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ON GEOMETRIC EXTENSION OF POLYNOMIALS ON BANACH SPACES

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We consider some questions related to Aron-Berner extensions of polynomials on infinitely dimensional complex Banach spaces, using natural extensions of their zeros.

Key words and phrases: homogeneous polynomials on Banach spaces, zeros of polynomials, Aron-Berner extension.

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INTRODUCTION AND PRELIMINARIES

Let X be a complex Banach space, $\mathcal{P}(X)$ be the algebra of all continuous complex valued polynomials on X and $\mathcal{P}(^nX)$ be a subspace of $\mathcal{P}(X)$ of n-homogeneous polynomials. Since every polynomial $P \in \mathcal{P}(X)$ admits an unique (up to a multiplicative constant) factorization $P = P_1^{m_1} \cdot \ldots \cdot P_k^{m_k}$ in $\mathcal{P}(X)$ into irreducible polynomials, where $m_1, \ldots, m_k \in \mathbb{N}$, $\deg P_j > 0$, $j = 1, \ldots, k$, we can denote the *radical* of P as $\mathrm{Rad}\,(P) = P_1 \cdot \ldots \cdot P_k$.

In the general case, let $J = (P_1, \ldots, P_n)$ be the ideal generated by $P_1, \ldots, P_n \in \mathcal{P}(X)$, that is

$$J = \left\{ \sum_{i=1}^{n} P_i Q_i : \ Q_i \in \mathcal{P}(X) \right\}, \tag{1}$$

then the set Rad J is called the *radical* of J, if $P^k \in J$ for some positive integer k implies $P \in \mathrm{Rad}\, J$. For a given ideal $J \subset \mathcal{P}(X)$, $V(J)$ denotes the *zero* of J, that is, the common set of zeros of all polynomials in J.

Let G be a subset of X. Then $I(G)$ denotes the hull of G, that is the set of all polynomials in $\mathcal{P}(X)$ which vanish on G.

Ideals of the form (1) is called *finitely generated* in $\mathcal{P}(X)$. The following theorem is an analogue of the well known Hilbert Nulstellensatz for the infinite-dimension case (see [4, 5]).

Theorem 1. *Let J be a finitely generated in $\mathcal{P}(X)$. Then $I[V(J)] = \mathrm{Rad}\, J$.*

This theorem implies, in particular, that every irreducible polynomial in $\mathcal{P}(X)$ is defined (up to a multiplicative constant) by its zeros.

In this paper we consider questions related to extensions of polynomials, using their zeros. Here we will use the fact in [3] that zero set of every homogeneous polynomial on a complex infinite-dimensional Banach space X contains an infinite-dimensional linear subspace. Also, we will use the well known Aron-Berner extension of polynomials in $\mathcal{P}(X)$ to the second dual X'' and the fact that the extension operator $P \rightsquigarrow \widetilde{P}$ is a topological homomorphism of algebras $\mathcal{P}(X)$ and $\mathcal{P}(X'')$ (see [1, 2]).

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1 GEOMETRIC EXTENSION OF POLYNOMIALS

Let $P \in \mathcal{P}(X)$, $\dim X = \infty$. Since Ker P consists of infinite-dimensional closed linear subspaces ([3]), we can present Ker $P = \cup_\alpha V_\alpha$, where V_α are *maximal* closed linear subspaces in Ker P and α goes over a set of indexes. As usually,

$$V_\alpha^\perp = \left\{ f \in X' : \ f(x) = 0, \forall x \in V_\alpha \right\}.$$

So, $V_\alpha^{\perp\perp}$ naturally contains V_α and

$$V_\alpha^{\perp\perp} = \left\{ \varphi \in X'' : \varphi(f) = 0, \forall f \in V_\alpha^\perp \right\}.$$

Proposition 1. $V_\alpha^{\perp\perp} = V_\alpha''$.

Proof. $V_\alpha^{\perp\perp} \supset V_\alpha''$. By the Goldstain's theorem for every $\varphi \in V_\alpha''$ there is a net $(x_\beta) \in V_\alpha$ such that $x_\beta \xrightarrow{*-weak} \varphi$ that is $f(x_\beta) \to \varphi(f)$ for all $f \in X'$. Hence if $f \in V_\alpha^\perp$, then $f(x_\beta) = 0$ for all β. So, we have $\varphi(f) = 0$ and $V_\alpha^{\perp\perp} \supset V_\alpha''$.

