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H. M. Hubal
Lutsk national technical university

SOME LINEAR MATHEMATICAL MODELS IN ECONOMICS

Hubal H. M. Some linear mathematical models in economics. A linear mathematical model of a diversified economy and a linear mathematical model of exchange are explored in this paper. Mathematical analysis of these models is carried out.
Keywords: model of a diversified economy, model of exchange, technological matrix, matrix of trade.
Bibl. 3.

Губаль Г. М. Деякі лінійні математичні моделі в економіці. У статті досліджено лінійну математичну модель багатогалузевої економіки і лінійну математичну модель обміну. Здійснено математичний аналіз цих моделей.
Ключові слова: модель багатогалузевої економіки, модель обміну, технологічна матриця, матриця торгівлі.
Літ. 3.

Губаль Г. Н. Некоторые линейные математические модели в экономике. В статье исследовано линейную математическую модель многоотраслевой экономики и линейную математическую модель обмена. Проведено математический анализ этих моделей.
Ключевые слова: модель многоотраслевой экономики, модель обмена, технологическая матрица, матрица торговли.
Лит. 3.

Introduction. Linear mathematical models of a diversified economy and exchange are explored in this paper. We use mathematical apparatus of the theory of matrices to construct and to solve systems of linear algebraic equations.

The main part. All linear mathematical models have the properties of additivity and homogeneity. Additivity means that if the variable x_1 creates the effect γ_1 on its single use, the variables x_1, x_2 create the effect $\gamma_1 + \gamma_2$ on their joint use. Homogeneity means that if the variable x_1 creates the effect γ_1, then, for an arbitrary real number λ, the variable λx_1 creates the effect $\lambda\gamma_1$.

A linear mathematical model of a diversified economy. Consider a simplified economic mathematical model of inerindustry balance [1, 2]. A linear mathematical model in an economy asuumes that an economy consists of a number of interacting industries each consuming the products, including its own, and producing another, and that all these industries are connected with the final demand for final consumption goods. This assumption is a simplified reflection of the labour division between successive phases of production, such as production and processing of raw materials, transportation of the finished product to the market. By using the linear equations and the coefficients of proportionality a_{ij} depending on industry technology we can define how many products every industry has to produce to satisfy its demand, the demand of another industries and population that are supposed to be known. Connection between industries is displayed in the tables of interindustry balance, a mathematical model that allows to analyze them being recommended to connect separate industries with consumer demand, to determine the optimal prices and to calculate the maximum profit.

We use the matrix method for solving systems of linear algebraic equations in a mathematical model of a diversified economy.

The purpose of balance analysis is to answer the question that arises in macroeconomics and connected with the efficiency of conducting the diversified economy: what ought the production balance of each of *n* industries to be for all needs for products of the industry given to be satisfied? However, every industry act, on the one hand, as a manufacturer of products and, on the other hand, as a consumer of its products and products produced by other industries. For example, the automotive industry by the steel in the steel undustry, the tires in the rubber industry, the electric power in the electric power industry, etc.

Consider a static mathematical model of the economy (the static means that in the review period we do not take into account any change in time). The model is linear. This allows us to determine the amount of products that is necessary to satisfy the market demand and to determine the prices for it through added magnitudes.

Assume that *n* types of products are are produced, purchased, consumed and invested, *n* types of industry each producing their products are considered in an economic system. One part of products is for consumption by this industry and other ones, the other part being for sale (consumption) in non-production sphere.

Let an economic production system consist of *n* industries, i.e. produce *n* types of products.

Consider the production process for a certain time period, for example, for a year.

We display the connection between industries by the scheme of production and distribution interindustry balance shown in table 1.

Industry production	Production distribution between industries						Sum for production needs $\sum$	The volume of final production	The gross volume of production
	1	2	…	j	…	n			
1	x_{11}	x_{12}	…	x_{1j}	…	x_{1n}	$\sum_{j=1}^{n} x_{1j}$	Y_1	X_1
2	x_{21}	x_{22}	…	x_{2j}	…	x_{2n}	$\sum_{j=1}^{n} x_{2j}$	Y_2	X_2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
i	x_{i1}	x_{i2}	…	x_{ij}	…	x_{in}	$\sum_{j=1}^{n} x_{ij}$	Y_i	X_i
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	x_{n1}	x_{n2}	…	x_{nj}	…	x_{nn}	$\sum_{j=1}^{n} x_{nj}$	Y_n	X_n

Table 1

Denote the total (gross) volume of the *i*-th industry production by X_i, the volume of the *i*-th industry products ($i=\overline{1,n}$) consumed (wasted) by the *j*-th industry in production process by x_{ij} $(i,\ j=\overline{1,n})$, the volume of the final products of the *i*-th industry for non-production consumption by Y_i $(i=\overline{1,n})$.

