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OPTIMIZATION OF PARAMETERS IN SELF-ORGANIZATING SYSTEMS

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In article there are reviewed models of systems with self-organization, which use system's conditions in previous moments of time. The conditions are included in system as linear combinations with coefficients, which are determined. Such linear combinations could be reviewed as stabilizing controls according to the principle of feedback. As the controls could be chosen not in one way, it is necessary to give additional conditions on controls' properties as criteria of optimization. There is offered a new criteria for parametric optimization of modeling of self-organizing nonlinear systems. There is shown an algorithm for constructing such parametric controls in the low-order systems. There are given examples.

Keywords: nonlinear dynamic systems, optimal stabilization, self-organization, modeling

Introduction

One of the fundamental properties of self-organization [1] (in technical, biological, economic, social systems, etc.) is the ability to copy or reproduce completed structures in the process of evolution. Such versatility suggests the copy process is caused by relatively simple properties of systems which generate it. It also gives hope to building a relatively simple model of the studied phenomenon by using nonlinear dynamical systems of a special type. Error can accumulate if templates are reproduced many times, this error leads to partial or complete loss of templates due to nonlinearity of dynamical system. We can achieve that the template will start to recover, if enter accounting of prehistory of system structure. Prehistory is taken into account as a linear combination or mixing with the given parameters in a special way. The template can be considered as the position of equilibrium of the system, and prehistory – as control according to principle of feedback which aims to stabilize the unknown position of equilibrium. One of the most efficient methods among many ones of local stabilization of unknown cycles or positions of equilibrium (e. g. the review [2]) is the Pyragas method [3]. However it has significant disadvantages [4]. A modification of the Pyragas method is proposed in [5]. Note that the set of admissible stabilizing controls generally consist in more than one element. That is why it is advisable to add criteria in the form of additional requirements for controls.

The goal of this paper is a constructing a mathematical model of systems with self-organization, in which disturbed (for whatever reason) process of reproducing stationary forms that would be restored with the greatest speed.

The task. Construct an algorithm for computing the model parameters for which the value of the recovery rate of original template adopted as the optimality criterion would be the maximum under given constraints. It is assumed that the evaluation of the rate of template loss is known.

Main part

We assume that the copy of template in the system without self-organization is carried out due to dynamic system of the form:

$$x_{n+1} = f(x_n), \tag{1}$$

where $x_n \in R^m$, $f : A \to A \subset R^m$, is a continuous vector function given on the invariant set A. It is known [6] that a set is called invariant one for system (1) if $x_0 \in A$ implies $f(x_0) \in A$.

Exact reproducing of template means that if $x_0 = X$ then $x_1 = f(x_0) = X$, $x_{i+1} = f(x_i) = X$ for all $i = 1,2,\ldots$. Thus, $x_n \equiv X$ is a position of equilibrium of system (1) and the correct copying means local asymptotic steadiness of this position of equilibrium. In this turn, local asymptotic steadiness of under review of position of equilibrium means that all of eigenvalues of the Jakobi matrix $f'(X)$, called as multipliers, are contained in the central unit circle. If this condition breaks down then any arbitrarily small error in determination of intermediate copies x_n after a few steps will lead to the fact that the system cannot reproduce the template. Moreover, this template can be lost totally.

To ensure the local asymptotic steadiness of the position of equilibrium in original system we can use the structure with self-organization which recovers the template by principle of feedback. For example:

$$x_{n+1} = f\left(\sum_{j=1}^{N} a_j x_{n-j+1}\right), a_j \ge 0,\ j = 1,\ldots,N, \sum_{j=1}^{N} a_j = 1. \tag{2}$$

If $x_{n-j+1} = X$, $j = 1,\ldots,N$, then $\sum_{j=1}^{N} a_j x_{n-j+1} = X$, as convex combination with weights a_j, $j = 1,\ldots,N$, и $f(X) = X$.

