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INTEGRAL DYNAMIC MODEL FOR HEAT CONDUCTION IN PIECEWISE HOMOGENEOUS MEDEUM FOR CASE OF MULTIDIMENSIONAL SPACE

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Solving a thermal conductivity equation for piecewise homogeneous medium for the case of multidimensional space were considered. An initial problem was reduced to an integral equation (the essence of dynamic model) which was solved in the class of increasing functions. Solvability conditions for the problem were determined.

Keywords: thermal conductivity equation, piecewise homogeneous medium, integral dynamic model, resolvent, functional space.

Introduction

A considerable number of field theory problems are reduced to solving a thermal conductivity equation [1-3]. These problems involve those related to the computation of thermal fields of motors and components of electronic equipment [4, 5]; optimization of thermal conditions in rolling mills [6-8]; and determination of thermal flows in heat-and-power equipment in gas (oil, etc.) network systems [9].

In practical applications, it is sometimes required to solve a thermal conductivity equation for piecewise homogeneous media (ie, a set of homogeneous material layers). Computation of thermal fields in composites or analyses of soil thermal conductivity are the examples of such problems. An analytical solution for the problem mentioned (for the case of one-dimensional space) was described previously [10]. This paper expands and generalizes the result of that work for the case of n-dimensional space.

Statement of Problem

It is required to find the solution of a thermal conductivity equation

$$\frac{\partial u}{\partial t} = a^2(x_i)\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}, \tag{1}$$

which meets the initial condition

$$u(x_1, x_2, ..., x_n, t)\big|_{t=0} = F(x_1, x_2, ..., x_n) \tag{2}$$

and boundary conditions (in the form of reduction conditions)

$$u(x_1, x_2, ..., x_n, t)\big|_{x_1=0} = u(0, x_2, ..., x_n, t), \tag{3}$$

$$
\begin{aligned}
u(x_1,x_2,...,x_n,t)\big|_{x_2=0} &= u(x_1,0,...,x_n,t),\\
&\dots,\\
u(x_1,x_2,...,x_n,t)\big|_{x_n=0} &= u(x_1,x_2,...,0,t),
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
k_1\,u'_{x_1}(x_1,x_2,...,x_n,t)\big|_{x_1=0} &= k_1\,u'_{x_1}(0,x_2,...,x_n,t),\\
k_2\,u'_{x_2}(x_1,x_2,...,x_n,t)\big|_{x_2=0} &= k_2\,u'_{x_2}(x_1,0,...,x_n,t),\\
&\dots,\\
k_n\,u'_{x_n}(x_1,x_2,...,x_n,t)\big|_{x_1=0} &= k_n\,u'_{x_n}(x_1,x_2,...,0,t),
\end{aligned}
\tag{4}
$$

where $k_i,\ i=\overline{1,n}$ are positive constants,

$$
a^2(x_i)=\begin{cases} a_{i_1}^2 \quad npu \quad x_i<0,\\ a_{i_2}^2 \quad npu \quad x_i>0; \quad i=\overline{1,n}.\end{cases}
\tag{5}
$$

Function $F(x_1,x_2,...,x_n)$ is subjected to the following limitation:

$$
|F(\cdot)|,\ |F'_{x_i}(\cdot)|<M\ \exp(\delta^2 r^2),
\tag{6}
$$

where $r^2=\sum_{i=1}^{n}x_i^2$, M , δ are positive.

We shall seek for a solution in the class of functions satisfying the inequality

$$
\max|u(x_1,x_2,...,x_n,t)|<M_0\ \exp(\delta_0^2 r^2),\ 0\le t\le t_0,
\tag{7}
$$

where

$$
\delta_0^2=\frac{\delta^2}{1-4\,a_0^2\,\delta^2\,t^2},
\tag{8}
$$

$$
a_0=\max\left[a_{i_1},a_{i_2}\right].
\tag{9}
$$

Integral representation of the solution

We shall seek for a solution represented as follows (for $x_i<0$):

$$
\begin{aligned}
u(x_1,x_2,...,x_n,t)=\int_0^t\Big\{\int\cdots\int_{-\infty}^{+\infty} G(x_1-\xi_1,x_2-\xi_2,...,x_n-\xi_n,t-\tau;a_0)\times\\
\times 2\,a_0^2\,\psi(\xi_1,\xi_2,...,\xi_n,\tau)\,d\xi_1\,d\xi_2...d\xi_n\Big\}d\tau+
\end{aligned}
\tag{10}
$$

$$+\int_0^t\left\{\int \overset{+\infty}{\underset{-\infty}{\cdots}}\int G(x_1-\xi_1, x_2-\xi_2,\ldots,x_n-\xi_n,t;a_0)\times F(\xi_1,\xi_2,\ldots,\xi_n)\,d\xi_1\,d\xi_2\ldots d\xi_n \right. \tag{10}$$

and as follows (for $x_i > 0$)

$$\begin{aligned} u(x_1,x_2,\ldots,x_n,t) = -\int_0^t\left\{\int \overset{+\infty}{\underset{-\infty}{\cdots}}\int G(x_1-\xi_1, x_2-\xi_2,\ldots,x_n-\xi_n,t-\tau;a_0)\times \right.\\ \times 2a_0^2\,\varphi(\xi_1,\xi_2,\ldots,\xi_n,\tau)\,d\xi_1\,d\xi_2\ldots d\xi_n\big\}d\tau + \\ +\int_0^t\left\{\int \overset{+\infty}{\underset{-\infty}{\cdots}}\int G(x_1-\xi_1, x_2-\xi_2,\ldots,x_n-\xi_n,t;a_0)\times F(\xi_1,\xi_2,\ldots,\xi_n)\,d\xi_1\,d\xi_2\ldots d\xi_n,\right. \end{aligned} \tag{11}$$

where

$$G(x_1,x_2,\ldots,x_n,t;a_0) = \frac{1}{\left(2\sqrt{\pi\,a_0^2\,t}\right)^n}\exp\left[-\left(\sum_{i=1}^n x_i^2\right)\Big/\left(4a_0^2\,t\right)\right]. \tag{12}$$

One can use a direct check to establish that this function satisfies equation (1) and initial conditions (2).