Note that the gross volume of production is breafly called the gross product. The volume of the final product for non-production consumption is breafly called the friendly product.

Indicators given above, namely X_i, x_{ij}, Y_i $(i,\ j=\overline{1,n})$, can be expressed in terms of natural units (thing, ton, litre, barrel, etc.), as well as value units. Depending on this there are natural or value interindustry balance. From an economic point of view, interindustry balance is more important in terms of value. In particular, it allows us to combine the industries ino groups or subgroups (for example, the oil industry and the gas industry combining into the oil-and-gas industry) that facilitates the preparation of product balances.

Thus, if the *i*-th indutry produces just the necessary amount of product, we obtain the balance equation for the *i*-th industry:

$$X_i = x_{i1} + x_{i2} + \cdots + x_{in} + Y_i = \sum_{j=1}^{n} x_{ij} + Y_i,$$

namely the gross volume of the *i*-th industry production is equal to the total volume of the products consumed by *n* industries and the final product. This equation shows the possibility that the *i*-th undustry uses some of its products (i.e. x_{ii}).

Thus, balance principle of communication in various industries is that the gross volume of any *i*-th industry production is equal to the sum of the consumprion volumes in production and non-production spheres, i.e.:

$$\begin{cases} X_1 = x_{11} + x_{12} + \cdots + x_{1n} + Y_1, \\ X_2 = x_{21} + x_{22} + \cdots + x_{2n} + Y_2, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ X_i = x_{i1} + x_{i2} + \cdots + x_{in} + Y_i, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ X_n = x_{n1} + x_{n2} + \cdots + x_{nn} + Y_n, \end{cases}$$

or

$$X_i = \sum_{j=1}^{n} x_{ij} + Y_i, \quad i = \overline{1, n}. \tag{1}$$

Let us consider value interindustry balance with all the magnituted involved in its relations have the value expression.

Let us introduce the coefficients of direct material costs. If the *j*-th industry plans to produce X_j production units, it is necessary to know how many production units the *i*-th industry will need. It is evident that the answer depends on the technology of the industry given. We assume that the volume of the *i*-th industry production required for manufactiring X_j production units is directly proportional to X_j, namely $x_{ij} = a_{ij} X_j$, whence the coefficient of proportionality

$$a_{ij} = \frac{x_{ij}}{X_j}, \quad i,\ j = \overline{1, n}, \tag{2}$$

where a_{ij} is the coefficient of direct maerial costs of the *i*-th industry production per unit of the gross volume of the *j*-th industry production. a_{ij} depends on the *j*-th industry technology. This means linear dependence of material costs x_{ij} on the gross volume of the products X_j. Therefore, the model of interindustry balance built on this basis is linear.

These coefficients form a square matrix of coefficients of direct material costs (the technological matrix):

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

The matrix A describes a technology in a single intensities of all industries. It is evident that the *j*-th industry works with intensity X_j, $j = \overline{1, n}$. We assume that the matrix A is a constant – technology is considered to be unchanged for some period (for example, for a year).

Taking into account (2), we rewrite the system of equations (1) in the form:

$$X_i = \sum_{j=1}^{n} a_{ij} X_j + Y_i, \quad i = \overline{1, n}$$

or in the matrix form:

$$X = AX + Y, \tag{3}$$

where *A* is the matrix of direct material costs, $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ is the single-column matrix of the gross volumes of products, $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ is the single-column matrix of the volumes of final products.

The equation (3) is the equation of linear interindustry balance.

The main task of interindustry balance is to find the single-column matrix of the gross volumes of products X that provides the given single-column matrix of the volumes of final products Y if the matrix of direct material costs A is known.

We write the equation (3) in the form:

$$(E-A)X=Y, \tag{4}$$

where E is the identity matrix.

If the matrix $E-A$ is non-singular, we can write the equation (4) in the form:

$$X=(E-A)^{-1}Y, \tag{5}$$

where $(E-A)^{-1}=S$ is the matrix of total material costs with each element s_{ij} that is the gross volume of the i-th industry production required to provide producing of the j-th industry final production unitsi ($i,\ j=\overline{1,n}$).

According to the economic content of the problem, the values of X_i ought to be non-negative if the values of Y_i and a_{ij} ($i,\ j=\overline{1,n}$) are non-negative.

The interindustry balance equation can be also used in the case when the single-column matrix of the gross volumes of the products X and the matrix of direct material costs A are known; we have to find the single-column matrix of the volumes of final products Y.