The values of parameters a_j and the value N are determined ambiguously and depend on system's multipliers. In [7] it is suggested an algorithm of this values' choice which minimize the number N. The article suggests an improvement of this algorithm such that maximizes the rate of process of template's recover. This algorithm is presented for case $N = 2$. I.e. when system (2) has form:

$$x_{n+1} = f(a_1 x_n + a_2 x_{n-1}),\ a_1 \ge 0, a_2 \ge 0,\ a_1 + a_2 = 1. \tag{3}$$

To illustrate the algorithm idea let us at first consider simple linear system of the form:

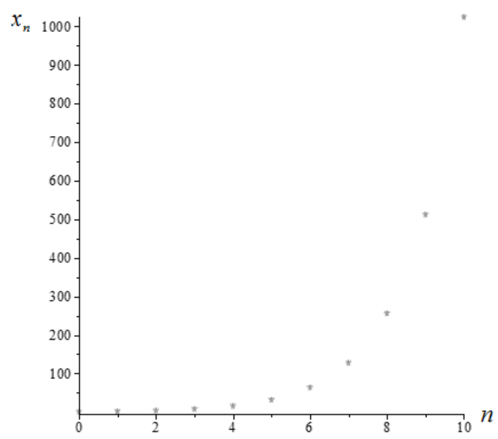
$$x_{n+1} = 2x_n. \tag{4}$$

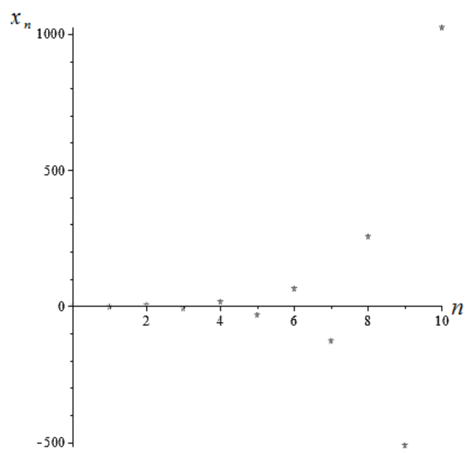
System (4) has the position of equilibrium $x_n \equiv 0$.

Without using of control solutions of this system form a divergent sequence $\{2^n x_0\}$ with rate of increasing is 2^n. (Fig. 1, a).

If we enter a control with parameters $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}$, then the system takes the form

$$x_{n+1} = x_n + x_{n-1}. \tag{5}$$





a b

Fig. 1. Behavior of dynamical system: a - $x_{n+1} = 2x_n$; b - $x_{n+1} = -2x_n$

I.e. solutions (5) form famous Fibonacci sequence which diverges with lower rate: $\left(\frac{\sqrt{5}-1}{2}\right)^n$, because $\frac{\sqrt{5}-1}{2} < 2$.

Choosing different values for a_1 and a_2 we can minimize the rate of divergence of process but we can't make it convergent.

The situation will change principally if system (1) has the form

$$x_{n+1} = -2x_n. \tag{6}$$

Let's find a common solution of equation (6): $x_n = (-2)^n x_0$. The sequence $\{(-2)^n x_0\}$ is illustrated in Fig. 1,(b).

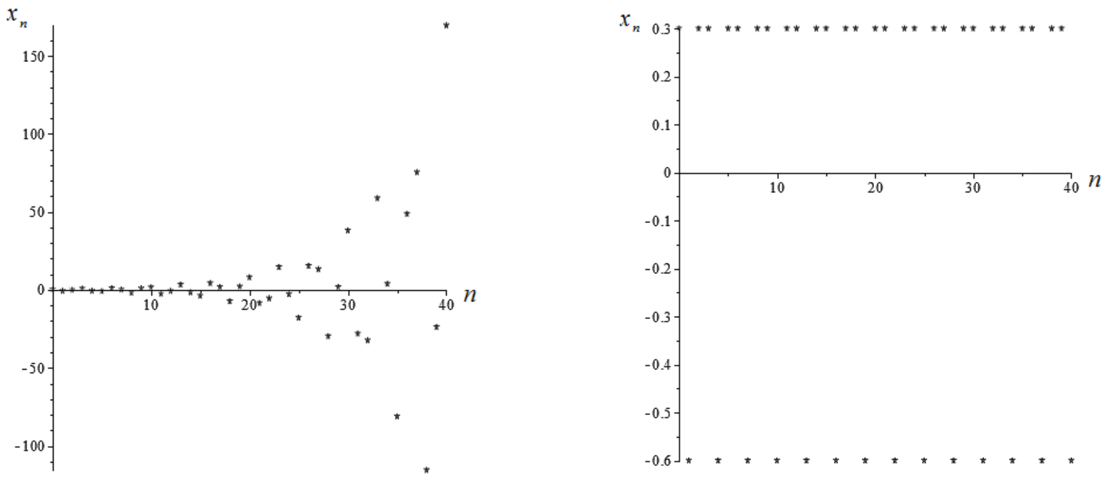
In this case, if we chose parameters $a_1 = \frac{1}{3}, a_2 = \frac{2}{3}$ then the sequence of solutions of equation $x_{n+1} = -2\left(\frac{1}{3}x_n + \frac{2}{3}x_{n-1}\right)$ will diverge, although more slowly, then original (Fig. 2, a). If we enter control with parameters $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}$, the sequence of solutions of the equation $x_{n+1} = -x_n - x_{n-1}$ will be bounded (Fig. 2, b). By choosing parameters $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$, we will get a sequence of solutions of the equation $x_{n+1} = -2\left(\frac{2}{3}x_n + \frac{1}{3}x_{n-1}\right)$ which will be convergent (Fig. 3).