It is necessary to select φ and ψ functions in such a way that $u(x_1,x_2,\ldots,x_n,t)$ function satisfies the conditions for convergence of (3) and (4). In such a case, if φ and ψ satisfy the following conditions:

$$\begin{gathered} |\varphi(x_2,\ldots,x_n,t)| \le M\,\exp\left[\delta^2 r^2(x_1)\right], \\ |\varphi(x_1,.x_3,..,x_n,t)| \le M\,\exp\left[\delta^2 r^2(x_2)\right], \\ \ldots, \\ |\varphi(x_2,\ldots,x_{n-1},t)| \le M\,\exp\left[\delta^2 r^2(x_n)\right], \end{gathered} \tag{13}$$

$$\begin{gathered} |\varphi(x_2',x_3',\ldots,x_n',t)| - |\varphi(x_2'',x_3'',\ldots,x_n'',t)| \le M\,\exp\left(\delta^2 n^2\right)\times r\left(|x_1'-x_1''|\right), \\ |\varphi(x_1',x_3',\ldots,x_n',t)| - |\varphi(x_1'',x_3'',\ldots,x_n'',t)| \le M\,\exp\left(\delta^2 n^2\right)\times r\left(|x_2'-x_2''|\right), \\ \ldots, \\ |\varphi(x_2',x_3',\ldots,x_{n-1}',t)| - |\varphi(x_2'',x_3'',\ldots,x_{n-1}'',t)| \le M\,\exp\left(\delta^2 n^2\right)\times r\left(|x_n'-x_n''|\right), \end{gathered} \tag{14}$$

$$\begin{gathered} |\psi(x_2,x_3,\ldots,x_n,t)| \le \left\{(M)\exp\left[\delta^2 r^2(x_1)\right]\right\}, \\ |\psi(x_1,x_3,\ldots,x_n,t)| \le \left\{(M)\exp\left[\delta^2 r^2(x_2)\right]\right\}, \\ \ldots, \\ |\psi(x_1,.x_2,\ldots,x_{n-1},t)| \le \left\{\left(M/\sqrt{t}\right)\exp\left[\delta^2 r^2(x_n)\right]\right\}, \end{gathered} \tag{15}$$

$$
\begin{gathered}
\left|\psi\left(x_2', x_3', \ldots, x_n', t\right)\right| - \left|\psi\left(x_2'', x_3'', \ldots, x_n'', t\right)\right| \le M \ \exp\left(\delta^2 n^2\right) \times r\left(\left|x_1' - x_1''\right|\right), \\
\left|\psi\left(x_1', x_3', \ldots, x_n', t\right)\right| - \left|\psi\left(x_1'', x_3'', \ldots, x_n'', t\right)\right| \le M \ \exp\left(\delta^2 n^2\right) \times r\left(\left|x_2' - x_2''\right|\right), \\
\ldots, \\
\left|\psi\left(x_1', x_3', \ldots, x_{n-1}', t\right)\right| - \left|\psi\left(x_2'', x_3'', \ldots, x_n'', t\right)\right| \le M \ \exp\left(\delta^2 n^2\right) \times r\left(\left|x_n' - x_n''\right|\right),
\end{gathered} \tag{16}
$$

and limitation (6) is met, then $u\left(x_1, x_2, \ldots, x_n, t\right)$ function satisfies inequality (7).

It is easy to show that the integrals that dominate the integrals in (10) and (11) are not convergent for any t value. The following condition should be satisfied to make these integrals convergent:

$$
0 \le t \le t_0 < \left[1/\left(4 a_0^2 \delta^2\right)\right] . \tag{17}
$$

In this case, δ_0 in expression (7) would be determined by expression (8). Therefore, further in this paper, we shall seek for φ and ψ functions in the class of functions satisfying inequalities (13) to (16).

Reduction of the problem to an integral equation

Now we proceed to determine φ and ψ functions in formulas (10), (11). Here it should be noted that the first term in formula (10) is a single-layer thermal potential, while the first term in formula (11) is a double-layer thermal potential.

Based on the conditions (3) and the properties of a double-layer thermal potential (i.e., component $x_1 = 0$), we may get the following:

$$
\begin{gathered}
\int_0^t \left\{ \int \cdots \int_{-\infty}^{+\infty} G\left(0, x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau; a_1\right) \times 2 a_1^2 \, \varphi\left(\xi_1, \xi_2, \ldots, \xi_n, \tau\right) d\xi_1 \, d\xi_2 \ldots d\xi_n \right\} d\tau + \\
+ \int \cdots \int_{-\infty}^{+\infty} G\left(\xi_1, x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau; a_1\right) \times F\left(\xi_1, \xi_2, \ldots, \xi_n, \tau\right) d\xi_1 \, d\xi_2 \ldots d\xi_n = \varphi\left(x_1, x_2, \ldots, x_n, t\right) + \\
+ \int \cdots \int_{-\infty}^{+\infty} G\left(\xi_1, x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau; a_2\right) \times F\left(\xi_1, \xi_2, \ldots, \xi_n, \tau\right) d\xi_1 \, d\xi_2 \ldots d\xi_n
\end{gathered}
$$