The same type of analysis can be applied to determine prices. The technological coefficient a_{ij} can be considered as the number of the i-th industry production units reqiured to produce the j-th industry production units. Let p_j be the price of the j-th industry production unit. Then the cost of material inputs required to manufacture the j-th industry production unit is equal to $a_{1j}p_1+\cdots+a_{nj}p_n$. The difference r_j between the price of the j-th industry production unit and the cost of material inputs required to manufacture the j-th industry production unit is the added value of the j-th industry. Thus, $r_j=p_j-\sum_{i=1}^{n}a_{ij}p_i$. The added value can include the cost of labour, depreciation, taxation, profit, etc.

If, for any non-negative single-column matrix Y and for a non-negative matrix A, there exists a non-negative solution X of the equation (4), then the non-negative matrix A is productive.

Note that the matrix A is non-negative if all its elements are non-negative.

If the maximum of the sums of column elements of the matrix A is not greater than one, at least for one of columns, the sum of elements is strictly less than one, then the matrix A is productive.

The interindustry balance equation is used to plan and forecast production.

The interindustry balance analysis is widely used by all leading economically developed countries.

For example, we consider a conventional production system that consists of three industries producing one type of products of volume X_i ($i=\overline{1,3}$) units, i.e. the single-column matrix of gross volumes of production is $X=\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}$. To ensure production, every industry uses a part of products produced by itself and by related industries. Let us find the single-colum matrix of the gross volumes of production X if

$$Y=\begin{pmatrix}60\\70\\30\end{pmatrix},\quad A=\begin{pmatrix}0,05 & 0,35 & 0,4\\0,1 & 0,1 & 0,4\\0,2 & 0,1 & 0,2\end{pmatrix},$$

where Y is the single-column matrix of the volumes of final production, A is the technological matrix.

To use the formula (5), we find the matrix $(E-A)^{-1}$.

$$E-A=\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}-\begin{pmatrix}0,05 & 0,35 & 0,4\\0,1 & 0,1 & 0,4\\0,2 & 0,1 & 0,2\end{pmatrix}=\begin{pmatrix}0,95 & -0,35 & -0,4\\-0,1 & 0,9 & -0,4\\-0,2 & -0,1 & 0,8\end{pmatrix}.$$

We calculate the determinant of the matrix $E-A$:

$$\det(E-A)=0,95\cdot 0,68+0,1\cdot(-0,28-0,04)-0,2\cdot(0,14+0,36)=$$
$$=0,646-0,032-0,1=0,514.$$

The determinant of the matrix $E-A$ being non-zero, the matrix $E-A$ is non-singular. Therefore, there exists a unique reciprocal of $E-A$.

Let us find the reciprocal of $E-A$:

$$(E-A)^{-1}=\frac{1}{0,514}\begin{pmatrix} \begin{vmatrix} 0,9 & -0,4 \\ -0,1 & 0,8 \end{vmatrix} & -\begin{vmatrix} -0,35 & -0,4 \\ -0,1 & 0,8 \end{vmatrix} & \begin{vmatrix} -0,35 & -0,4 \\ 0,9 & -0,4 \end{vmatrix} \\ -\begin{vmatrix} -0,1 & -0,4 \\ -0,2 & 0,8 \end{vmatrix} & \begin{vmatrix} 0,95 & -0,4 \\ -0,2 & 0,8 \end{vmatrix} & -\begin{vmatrix} 0,95 & -0,4 \\ -0,1 & -0,4 \end{vmatrix} \\ \begin{vmatrix} -0,1 & 0,9 \\ -0,2 & -0,1 \end{vmatrix} & -\begin{vmatrix} 0,95 & -0,35 \\ -0,2 & -0,1 \end{vmatrix} & \begin{vmatrix} 0,95 & -0,35 \\ -0,1 & 0,9 \end{vmatrix} \end{pmatrix}=$$

$$=\frac{1}{0,514}\begin{pmatrix} 0,68 & 0,32 & 0,5 \\ 0,16 & 0,68 & 0,42 \\ 0,19 & 0,165 & 0,82 \end{pmatrix}.$$

Thus the single-column matrix of the gross volumes of production has the form

$$X=\frac{1}{0,514}\begin{pmatrix} 0,68 & 0,32 & 0,5 \\ 0,16 & 0,68 & 0,42 \\ 0,19 & 0,165 & 0,82 \end{pmatrix}\begin{pmatrix} 60 \\ 70 \\ 30 \end{pmatrix}=$$

$$=\frac{1}{0,514}\begin{pmatrix} 40,8+22,4+15 \\ 9,6+47,6+12,6 \\ 11,4+11,55+24,6 \end{pmatrix}\approx\begin{pmatrix} 152,1 \\ 135,8 \\ 92,5 \end{pmatrix}.$$

A linear mathematical model of exchange (a mathematical model of international trade). Let us consider a linear mathematical model of exchange which is interpreted as a model of international trade that allows us to determine trade income of countries (or their ratio) for balanced trade [3]. Let $K_1, K_2, \ldots, K_n$ be the group of *n* countries that are trading. We denote by Z_j the trading income (the national income) of the *j*-th country generated from sale of own goods in both domestic and foreign markets. The structure of trade relations between the countries is considered to be known: the share q_{ij} of the trading income Z_j that the *j*-th country spends on the purchase of goods (imports) from the *i*-th country is a constant; in particular, q_{ij} does not depend on the value of Z_j. This hypothesis is an assumption about linearity of the model.