Thus, the use of averaging over time of the linear system (path) reduces the rate of divergence and for (6) makes a divergent process from vergent.

For realization of suggested idea of averaging over system's path, in common case of a nonlinear system, let us apply a method of linearization [6]. For research of stability of position of equilibrium $x_n \equiv X$ of system (2) let us consider the linearized system

$$x_{n+1} = f'(x)\sum_{j=1}^{N} a_j x_{n-j+1}, \tag{7}$$

where $f'(X)$ is the Jakobi matrix, which is calculated at position of equilibrium X.



a b

Fig. 2. Using of control with parameters for system (6): a - $a_1 = \frac{1}{3}, a_2 = \frac{2}{3}$; b - $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}$

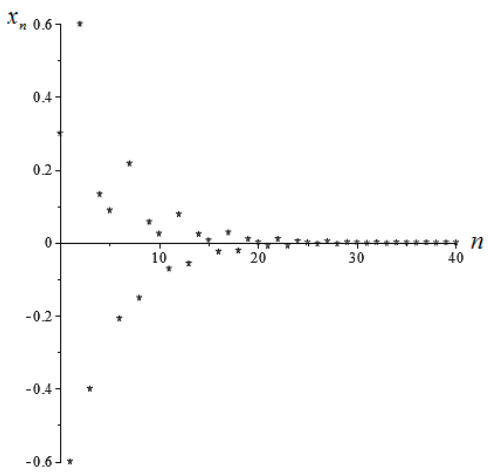


Fig. 3. Using of control with parameters $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ for system (6)

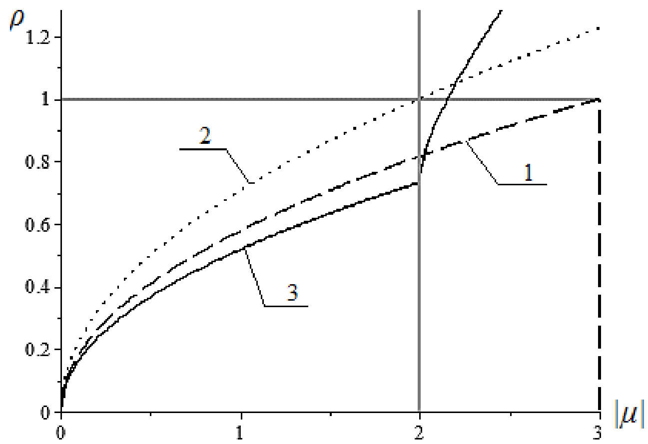


Fig. 4. Comparison of dependency ρ on $|\mu|$ with different values a_1, a_2: 1 – with $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$, 2 – with $a_1 = a_2 = \frac{1}{2}$, 3 – with $a_1 = \sqrt{3} - 1, a_2 = 2 - \sqrt{3}$.

The characteristic equation for linearized system in a neighborhood of the equilibrium position X of system (3) can be written as [8]:

$$\det\left[\lambda^N E - f'(X)\sum_{j=1}^{N} a_j \lambda^{N-j}\right] = 0. \qquad (8)$$

For our case, $N = 2$, we get

$$\prod_{j=1}^{m}\left[\lambda^2 - \mu_j(a_1\lambda + a_2)\right] = 0. \qquad (9)$$

where $\mu_j, j = 1,\ldots,m$ are the eigenvalues of matrix $f'(X)$.