or

$$
\begin{gathered}
\varphi\left(x_1, x_2, \ldots, x_n, t\right) = \\
= \int_0^t \left\{ \int \cdots \int_{-\infty}^{+\infty} G\left(0, x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau; a_1\right) \times 2 a_1^2 \, \varphi\left(\xi_1, \xi_2, \ldots, \xi_n, \tau\right) d\xi_1 \, d\xi_2 \ldots d\xi_n \right\} d\tau + \\
+ \sum_{j=1}^{2} (-1)^{j-1} \times \int \cdots \int_{-\infty}^{+\infty} G\left(\xi_1, x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau; a_2\right) \times F\left(\xi_1, \xi_2, \ldots, \xi_n, \tau\right) d\xi_1 \, d\xi_2 \ldots d\xi_n,
\end{gathered} \tag{18}
$$

where a_1 and a_2 are determined from (5) when $i = 1$.

Based on conditions (4) and the properties of the normal derivative of the single-layer thermal potential, it follows that:

$$k_1\left\{\varphi(x_1,x_2,...,x_n,t)-\int\cdots\int_{-\infty}^{+\infty} G'_{\xi_1}(\xi_1,x_2-\xi_2,...,x_n-\xi_n,t-\tau;a_1)\times\right.$$
$$\times F(\xi_1,\xi_2,...,\xi_n,\tau)d\xi_1\,d\xi_2\,...d\xi\} = k_2\left\{\lim_{x_1\to+0}\frac{\partial}{\partial x_1}\int\cdots\int_{-\infty}^{+\infty} G'_{x_1}(x_1,x_2-\xi_2,...,x_n-\xi_n,t-\tau;a_2)\times\right.$$
$$\times 2\,a_2^2\,\varphi(\xi_1,\xi_2,...,\xi_n,\tau)d\xi_1\,d\xi_2\,...d\xi_n -$$
$$\left.-\int\cdots\int_{-\infty}^{+\infty} G'_{\xi_1}(\xi_1,x_2-\xi_2,...,x_n-\xi_n,t-\tau;a_2)\times F(\xi_1,\xi_2,...,\xi_n,\tau)d\xi_1\,d\xi_2\,...d\xi_n\right\}. \tag{19}$$

After eliminating the function φ from the two last equations:

$$k_1\,\varphi(x_1,...,x_n,t)-k_1\int\cdots\int_{-\infty}^{+\infty} G'_{\xi_1}(\xi_1,x_2-\xi_2,...,x_n-\xi_n,t-\tau;a_1)\times$$
$$\times F(\xi_1,\xi_2,...,\xi_n,\tau)d\xi_1\,d\xi_2\,...d\xi\} =$$
$$=-k_2\lim_{x_1\to+0}\frac{\partial}{\partial x_1}\int_0^t d\tau\int\cdots\int_{-\infty}^{+\infty}\left[\int_\tau^t dt_1\int\cdots\int_{-\infty}^{+\infty} G'_{x_1}(x_1,x_2-\xi'_2,...,x_n-\xi'_n,t-t_1;a_2)\times\right.$$
$$\left.\times G(0,\xi'_2-\xi_2,...,\xi'_n-\xi_n,t-\tau;a_1)d\xi'_2\,...\,d\xi'_n\right]4\,a_1^2\,a_2^2\,\psi(\xi_2,\xi_3,...,\xi_n,\tau)d\xi_2,d\xi_3,...,d\xi_n+I, \tag{20}$$

where

$$I=-k_2\lim_{x_1\to+0}\frac{\partial}{\partial x_1}\int_0^t d\tau\int\cdots\int_{-\infty}^{+\infty} G'_{x_1}(x_1,x_2-\xi_2,...,x_n-\xi_n,t-\tau;a_2)\times$$
$$\times 2\,a_2^2\left[\sum_{j=1}^{2}(-1)^{j-1}\int\cdots\int_{-\infty}^{+\infty} G'(\xi_1,\xi'_2-\xi_2,...,\xi'_n-\xi_n,,\tau;a_j)\times\right.$$
$$\left.\times F(\xi_1,\xi_2,...,\xi_n)d\xi_1\,d\xi_2\,...d\xi_n\right]d\xi'_2...d\xi'_n -$$
$$-\int\cdots\int_{-\infty}^{+\infty} G'_{\xi_1}(\xi_1,x_2-\xi_2,...,x_n-\xi_n,t;a_2)\times F(\xi_1,\xi_2,...,\xi_n,\tau)d\xi_1\,d\xi_2\,...d\xi_n. \tag{21}$$

By direct calculation we can show that:

$$\pm 4\,a_1^2a_2^2\int_\tau^t dt_1\int\cdots\int_{-\infty}^{+\infty} G'_{x_1}(x_1,x_2-\xi'_2,...,x_n-\xi'_n,t-t_1;a_2)\times$$
$$\times G(0,\xi'_2-\xi_2,...,\xi'_n-\xi_n,t-\tau;a_1)d\xi'_2\,...\,d\xi'_n =$$
$$=\pm\frac{a_1}{a_2^{n-1}\pi^{3/2}\left(2\sqrt{\pi}\right)^{n-2}(t-\tau)^{\pi/2}}\int_0^\infty \exp\left[-z^2-\frac{x_1^2}{4\,a_1^2(t-\tau)}-\right.$$
$$\left.-\frac{\sum_{l=1}^{n}(x_l-\xi_l)^2}{4\,a_2^2(t-\tau)}\cdot\frac{x_1^2+4\,a_2^2(t-\tau)z^2}{x_1^2+4\,a_1^2(t-\tau)z^2}\right]\times\left[\frac{x_1^2+4\,a_2^2(t-\tau)z^2}{x_1^2+4\,a_1^2(t-\tau)z^2}\right]^{\frac{n-1}{2}}dz. \tag{22}$$

In this formula, the sign of the right side matches the sign of x_1 in the left side.