Let us consider the structural matrix of trade (the matrix of exchange):

$$Q=\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ . & . & . & . \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}.$$

Starting trading according to the matrix of exchange *Q*, countries will have the trading income with magnitude described by the single-column matrix *QZ* after one round.

We consider that all the trading income is spent either on the purchase of goods on its territory or on imports from other countries, namely the sum of elements of any column of the matrix *Q* is equal to unity:

$$\sum_{i=1}^{n} q_{ij}=1, \quad j=\overline{1,n}.$$

For the country K_i, the income from domestic and foreign trade is

$$Z_i=\sum_{j=1}^{n} q_{ij}Z_j, \quad i=\overline{1,n}.$$

For the balanced trade, it is necessary to find such a matrix of the trading income

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix},$$

for which the matrix equation:

$$Z = QZ, \qquad (6)$$

is fulfilled. This equation can be solved for Z.

It follows from mathematical analysis of the model that if the system operates k rounds of the trade with the matrix of exchange Q , after each round, we have the single-column matrices of the trading income: $QZ, Q^2Z, \ldots, \; Q^kZ$. Indeed, substituting QZ for the one-round trade for Z in the right-hand side of the formula (6), we obtain $Q(QZ) = (QQ)Z = Q^2Z$ for the two-round trade; substituting Q^2Z for the two-round trade for Z in the right-hand side of the formula (6), we obtain $Q(Q^2Z) = Q^3Z$ for the three-round trade, etc. For example, for the three-round trade, we substitute the matrix of the trading income for the two-round trade in the formula (6).

For example, we take three countries (denoting them by K_1, K_2, K_3), participants in the trade with the trading income respectively Z_1, Z_2, Z_3. Let the country K_1 spends a half of the trading income on the purchase of goods on its territory, a quarter of the trading income on the purchase of goods from the country K_2 and a quarter on the purchase of goods from the country K_3. The country K_2 equally spends the trading income on the purchase of goods from the country K_1 on its territory and from the country K_3. The country K_3 spends a half of the trading income on the purchase of goods from the country K_1, it spends the other half of the trading income on the purchase of goods from the country K_2 and does not buy anything on its territory. Let us find the national income of the countries that would satisfy the balanced trade without a deficite if the amount of their national income is 9000 conditional monetary units a year.

Let us write the structural matrix of trade:

$$\begin{array}{cc} & \begin{array}{ccc} K_1 & K_2 & K_3 \end{array} \\ Q = & \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & 0 \end{pmatrix}. \end{array}$$

Let q_{ij} be the share of the trading income that the j-th country spends on the purchase of the i-th country's goods. Note that the sum of elements in every column of the matrix Q is equal to unit.

After summarizing the results of trading for the year, the i-th country gets the income (the one-round trade doing for the year):

$$Z_i = \sum_{j=1}^{3} q_{ij} Z_j, \quad i = \overline{1, 3}.$$

Let us write the matrix equation to find the matrix Z:

$$Z = QZ \quad \text{або} \quad (Q - E)Z = O,$$

i.e.

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & -\frac{2}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution of the system of equations:

$$Z_1 = 2Z_3, \quad Z_2 = \frac{3}{2}Z_3, \quad Z_3 \in \mathrm{R}.$$

The result obtained means that the given countries’ trade balance is achieved at the ratio of their national incomes $2 : \frac{3}{2} : 1$.

Let us find the countries’ national incomes for the year that would satisfy the balanced trade without a deficite under the condition that the sum of incomes is equal to $Z_1 + Z_2 + Z_3 = 9000$ conditional monetary units. Substitute the values $Z_1 = 2C$, $Z_2 = \frac{3}{2}C$, $Z_3 = C$ in this equality, where $C = \mathrm{const}$. Then we obtain

$$2C + \frac{3}{2}C + C = 9000,$$

whence $C = 2000$. Thus, $Z_1 = 4000$, $Z_2 = 3000$, $Z_3 = 2000$ conditional monetary units.

Note that here are simplified versions of a mathematical model of interindustry balance and foreign trade.

Conclusions. A linear mathematical model of a diversified economy and a linear mathematical model of exchange are explored in this paper. Mathematical analysis of these models is carried out.

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