Let any of the multipliers $f'(X)$ be more than 1. Let us chose it as μ_1. In this case the equation

$$\lambda^2 - \mu_1(a_1\lambda + a_2) = 0 \tag{10}$$

has necessarily a root greater than 1, because the left side of equation at $\lambda = 1$ that is less than 0, and at $\lambda = 2\mu_1$ we can estimate the left side in the following way: $4\mu_1{}^2 - \mu_1(a_1 2\mu_1 + a_2) \geq 2\mu_1^2 - \mu_1$, because $a_1 2\mu_1 + a_2 < 2\mu_1 + 1$.

Then $2\mu_1^2 - \mu_1 > 0$ at $\mu_1 > 1$ and, by Rolle's theorem, we get that on the interval $(1.2\mu_1)$ the equation (10) has a root greater than 1. That's why it is inexpedient for this task to consider the case $\mu_j > 1$. If all of $\mu_j \in (-1,1)$, then the position of equilibrium is stable without the use of control. If $\mu_j \in (-3,-1)$, $j = 1,\ldots,m$, then there exist [6] a_1, a_2, such that the equation (3) at these a_1, a_2 has all of its roots in the central unit circle, i.e. there exist a_1, a_2 such that if $\lambda^2 - \mu_j(a_1\lambda + a_2) = 0$, $\mu_j \in (-3,-1)$, then $|\lambda| < 1$. It will, for example, at $a_1 = \dfrac{2}{3}, a_2 = \dfrac{1}{3}$.

Let μ^* be the largest among all absolute values of eigenvalues of Jakobi matrix $f'(X)$, which is calculated in position of equilibrium X. μ^* is called the spectral radius of matrix $f'(X)$. The value μ^* defines the template loss rate.

The convergence rate of perturbed solutions to position of equilibrium is defined by the maximum value of the modules of the roots of characteristic equation (9). If $\mu^* \approx -3$ then the coefficients are $a_1 \approx \dfrac{2}{3}, a_2 \approx \dfrac{1}{3}$. Then the maximum root of characteristic equation (9) at module, for example, at $\mu^* = -2.99$ equals 0.998, and the convergence rate therefore equals $\dfrac{1}{0.998} \approx 1.02$.

Let us define ρ as distance from the origin to the most distant root of characteristic equation (9) then let us accept that the rate of template recover equals ρ^{-1}.

Let us define a dependence of this value on value of the template loss rate μ^* and parameters of the system a_1, a_2 ($a_1 + a_2 = 1$). In case $N = 2$ we will get a quadratic equation which we can solve explicitly and find a function $\rho(\mu)$. Considering that $\mu < 0$ we will use the function $\rho(|\mu|)$.

We have

$$\rho(|\mu|) = \begin{cases} \sqrt{|\mu|(1-a_1)}, & |\mu| < \dfrac{4(1-a_1)}{a_1^2} \\ \dfrac{1}{2}\left(a_1|\mu| + \sqrt{a_1^2\mu^2 - 4(1-a_1)|\mu|}\right), & |\mu| \geq \dfrac{4(1-a_1)}{a_1^2}. \end{cases} \tag{11}$$

Let us note that for N larger than 4, it is impossible to write out the dependency $\rho(|\mu|)$ explicitly.

Figure 6 shows a chart $\rho(|\mu|)$ at $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ by a dotted line, at $a_1 = a_2 = \frac{1}{2}$ by points and at $a_1 = \sqrt{3} - 1, a_2 = 2 - \sqrt{3}$ by a solid line.

Let us note that if $|\mu| > 3$ then at $N = 2$ it will necessarily need $\rho(|\mu|) > 1$, that's why in this case in a system with self-organization N must be larger than 2. The coefficients $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ provide the template recover for any of $|\mu^*| < 3$ but not always with maximum rate (Fig. 4).

Let us formulate the task in the following way: the evaluation of template loss rate $\mu \in (-\mu^*, -1]$ is given and desired recovery rate ρ^{-1} is given too. Find the minimum N and coefficients $a_1, \cdots, a_N$ which provide the template recovery with the recovery rate ρ^{-1} at known loss rate of a template μ^* and for any of $\mu \in (-\mu^*, -1]$.