We denote by I_1 the first term in formula (20). Then

$$I_1 = -\frac{a_1 k_2}{a_2}\psi(x_2, x_3, ..., x_n, t) + \frac{a_1 k_2 (a_1^2 - a_2^2)}{2\pi^{3/2} a_2} \times$$
$$\times \int_0^t d\tau \int \cdots \int_{-\infty}^{+\infty} K_0(x_2 - \xi_2, ..., x_n - \xi_n, t - \tau)\psi(\xi_2, \xi_3, ..., \xi_n, \tau) d\xi_2 ... d\xi_n, \tag{23}$$

where

$$K_0(x_2 - \xi_2, ..., x_n - \xi_n, t - \tau) = \frac{1}{(2\sqrt{\pi})^{n-2}(t-\tau)^{\frac{n+1}{2}}} \times$$
$$\times \int_0^\infty \exp\left[\frac{\sum_{l=1}^n (x_l - \xi_l)^2}{4a^2(z)(t-\tau)}\right] \times \frac{\rho(z)}{a^{n-2}(z)}\left\{(n-1) - \frac{\sum_{l=1}^n (x_l - \xi_l)^2}{2a^2(z)(t-\tau)}\right\} dz, \tag{24}$$

$$a^2(z) = a_2^2 \frac{z^2 + a_1^2}{z^2 + a_2^2}, \ \rho(z) = \frac{1}{(z^2 + a_1^2)^{3/2}(z^2 + a_2^2)^{3/2}}. \tag{25}$$

After substitution of I_1 into (20), we obtain the following integral equation:

$$\psi(x_2, x_3, ..., x_n, t) = \frac{a_1 k_2 (a_1^2 - a_2^2)}{2\pi^{3/2}(a_1 k_2 + a_2 k_1)} \int_0^t d\tau \int \cdots \int_{-\infty}^{+\infty} K_0(x_2 - \xi_2, ..., x_n - \xi_n, t - \tau) \times$$
$$\times \psi(\xi_2, \xi_3, ..., \xi_n, \tau) d\xi_2 ... d\xi_n + f(x_2, x_3, ..., x_n, t). \tag{26}$$

After the limit of the derivative of I_1 is computed, we determine f function finally using the following formula:

$$f(x_2, x_3, ..., x_n, t) = -\frac{a_2}{a_1 k_2 + a_2 k_1}\left\{\sum_{j=1}^2 (-1)^{j-1} \int \cdots \int_{-\infty}^{+\infty} G'(\xi_1, \xi_2' - \xi_2, ..., \xi_n' - \xi_n, , \tau; a_j) \times\right.$$
$$\times F(\xi_1, \xi_2, ..., \xi_n) d\xi_1 d\xi_2 ... d\xi_n + \frac{2k_2}{(2\sqrt{\pi})^{n+1} a_2 t^{\frac{n+2}{2}}} \sum_{j=1}^2 \int \cdots \int_{-\infty}^{+\infty} \frac{|\xi_1| F(\xi_1, \xi_2, ..., \xi_n)}{a_j^{n+1}} \times$$
$$\times \left[\int_0^\infty \exp\left(-z^2 - \frac{\xi_1^2}{4a\, j^2 t} - \frac{\sum_{l=2}^n (x_l - \xi_l)^2}{4a\, j^2 t} \cdot \frac{\xi_1^2 + 4a_1^2 t z^2}{\xi_1^2 + 4a_2^2 t z^2}\right) \times\right.$$
$$\left.\times \left(\frac{\xi_1^2 + 4a_1^2 t z^2}{\xi_1^2 + 4a_2^2 t z^2}\right)^{\frac{n-1}{2}} dz\right] d\xi_1 d\xi_2 ... d\xi_n + \frac{2k_2(a_1^2 - a_2^2)}{(2\sqrt{\pi})^{n+1} a_1 a_2 t^{\frac{n+2}{2}}} \int \cdots \int_{-\infty}^{+\infty} F(\xi_1, \xi_2, ..., \xi_n) \times \tag{27}$$
$$\times \left\{\int_0^\infty \exp\left(-\frac{\xi_1^2}{4a_1^2 t} - \frac{\xi_1^2}{4a_1^2 a_2^2 t z^2} - \frac{\sum_{l=2}^n (x_l - \xi_l)^2}{4a\, j^2 t}\right) \times\right.$$

$$\times\left[(n-1)-\frac{\sum_{l=2}^{n}(x_l-\xi_l)^2}{4a^2(z)t}\right]dz\Bigg\}d\xi_1\,d\xi_2\ldots d\xi_n\Bigg\}. \tag{27}$$

Solving an integral equation

Let us consider the following integral equation:

$$\psi(x_2,x_3,\ldots,x_n,t)=f(x_2,x_3,\ldots,x_n,t)+\lambda\int_0^t\left\{\int\cdots\int_{-\infty}^{+\infty}K_0(x_2-\xi_2,\ldots,x_n-\xi_n,t-\tau)\times\right.$$
$$\left.\times\psi(\xi_2,\xi_3,\ldots,\xi_n,\tau)d\xi_2\ldots d\xi_n\right\}d\tau. \tag{28}$$

