as

$$(u_n)' = u_{n+1} - u_n. \quad (3)$$

The proof of this formula is based on mean value Lagrange's theorem [2]. Thus we suppose the sequence $\{u_n\}$ is monotonically decreasing at $n \to \infty$. Only in this case the derivative is existed.

After of the substitution of the expression (3) to the previous inequality we will receive:

$$\lim_{n\to\infty} \frac{u_{n+1} - u_n}{u_n} \le 0 \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} - \lim_{n\to\infty} \frac{u_n}{u_n} \le 0 \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} \le 1.$$

It is possible the reverse solution of the problem. Starting from D'alembert's test (2) it is not difficult to get Cauchy's test. We make a conclusion about of the equivalence of these tests.

In any of the cases of the proof of the equivalence of the tests we receive an intermediate product - the inequality $\lim_{n\to\infty} (\ln u_n)' \le 0$.

A new test of a convergence of series with positive terms

From the scheme of the proof of equivalence of D'alembert's and Cauchy's tests we have an inequality which can be considered as a new test

$$\lim_{n\to\infty} (\ln u_n)' \le 0 \ , \ u_n > 0. \quad (4)$$

The sign '=' in the condition (4) means uncertainty as well as in Cauchy-D'alembert theory. The problem of a convergence needs in additional research, for example, to apply any other test of convergence.

This remark in accuracy coincides with the cases in the tests (1) and (2).

The proof of the test (4) can be done independently by using geometrical series $\sum_{n=0}^{\infty} q^n$, $q < 1$. Compare the given series $\sum_{n=0}^{\infty} u_n$, $u_n > 0$ with the geometrical series

$$u_n < q^n \Rightarrow \ln u_n < \ln q^n = n \ln q \Rightarrow \frac{\ln u_n}{n} < \ln q.$$

The right-hand side of the inequality is negative and in the limit at $n \to \infty$ the rigorous sign in the inequality must be changed on non-rigorous one:

$$\lim_{n\to\infty} \frac{\ln u_n}{n} \le \ln q \Rightarrow \lim_{n\to\infty} \frac{\ln u_n}{n} \le 0$$

If we apply L'Hospital's rule to the left-hand side of the inequality, we get the test (4).

The test (4) contains derivative, therefore we can call the test as the differential form of D'alembert-Cauchy's tests, or shortly, the differential test. This test can be applied to any series. Therefore we will consider some examples.

Example 1. To investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2}{2^{n+3}\sqrt{n^2+3n+5}}$.

Here, $\ln u_n = \ln(n+2) - (n+3)\ln 2 - \frac{1}{2}\ln(n^2+3n+5)$,

$(\ln u_n)' = \frac{1}{n+2} - \ln 2 - \frac{2n+3}{2(n^2+3n+5)}$, $\lim_{n\to\infty} (\ln u_n)' = -\ln 2 < 0$. The series is convergent.

Example 2. To investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2}{\sqrt[3]{n^7+3n^4+5}}$.

Here, $\ln u_n = \ln(n+2) - \frac{1}{3}\ln(n^7 + 3n^4 + 5)$,

$(\ln u_n)' = \frac{1}{n+2} - \frac{7n^6 + 12n^3}{3(n^7 + 3n^4 + 5)}$, $\lim_{n\to\infty}(\ln u_n)' = 0$. The test does not work.

Example 3. To investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n e^{-n}}{n!}$.

Here, $\ln u_n = n\ln n - n\ln e - \ln n!$,

$(\ln u_n)' = \ln n + 1 - 1 - \frac{(n!)'}{n!} = 0$, $\lim_{n\to\infty}(\ln u_n)' = 0$. The test does not work.

Here the asymptotic formula ($(n!)' = n!\ln n$ is used (APPENDIX).

The first example shows an application of our test, the second shows, when the test has the same restrictions that D'alembert's and Cauchy's tests. The third example shows possibility of application of the test to series, which contain the factorial function $n!$.

Conclusions

In the paper two main results are received. First, the equivalence of D'alembert's and Cauchy's tests is proved for monotonically series. It is shown, how from one of the test follows the other.

In the process of the proof it has formulated a new test which is equivalent to D'alembert's and Cauchy's tests. We also have shown how the differential test arises from the comparison test with respect to the geometrical series.

Now the scheme that was represented in the Fig. 1 becomes as it shown in the Figure. 2.

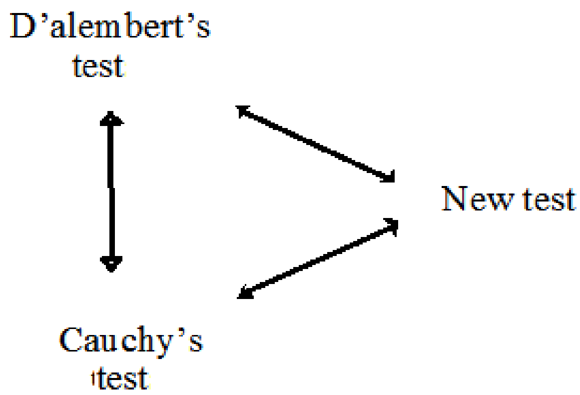

Figure. 2. – The scheme demonstrates three equivalent tests

As a rule, D'alembert's test is applied when series' terms contain the factorial $n!$. Cauchy's test does not work in this case. The operation $\sqrt[n]{n!}$ is not defined. From this point of view D'alembert's and Cauchy's tests supplement each other. Our test can be applied in both situations as in the case $n!$ as in any cases as well. Our differential test is more universal than D'alembert's and Cauchy's tests, but it has a very important restriction. It can be applied only to series with monotonically decreasing terms.

In the case of $n!$ we propose to use the approximate formula $(n!)' \underset{n>>1}{\approx} n!\ln n$. This formula allows applying our test to such series for which there is the problem of a choice what test should be used: D'alembert's or Cauchy's test.

Application of our test is not more difficult than D'alembert's and Cauchy's tests and in many cases it is easier them. Once more we will emphasize that our test has no alternative in the case of application, because logarithm of a product or division of functions is equal to their sum and difference. Derivative of such objects is calculated simply even it is easier than for product and division.

APPENDIX**.** The derivative of the factorial function $n!$.

Let's begin from Stirling's formula

$$n! = \sqrt{2\pi n}n^n e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\frac{139}{51840n^3}+o(n^{-4})\right),$$

We will restrict ourselves only with of two first terms of the sum. We take logarithm of the equality

$$\ln u! = \ln\sqrt{2\pi n}+\ln n^n+\ln e^{-n}+\ln\left(1+\frac{1}{12n}\right).$$

Let's accept approximation $\ln(1+1/12n)\approx 1/12n,$ then

$$\ln u! \approx \ln\sqrt{2\pi}+\frac{1}{2}\ln n+n\ln n-n+\frac{1}{12n},\quad \frac{(\ln n!)'}{n!}\approx \ln n+\frac{1}{2n}-\frac{1}{12n^2}.$$

Then it follows that $(n!)' \approx n!\left(\ln n+\frac{1}{2n}-\frac{1}{12n^2}\right) \underset{n>>1}{\approx} n!\ln n$.

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RESUME

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In the paper it is shown that D'alembert's test and Cauchy's radical test are not independent one from another. It is proposed a transition scheme from one test to another and vice versa. The equivalence of the tests is proved for series with the monotonically decreasing terms. This fact is used to formulate a new test of the convergence for series with positive terms. The test is equivalent to Dalembert and Cauchy's radical tests, but it has some advantages. It can be applied to any series within Cauchy-D'alembert's theory.

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