We can give an alternative formulation of the problem. Let μ^* and N be given. It is necessary to maximize the template recovery rate ρ^{-1} through coefficients $a_1, \cdots, a_N$ choosing them from the set $\left\{ a_j \geq 0, j = 1, \cdots, N, \sum_{j=1}^{N} a_j = 1 \right\}$.

Let us proceed to solving problems.

So, let $\rho(|\mu|)$ be modulo the maximal root of the equation

$$\lambda^2 - \mu(a_1\lambda + a_2) = 0, \ \mu < 0. \tag{12}$$

Let us define the value

$$\|\rho(\mu^*)\| = \max_{\mu \in [0, \mu^*]} \{\rho(|\mu|)\}.$$

It is required to minimize $\|\rho(\mu^*)\|$ by parameters a_1, a_2 choosing them from the set $\{(a_1, a_2) : a_1 + a_2 = 1, a_1 \geq 0, a_2 \geq 0\}$.

At $N = 2$ $\|\rho(\mu^*)\| = \rho(\mu^*)$ we get the minimum of the value $\rho(\mu^*)$ at condition of equality of equation roots (12), i.e. when the correlation $\mu^* = \frac{4(1 - a_1)}{a_1^2}$ is performed.

Then

$$\rho^* = \min_{a_1, a_2} \rho(\mu^*) = \sqrt{\frac{4(1 - a_1)^2}{a_1^2}} = \frac{2(1 - a_1)}{a_1}, \tag{13}$$

where from

$$a_1 = \frac{2}{\rho^* + 2}. \tag{14}$$

Since

$$a_2 = 1 - a_1, \text{ then} \tag{15}$$

$$a_2 = \frac{\rho^*}{\rho^* + 2}. \quad (16)$$

Put (14) in (11):

$$\rho^* = \sqrt{\mu^*(1 - \frac{2}{\rho^* + 2})}, \text{ where from } \rho^{*2} = |\mu^*| \frac{\rho^*}{\rho^* + 2},$$

then:

$$\mu^* = \rho^{*2} + 2\rho^*. \quad (17)$$

From (17) follow that

$$\rho^* = -1 + \sqrt{1 + \mu^*}. \quad (18)$$

Thus we explicitly get dependency of the maximal template recovery rate $\frac{1}{\rho^*}$ on loss rate of template μ^*

$$\frac{1}{\rho^*} = \frac{1}{-1 + \sqrt{1 + \mu^*}} = \frac{1}{\mu^*}\left(1 + \sqrt{1 + \mu^*}\right), \quad (19)$$

and formulas for optimal coefficients too

$$a_1^* = \frac{2(-1 + \sqrt{1 + \mu^*})}{\mu^*}, \; a_2^* = \frac{\mu^* - 2(-1 + \sqrt{1 + \mu^*})}{\mu^*}. \quad (20)$$

Examples

As an example of applying the algorithm with the use of self-organized system let us consider the system with logistic function $f(x) = x \to 4x(1 - x)$ which keeps point $X = 0.75$ on any step of the copying (Fig. 5). Herein $\mu^* = |f'(0.75)| = |-2| = 2$. Let us consider that the value of point step i was defined with on error, i.e. for example $x_i = 0.7500001$. Then last copies will be much different than the template.

The system of self-organization must recover the lost template. If we accept parameters' values as $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ then from (11) it follows that the rate of template recovery equals $\frac{1}{\rho} \approx 1.225$. Let us find values of parameters a_1, a_2, in which the template is recovering with maximal rate. From (18), (19) in $\mu^* = 2$ we will get $a_1 = \sqrt{3} - 1 \approx 0.732$, $a_2 = 2 - \sqrt{3} \approx 0.238$, $\rho^* = 0.732$, $\frac{1}{\rho^*} = \frac{1}{2}(1 + \sqrt{3}) \approx 1.366 > 1.225$.

The exact recovery of template occurs on the interval $n \in [1,10]$. It is expected that the value of x_{10} is defined with an error of 10^{-7}. Thus on interval $n \in [10, 40]$ the process of full template loss is demonstrated. The control is activated at the moment $n = 40$ in form of system of self-organization. On interval $n \in [40, 60]$ the process of template recovery occurs. For self-organization the following parameters are used $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ (process is shown by *asterisk*) and $a_1 = \sqrt{3} - 1$, $a_2 = 2 - \sqrt{3}$ (process is shown by *circle*). It is seen that parameters' values $a_1 = \sqrt{3} - 1$, $a_2 = 2 - \sqrt{3}$ are more efficient than values $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ in the sense of the rate of convergence.