We denote by H_0 the class of functions $\varphi(x_1,x_2,\ldots,x_n,t)$ that can be subjected to Fourier and Laplace transformations. Denote

$$\int\cdots\int_{-\infty}^{+\infty}e^{-2\pi i\sum_{j=1}^{n}u_jx_j}\varphi(x_2,x_3,\ldots,x_n,t)dx_2\ldots dx_n=\widetilde{\varphi}(u_2,u_3,\ldots,u_n,t), \tag{29}$$

$$p\int_0^\infty e^{-pt}\widetilde{\varphi}(u_2,u_3,\ldots,u_n,t)=\widetilde{\varphi}(u_2,u_3,\ldots,u_n,p). \tag{30}$$

Initially, we assume that $f\in H_0$, and shall seek for the solution of equation (28) in H_0. We shall subject both sides of equation (28) to Fourier transformation. Then

$$\widetilde{\psi}(u_2,u_3,\ldots,u_n,t)=\widetilde{f}(u_2,u_3,\ldots,u_n,t)+\lambda\int_0^t\widetilde{K}_0(u_2,u_3,\ldots,u_n,t-\tau)\widetilde{\psi}(u_2,u_3,\ldots,u_n,\tau)d\tau. \tag{31}$$

By direct calculation it is easy to show that:

$$\int_0^t\widetilde{K}_0(u_2,u_3,\ldots,u_n,t-\tau)=\int\cdots\int_{-\infty}^{+\infty}K_0(x_2,x_3,\ldots,x_n,t-\tau)\exp\left[-2\pi\sum_{j=2}^{n}u_jx_j\right]dx_2\ldots dx_n=$$
$$=16\pi^{3/2}\sum_{j=2}^{n}u_j^2\int_0^\infty\rho(z)a^3(z)\exp\left[-4\pi^2a^2(z)(t-\tau)\sum_{j=2}^{n}u_j^2\right]dz. \tag{32}$$

Now we shall subject both sides of equation (31) to a Laplace transformation. Then:

$$\widetilde{\psi}_1(u_2,u_3,\ldots,u_n,p)=\widetilde{f}_1(u_2,u_3,\ldots,u_n,p)+\lambda\widetilde{K}_{01}(u_2,u_3,\ldots,u_n,p)\widetilde{\psi}_2(u_2,u_3,\ldots,u_n,p), \tag{33}$$

where

$$\widetilde{K}_{01}(u_2,u_3,...,u_n,p)=\int_0^\infty e^{-pt}K_0(u_2,u_3,...,u_n,t)dt=\frac{2\pi^{3/2}}{\left(a_2^2-a_1^2\right)}\left[\sqrt{\frac{p+4a_2^2\pi^2\sum_{j=2}^n u_j^2}{p+4a_1^2\pi^2\sum_{j=2}^n u_j^2}}-1\right]. \quad (34)$$

Therefore, from expression (31), we get:

$$\widetilde{\psi}_1(u_2,u_3,...,u_n,p)=\mu\,\widetilde{f}_1(u_2,u_3,...,u_n,p):\left(\nu+\sqrt{\frac{p+4a_2^2\pi^2\sum_{j=2}^n u_j^2}{p+4a_1^2\pi^2\sum_{j=2}^n u_j^2}}\right), \quad (35)$$

where $\mu=\dfrac{a_1^2-a_2^2}{2\pi^{3/2}\lambda}$, $\nu=\dfrac{a_1^2-a_2^2}{2\pi^{1/2}\lambda}$.

Applying the convolution formula to (35) gives the original function:

$$\begin{aligned}\widetilde{\psi}(u_2,u_3,...,u_n,t)&=\widetilde{f}(u_2,u_3,...,u_n,t)+\frac{4\pi^2\mu\left(|\nu|-\nu\right)\left(a_1^2-a_2^2\right)}{\left(\nu^2-1\right)^2}\times\\&\times\int_0^t\sum_{j=2}^n u_j^2\widetilde{f}\exp\left[-4\pi^2 d_0(t-\tau)\sum_{j=2}^n u_j^2\right]d\tau+8\pi a_1\mu\left(a_1^2-a_2^2\right)\times\\&\times\int_0^t\left\{\int_0^\infty\frac{z^2a^2(z)\sum_{j=2}^n u_j^2\exp\left[-4\pi^2a^2(z)(t-\tau)\sum_{j=2}^n u_j^2\right]}{\left(z^2-a_1^2\right)\left(z^2-a_2^2\right)\left(\nu^2z^2-a_1^2\right)}dz\right\}\times\widetilde{f}(u_2,u_3,...,u_n,\tau)d\tau,\end{aligned} \quad (36)$$

where

$$d_0=\frac{a_1^2\nu^2-a_2^2}{\nu^2-1}. \quad (37)$$

It is apparent that the right side of the last equality increases with the increase in u, if $d_0\le 0$ and $\nu<0$. Therefore, in this case we cannot apply the formula of Fourier transformation. Assuming that $d_0=b^2>0$ or $\nu=-\nu\ge 0$, we shall apply the formula of Fourier transformation to (36). Then