On the other hand, if we have $\varphi \in V_\alpha^{\perp\perp}$, then $\varphi \in X''$. For any $g \in V_\alpha'$ by the Hahn-Banach theorem there exists an extension $\widetilde{g} \in X'$. Let $\widetilde{g_1}$, $\widetilde{g_2}$ be extensions of g to some elements in X', then $\widetilde{g_1} - \widetilde{g_2} = 0$ on V_α, i.e. $\widetilde{g_1} - \widetilde{g_2} \in V_\alpha^\perp$. Since $\varphi \in V_\alpha^{\perp\perp}$, then $\varphi(\widetilde{g_1} - \widetilde{g_2}) = 0$ and so, $\varphi(\widetilde{g_1}) = \varphi(\widetilde{g_2})$.

Therefore we can define

$$\varphi(g) := \varphi(\widetilde{g}),$$

where $\widetilde{g}$ is an arbitrary extension.

We have proved that $\varphi \in V_\alpha''$, which means that $V_\alpha^{\perp\perp} \subset V_\alpha''$. It follows that $V_\alpha^{\perp\perp} = V_\alpha''$. □

The main question of this paper is: *does Ker $\widetilde{P} = \cup_\alpha V_\alpha^{\perp\perp}$, where $\widetilde{P}$ is the Aron-Berner extension of P?*

We have an affirmative answer for some partial cases.

Proposition 2. *Ker $\widetilde{P} \supset \cup_\alpha V_\alpha^{\perp\perp}$.*

Proof. Since each $V_\alpha^{\perp\perp}$ is naturally identified with V_α'', then the restriction of $\widetilde{P}$ on $V_\alpha^{\perp\perp}$ coincide with the Aron-Berner extension of restriction of P onto V_α. But P vanishes on V_α and so, $V_\alpha^{\perp\perp} \subset$ Ker $\widetilde{P}$. □

Corollary 1. *If $\cup_\alpha V_\alpha^{\perp\perp}$ is a zero-set of a polynomial R on X'', then Ker $\widetilde{P} = \cup_\alpha V_\alpha^{\perp\perp}$.*

Proof. Without lost of the generality, we can assume that R is radical. By the Proposition 2, Ker $\widetilde{P} \supset$ Ker R, that is $\widetilde{P} \in I[V(R)]$. By Theorem 1, $\widetilde{P} = RQ$ for some $Q \in \mathcal{P}(X'')$. Note that Ker $P =$ Ker $R|_X$ and Ker $P =$ Ker $R|_X \cup$ Ker $Q|_X$. So, Ker $Q|_X \subset$ Ker $P|_X$ and we have that Ker $Q \subset$ Ker R. Hence Ker $\widetilde{P} =$ Ker $RQ =$ Ker $R = \cup_\alpha V_\alpha^{\perp\perp}$. □

Corollary 2. *If X is a complemented subspace in X'', then Ker $\widetilde{P} = \cup_\alpha V_\alpha^{\perp\perp}$.*

Proof. Let $T : X'' \to X$ be a projection. Then T maps $V_\alpha^{\perp\perp}$ onto V_α. Let $R = P \circ T$. So $\cup_\alpha V_\alpha^{\perp\perp} \subset$ Ker R.

On the other hand, if $z \notin \cup_\alpha V_\alpha^{\perp\perp}$, then $T(z) \notin \cup_\alpha V_\alpha$ and so $R(z) \neq 0$, that is $\cup_\alpha V_\alpha^{\perp\perp} = R$. By the Corollary 1, Ker $\widetilde{P} = \cup_\alpha V_\alpha^{\perp\perp}$. □

Since every dual Banach space is complemented in its second dual we have the following corollary.

Corollary 3. *If X is a dual space to a Banach space, then Ker $P = \cup_\alpha V_\alpha^{\perp\perp}$.*

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У роботі розглянуті деякі питання, пов'язані з продовженням Арона-Бернера поліномів на нескінченно вимірних комплексних банахових просторах, використовуючи природне продовження їхніх нулів.

Ключові слова і фрази: однорідні поліноми на банахових просторах, нулі поліномів, продовження Арона-Бернера.

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У роботе рассмотрены некотрые вопросы, связанные с продолжением Арона-Бернера полиномов на бесконечно измеримых комплексных банаховых пространствах, используя естественное продолжение их нулей.

Ключевые слова и фразы: однородные полиномы на банаховых пространствах, нули полиномов, продолжение Арона-Бернера.