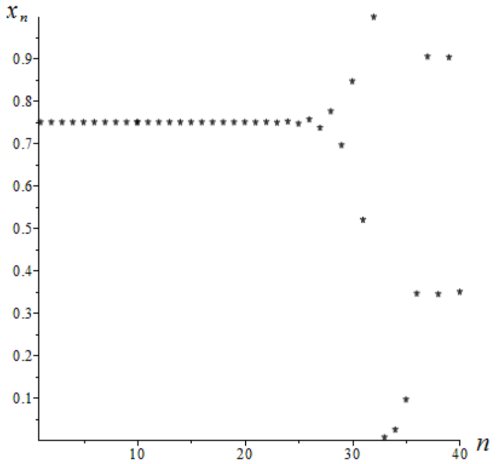


Fig. 5. Result of the copying without errors and with the error 10^{-7} which occurred on 10 step of the copying

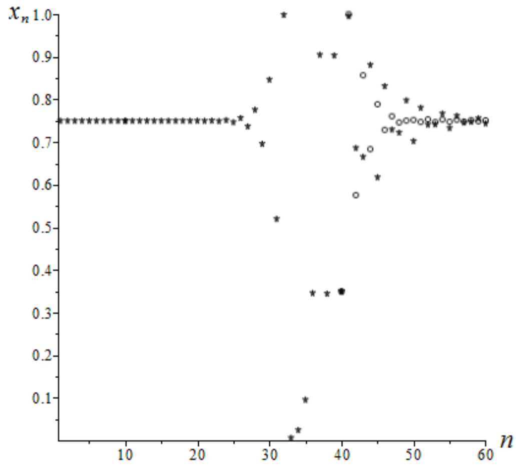


Fig. 6. The dynamic of copy process which described by the equations (1) and (2)

In the next example let us consider the tent map [9] $f(x) = x \to -2\left|x - \frac{1}{2}\right| + 1$ in which the point $X = \frac{2}{3}$ is kept (Fig. 7). Wherein $\mu = -2$, the point's value is defined with an error e.g. $x_i = X + 0.0000001$.

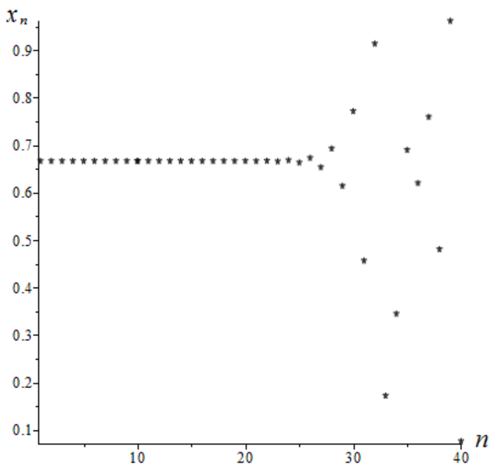


Fig. 7. The result of the copying without errors and with the error 10^{-7} which occurred on 10 step of the copying for the tent map

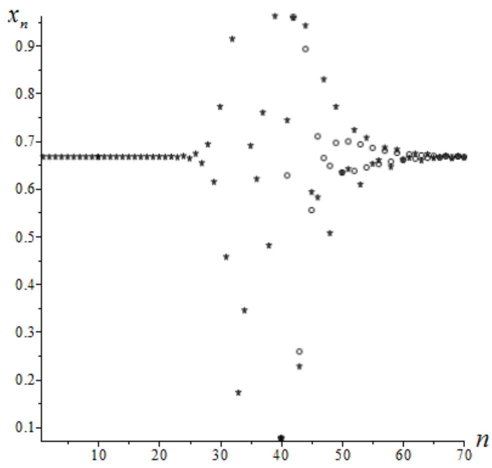


Fig. 8. The dynamic of copy process which described by the equations (1) and (2) for the tent map

Let's consider the vector case model of Markov-Ferhulst [10]

$$\begin{cases} x_{n+1} = c_{11} f(x_n) + c_{12} f(y_n), \\ y_{n+1} = c_{21} f(x_n) + c_{22} f(y_n), \end{cases} \tag{21}$$

where $f(x) = 4x(1-x)$, c_{ij} are elements of matrix $C = \begin{pmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{pmatrix}$. The vector $\begin{pmatrix} 0.75 \\ 0.75 \end{pmatrix}$ is the position of equilibrium of the system (21). Wherein $\mu^* = |f'(0.75)| = |-2| = 2$. We use the control with parameters for self-organization $a_1 = \sqrt{3} - 1, a_2 = 2 - \sqrt{3}$ (Fig. 9) – process is shown by symbols • and the control with parameters for self-organization $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$ – process is shown by symbols *.