$$\begin{aligned}\psi(x_2,x_3,...,x_n,t)&=f(x_2,x_3,...,x_n,t)+\frac{4\pi^2\mu\left(|\nu|-\nu\right)\left(a_1^2-a_2^2\right)}{\left(\nu^2-1\right)^2}\times\\&\times\int_0^t d\tau\int_{-\infty}^{+\infty}\!\!\cdots\!\int\sum_{j=2}^n u_j^2 f(u_2,u_3,...,u_n,\tau)\exp\left[-2\pi\sum_{j=2}^n u_j x_j-4\pi^2b^2(t-\tau)-\right.\\&\left.-2\pi\sum_{j=2}^n u_j\right]du_2...du_n+8\pi a_1\mu\left(a_1^2-a_2^2\right)\int_0^t d\tau\int_{-\infty}^{+\infty}\!\!\cdots\!\int\widetilde{f}(u_2,u_3,...,u_n,\tau)\exp\left[2\pi\sum_{j=2}^n u_j x_j\right]\times\end{aligned} \quad (38)$$

$$\times\left\{\int_0^{\infty}\frac{z^2a^2(z)\sum_{j=2}^{n}u_j^2\exp\left[-4\pi^2a^2(z)(t-\tau)\sum_{j=2}^{n}u_j^2\right]}{\left(z^2-a_1^2\right)\left(z^2-a_2^2\right)\left(v^2z^2-a_1^2\right)}dz\right\}du_2...du_n, \tag{38}$$

but

$$\tilde{f}_1(u_2,u_3,...,u_n,p)=\int\cdots\int_{-\infty}^{+\infty} f(\xi_2,\xi_3,...,\xi_n,\tau)\exp\left[-2\pi\sum_{j=2}^{n}\xi_j\,x_j\right]d\xi_2...d\xi_n. \tag{39}$$

We shall substitute (39) in equation (38). Because $f\in H_0$, we can interchange the integrals (i.e., interchange the order of integration) to get:

$$\begin{aligned}&\psi(x_2,x_3,...,x_n,t)=f(x_2,x_3,...,x_n,t)+\frac{4\pi^2b^2\mu(|v|-v)\left(a_1^2-a_2^2\right)}{\left(v^2-1\right)^2}\times\\&\times\int_0^{\infty}\left\{\int\cdots\int_{-\infty}^{+\infty} f(u_2,u_3,...,u_n,\tau)\times\int\cdots\int_{-\infty}^{+\infty}\sum_{j=2}^{n}u_j^2\exp\left[-4\pi^2b^2(t-\tau)\sum_{j=2}^{n}u_j^2+2\pi\sum_{j=2}^{n}(x_j-\right.\right.\\&\left.\left.-\xi_j)u_j\right]d\xi_2...d\xi_n\right\}d\tau++8\pi\,a_1\,\mu\left(a_1^2-a_2^2\right)\int_0^t\left\{\int\cdots\int_{-\infty}^{+\infty} f(\xi_2,\xi_3,...,\xi_n,\tau)\times\right.\\&\times\left\{\int_0^{\infty}\frac{z^2a^2(z)}{\left(z^2-a_1^2\right)\left(z^2-a_2^2\right)\left(v^2z^2-a_1^2\right)}\times\int\cdots\int_{-\infty}^{+\infty}\sum_{j=2}^{n}u_j^2\exp\left[-4\pi^2a^2(t-\tau)\sum_{j=2}^{n}u_j^2+2\pi\sum_{j=2}^{n}(x_j-\right.\right.\\&\left.\left.\left.-\xi_j)u_j\right]du_2...du_n\,dz\right\}d\xi_2...d\xi_n\right\}d\tau.\end{aligned} \tag{40}$$

Inner integrals in (40) can be computed exactly. Sequentially computing these integrals gives:

$$\begin{aligned}&\psi(x_2,x_3,...,x_n,t)=f(x_2,x_3,...,x_n,t)+\\&+\lambda\int_0^t\left\{\int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2,\,x_3-\xi_3,...,x_n-\xi_n,t-\tau)f(\xi_2,\,\xi_3,...,\xi_n,)d\xi_2...d\xi_n\right\}d\tau,\end{aligned} \tag{41}$$

where

$$\begin{aligned}&R(x_2-\xi_2,\,x_3-\xi_3,...,x_n-\xi_n,t)=\\&=\frac{4\pi^{3/2}(|v|-v)}{\left(v^2-1\right)^2}\left(\frac{1}{2b\sqrt{\pi t}}\right)^{n-1}\left[(n-1)-\frac{\sum_{j=2}^{n}x_j^2}{2b^2t}\right]\exp\left[-\frac{\sum_{j=2}^{n}u_j^2}{4b^2t}\right]+\\&+\frac{8\pi a_2^2\,\mu\pi^{3/2}}{\left(2b\sqrt{\pi t}\right)^{n+1}}\int_0^{\infty}\frac{z^2}{\left(z^2-a_2^2\right)\left(v^2z^2-a_2^2\right)a^{n+1}(z)}\left[(n-1)-\frac{\sum_{j=2}^{n}x_j^2}{2b^2t}\right]\exp\left[-\frac{\sum_{j=2}^{n}x_j^2}{4a^2(z)t}\right]dz.\end{aligned} \tag{42}$$

When solving integral equation (28), we have assumed that $d_0>0$ or $v\geq 0$. Solving these conditions with respect to λ gives the following solvability conditions:

$$\lambda < \frac{a_1(a_1 + a_2)}{2\pi^{3/2}} = \lambda_0 . \tag{43}$$

One can show that integral equation (28) cannot be solved for any f function, if $\lambda \geq \lambda_0$. Moreover, this equation always has a solution in the class of generalized functions, for arbitrary λ. It is a necessary and sufficient condition for the generalized function which is the solution of equation (28) to be an ordinary function, that λ satisfies (43).