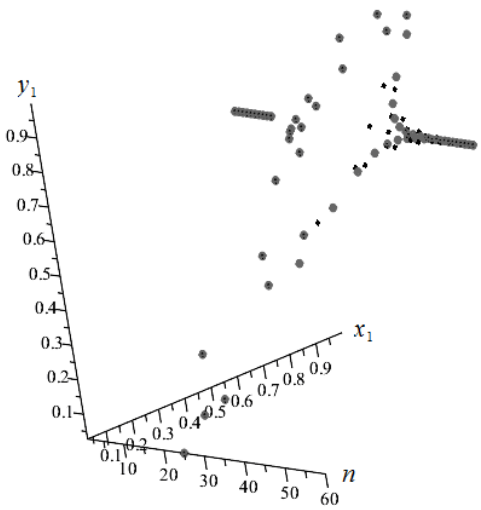


Fig. 9. The dynamic of the copy process for a vector case: • - with parameters $a_1 = \sqrt{3} - 1$, $a_2 = 2 - \sqrt{3}$; * - with parameters $a_1 = \frac{2}{3}, a_2 = \frac{1}{3}$

The convergence of the template recovery process occurs with a larger rate for values of parameters $a_1 = \sqrt{3} - 1, a_2 = 2 - \sqrt{3}$.

Conclusion

This paper introduces an algorithm of definition of parameters' values in system of self-organization, which provide the maximal rate of recovering of the damaged template. The structure of a mathematical model is suggested for describing of systems of self-organization. For choosing of model's parameters is suggested a criteria: the maximum of rate of the convergence of recovering the template. The limit value of convergence rate is defined by scatter of system's multipliers. The algorithm is formulated for definition of system's parameters which optimize the rate of recovering the template with given value of scatter of multipliers. In particular for the logistic map we found the optimal coefficients and we show that the rate of recovering the template could be increased by 11.5%. The same result was obtained the Markov-Ferhulst model.

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ОПТИМІЗАЦІЯ ПАРАМЕТРІВ В СИСТЕМАХ ОПТИМИЗАЦІЇ З САМООРГАНІЗАЦІЄЮ

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У статті розглядаються моделі систем з самоорганізацією, в яких використовуються стани системи в попередні моменти часу. Ці стани входять в систему у вигляді лінійних комбінацій з коефіцієнтами, що підлягають означенню. Ці лінійні комбінації можна розглядати як стабілізуючі управління за принципом зворотнього зв'язку. Так як ці управління можна обирати не єдиним способом, необхідно накладати допоміжні умови на властивості управлінь у вигляді критеріїв оптимізації. Запропоновано новий критерій параметричної оптимізації моделювання нелінійних систем, що саморганізуються. Вказано алгоритм конструювання таких параметричних управлінь в системах малих порядків. Приведені приклади.

Ключові слова: нелінійні динамічні системи, оптимальна стабілізація, самоорганізація, моделювання

ОПТИМИЗАЦИЯ ПАРАМЕТРОВ В САМООРГАНИЗУЮЩИХСЯ СИСТЕМАХ

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В статье рассматриваются модели систем с самоорганизацией, в которых используются состояния системы в предшествующие моменты времени. Эти состояния входят в систему в виде линейных комбинаций с коэффициентами, подлежащими определению. Эти линейные комбинации можно рассматривать как стабилизирующие управления по принципу обратной связи. Так как эти управления можно выбирать не единственным образом, необходимо накладывать дополнительные условия на свойства управлений в виде критериев оптимизации. Предложен новый критерий параметрической оптимизации моделирования самоорганизующихся нелинейных систем. Указан алгоритм конструирования таких параметрических управлений в системах малых порядков. Приведены примеры.

Ключевые слова: нелинейные динамические системы, оптимальная стабилизация, самоорганизация, моделирование