Resolvent integral equation

We shall apply the name resolvent to the function defined by (42). Resolvent satisfies the following integral equations:

$$R(x_2, x_3,...,x_n,t) = K_0(x_2,x_3, ...,x_n,t) + \lambda\int_0^t d\tau \int\cdots\int_{-\infty}^{+\infty} K_0(x_2 - \xi_2,...,x_n - \xi_n,t-\tau) \times \\ \times R(\xi_2, \xi_3,...,\xi_n,\tau) d\xi_2 ... d\xi_n, \tag{44}$$

$$R(x_2, x_3,...,x_n,t) = K_0(x_2,x_3, ...,x_n,t) + \lambda\int_0^t d\tau \int\cdots\int_{-\infty}^{+\infty} R(x_2 - \xi_2,...,x_n - \xi_n,t-\tau) \times \\ \times K_0(\xi_2, \xi_3,...,\xi_n,\tau) d\xi_2 ... d\xi_n. \tag{45}$$

Since a resolvent is defined if and only if $d_0 > 0$ or $v \geq 0$, equations (44), (45) are true if and only if $\lambda \geq \lambda_0$. To prove equation (44), we begin with substituting expression (41) into equation (28). After interchanging the integrals and using the arbitrariness of function $f \in H$, one can arrive at equation (44). Equation (45) can be arrived at in a similar way.

Solving the integral equation in the class of increasing functions

We denote by H_2 the function space with functions $\varphi(x_2,x_3,...,x_n,t)$ that satisfy the following inequalities within $-\infty < x_1 < +\infty$, $0 < t \leq T_0$:

$$|\varphi(x_2,x_3,...,x_n,t)| \leq M \exp[\delta^2 r_1^2(x)] , \tag{46}$$

$$|\varphi(x_2',x_3',...,x_n',t) - \varphi(x_2'',x_3'',...,x_n'',t)| \leq M \exp[\delta^2 m^2]\ r_1^2(|x' - x''|), \tag{47}$$

where we denote by H_2, $r_1(x) = \sqrt{\sum_{i=1}^{n} x_i^2}$, $m = \max[|x_2'|,|x_3'|,...,|x_n'|,|x_2''|,|x_3''|,...,|x_n''|]$.

It is apparent that $H_0 \in H_2$ and that the functions that satisfy inequalities (15), (16) belong to H_2. It is easy to show that if $F(x_2,x_3,...,x_n)$ is a function which satisfies condition (6), then $f(x_2,x_3,...,x_n,t)$ defined by (27) belongs to function space H_2. Now let us consider integral equation (28), when $f(x_2,x_3,...,x_n,t) \in H_2$. We shall seek for a solution of equation (28) in class H_2. Let $\gamma(x_2,x_3,...,x_n,t) \in H_2$. We shall introduce the following notation:

$$K^0\gamma = \gamma - \lambda\int_0^t d\tau \int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2,...,x_n-\xi_n,t-\tau)\,\gamma(\xi_2,...,\xi_n,t-\tau)d\xi_2...d\xi_n, \tag{48}$$

$$K^{0^{-1}}\gamma = \gamma + \lambda\int_0^t d\tau \int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2,...,x_n-\xi_n,t-\tau)\,\gamma(\xi_2,...,\xi_n,t-\tau)d\xi_2...d\xi_n. \tag{49}$$

One can state that the following expressions are true:

$$\begin{aligned} K^0K^{0^{-1}}\gamma &= \gamma + \lambda\int_0^t d\tau \int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2,...,x_n-\xi_n,t-\tau)\,\gamma(\xi_2,...,\xi_n,t-\tau)d\xi_2...d\xi_n - \\ &-\lambda\int_0^t dt_1\int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2',...,x_n-\xi_n',t-t_1)\,\gamma(\xi_2,...,\xi_n,t-t_1)\big[\gamma(\xi_2',...,\xi_n',t_1)+ \\ &+\int_0^{t_1} d\tau\int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2,...,x_n-\xi_n,t_1-\tau)\,\gamma(\xi_2,...,\xi_n,t_1-\tau)d\xi_2...d\xi_n\Big]d\xi_2'...d\xi_n' = \\ &= \gamma + \lambda\int_0^t d\tau \int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2,...,x_n-\xi_n,t-\tau)\,\gamma(\xi_2,...,\xi_n,t-\tau)d\xi_2...d\xi_n - \\ &-\lambda\int_0^t dt_1\int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2',...,x_n-\xi_n',t-t_1)\,\gamma(\xi_2',...,\xi_n',t-t_1)\,d\xi_2',...,d\xi_n' - \\ &-\lambda^2\int_0^t dt_1\int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2',...,x_n-\xi_n',t-t_1)\times\Big[\int_0^t d\tau\int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2',...,x_n-\xi_n',t_1-\tau)\times \\ &\times\gamma(\xi_2,...,\xi_n,\tau)d\xi_2...d\xi_n\big]d\xi_2'...d\xi_n'. \end{aligned} \tag{50}$$

Taking into account that

$$\int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2,...,x_n-\xi_n,t-\tau)d\xi_2...d\xi_n = 0, \tag{51}$$

$$\int\cdots\int_{-\infty}^{+\infty} R(x_2-\xi_2,...,x_n-\xi_n,t-\tau)d\xi_2...d\xi_n = 0, \tag{52}$$

one may prove that:

$$\begin{aligned} &\int_0^t dt_1\int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2',...,x_n-\xi_n',t-t_1)\Big[\int_0^{t_1} d\tau\int\cdots\int_{-\infty}^{+\infty} R_0(\xi_2'-\xi_2,...,\xi_n'-\xi_n,t_1-\tau)\times \\ &\times\gamma(\xi_2,...,\xi_n,\tau)d\xi_2...d\xi_n\big]d\xi_2'...d\xi_n' = \int_0^t dt_1\int\cdots\int_{-\infty}^{+\infty}\gamma(\xi_2,...,\xi_n,\tau)\times \\ &\times\Big[\int_0^t dt_1\int\cdots\int_{-\infty}^{+\infty} K_0(x_2-\xi_2',...,x_n-\xi_n',t-t_1)\,R(\xi_2'-\xi_2,...,\xi_n'-\xi_n,t_1-\tau)\Big]d\xi_2...d\xi_n. \end{aligned} \tag{53}$$

The last expression, actually, represents the interchange formula for two singular integrals. Interchanging the order of integration for two singular integrals usually results in a separation of an additional term. However, this is not true for the case considered here due to the presence of two equalities, (51) and (52).

After substituting expression (53) into formula (50) we get the following:

$$K^0 K^{0^{-1}} \gamma = \gamma + \lambda \int_0^t d\tau \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int \gamma(\xi_2, \ldots, \xi_n, \tau) [R(x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau) -$$

$$- K_0(x_2 - \xi_2, \ldots, x_n - \xi_n, t - \tau) - \lambda \int_0^t dt_1 \int \overset{+\infty}{\underset{-\infty}{\cdots}} \int K_0(x_2 - \xi_2', \ldots, x_n - \xi_n', t - t_1) \times$$

$$\times R(\xi_2' - \xi_2, \ldots, \xi_n' - \xi_n, t_1 - \tau) d\xi_2' \ldots d\xi_n'] d\xi_2 \ldots d\xi_n$$

However, on the basis of resolvent equation (44), the integrand is identically equal to zero. Therefore,

$$K^0 K^{0^{-1}} \gamma = \gamma . \tag{54}$$

Similarly,

$$K^{0^{-1}} K^0 \gamma = \gamma . \tag{55}$$

After establishing the last two equalities, it is easy to solve equation (28), if $f(x_2, x_3, \ldots, x_n, t)$ is an increasing function. Let us rewrite equation (28) using introduced notation (48):

$$K^0 \psi = f(x_2, x_3, \ldots, x_n, t). \tag{56}$$

We shall apply operator $K^{0^{-1}}$ to both sides of the equation. Then, taking into account (54):

$$\psi = K^{0^{-1}} f . \tag{57}$$

Note that (57) represents formula (41) in symbolic notation. Therefore, a solution of equation (28) is determined by formula (41) also when $f \in H_2$, if $\lambda < \lambda_0$.

Conclusion

The basic problem was reduced to determining functions φ and ψ in expressions (10) to (12). As we have already noted, if $f(x_1, x_2, \ldots, x_n)$ satisfies condition (6), then function $F(x_2, x_3, \ldots, x_n, t)$, which is determined by expression (27), belongs to H_2. Therefore, the investigation of the solvability conditions for the problem is reduced to the analysis of the coefficient of equation (25). However, it is easy to see that:

$$\left(a_1 k_2 \left(a_1^2 - a_2^2\right)\right) / \left(2\pi^{3/2} \left(a_1 k_2 - a_2 k_1\right)\right) < \left(a_1 \left(a_1 + a_2\right)\right) / \left(2\pi^{3/2}\right). \tag{58}$$

Therefore, function $\psi(x_2, x_3, \ldots, x_n, t)$ is determined by formula (41), and function $\varphi(x_2, x_3, \ldots, x_n, t)$ is determined by formula (18). On this basis, a solution of the initial problem is completely determined.

It should be noted that in the case considered:

$$v = \left(\left(a_1^2 - a_2^2\right) - 2\pi^{3/2} \lambda\right) / \left(2\pi^{3/2} \lambda\right) = \left(a_2 k_1\right) / \left(a_1 k_2\right) > 0 . \tag{59}$$

Then the resolvent can be represented with formula (42), and the first term in this expression becomes equal to zero. Correspondingly, seeking the function $\psi(x_2, x_3, ..., x_n, t)$ is simplified.

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ІНТЕГРАЛЬНА ДИНАМІЧНА МОДЕЛЬ РОЗПОВСЮДЖЕННЯ ТЕПЛА У ШМАТКОВО-ОДНОРІДНИХ СЕРЕДОВИЩАХ ДЛЯ ВИПАДКУ БАГАТОВИМІРНОГО ПРОСТОРУ

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Розглянуто розв’язання рівняння теплопровідності для шматково-однорідних середовищ у випадку багатовимірного простору. Вихідну задачу було зведено до інтегрального рівняння (суть динамічної моделі), розв’язок якого отримано в класі зростаючих функцій. Визначено умови, за яких задача має розв’язок.

Ключові слова: рівняння теплопровідності, шматково-однорідні середовища, інтегральна динамічна модель, резольвента, функціональний простір.

ИНТЕГРАЛЬНАЯ ДИНАМИЧЕСКАЯ МОДЕЛЬ РАСПРОСТРАНЕНИЯ ТЕПЛА В КУСОЧНО-ОДНОРОДНЫХ СРЕДАХ ДЛЯ СЛУЧАЯ МНОГОМЕРНОГО ПРОСТРАНСТВА

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Рассмотрено решение уравнения теплопроводности для кусочно-однородных сред в случае многомерного пространства. Исходная задача сведена к интегральному уравнению (суть динамической модели), решение которого получено в классе возрастающих функций. Определены условия разрешимости задачи.

Ключевые слова: уравнение теплопроводности, кусочно-однородная среда, интегральная динамическая модель, резольвента, функциональное пространство.