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# ON CONTINUITY OF HOMOMORPHISMS BETWEEN TOPOLOGICAL CLIFFORD SEMIGROUPS

Generalizing an old result of Bowman we prove that a homomorphism $f : X \to Y$ between topological Clifford semigroups is continuous if

- the band $E_X = \{x \in X : xx = x\}$ of $X$ is a $U$-semilattice;
- the topological Clifford semigroup $Y$ is ditopological;
- the restriction $f|E_X$ is continuous;
- for each subgroup $H \subset X$ the restriction $f|H$ is continuous.

*Key words and phrases:* ditopological unosemigroup, Clifford semigroup, topological semilattice.

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## INTRODUCTION

This paper was motivated by the following old result of Yeager [6] who generalized an earlier result of Bowman [3].

**Theorem 1.** *A homomorphism $h : X \to Y$ between compact topological Clifford semigroups is continuous if and only if for any subgroup $H \subset X$ and any subsemilattice $E \subset X$ the restrictions $h|H$ and $h|E$ are continuous.*

In this paper we shall extend this result of Yeager beyond the class of compact topological Clifford semigroups. Let us define a homomorphism $h : X \to Y$ between topological semigroups to be *EH-continuous* if

- the restriction $h|E_X$ to the set of idempotents of $X$ is continuous;
- for every subgroup $H \subset X$ the restriction $h|H$ is continuous.

In terms of EH-continuity, Theorem 1 says that each EH-continuous homomorphism $h : X \to Y$ between compact topological Clifford semigroups is continuous. For compact topological Clifford semigroup $X$ with Lawson maximal semilattice $E_X = \{x \in X : xx = x\}$ this result of Yeager was proved by Bowman [3] in 1971. Generalizing the Bowman's result, in Theorem 3 we shall prove that each EH-continuous homomorphism $h : X \to Y$ from a topological Clifford $U$-semigroup $X$ to a ditopological Clifford semigroup $Y$ is continuous. Topological $U$-semigroups will be introduced and studied in Section 2. Section 1 presents some preliminaries. Section 4 contains our main result and some its corollaries.

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## 1 PRELIMINARIES

**1.1. Semigroups.** *A semigroup* is a non-empty set endowed with an associative binary operation. A semigroup $S$ is said to be

- *inverse* if for every $x \in S$ there is a unique element $x^{-1} \in S$ such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$;
- *Clifford* if it is inverse and $xx^{-1} = x^{-1}x$ for every $x \in S$;
- *a semilattice* if it is commutative and every element $x \in S$ is an *idempotent*, that is $xx = x$.

For a semigroup $S$ by $E_S = \{e \in S : ee = e\}$ we denote the set of idempotents of $S$ and for each idempotent $e \in E_S$ let

$$H_e = \{x \in S : \exists y \in S\ xy = e = yx,\ xe = x = ex,\ ye = y = ey\}$$

denote *the maximal subgroup* of $S$ containing $e$. If the semigroup $S$ is inverse, then the maximal group $H_e$ can be written as $H_e = \{x \in S : xx^{-1} = e = x^{-1}x\}$.

Each semilattice $E$ carries the natural partial order $\le$ defined by $x \le y$ iff $xy = yx = x$. For a point $x \in E$ let ${\downarrow}x = \{y \in E : y \le x\}$ and ${\uparrow}x = \{y \in E : x \le y\}$ be the lower and upper cones of $x$, respectively. By ${\Uparrow}x$ we shall denote the interior of the upper cone ${\uparrow}x$ in $E$.

A *homomorphism* between semigroups $X, Y$ is a function $h : X \to Y$ preserving the operation in the sense that $h(x \cdot y) = h(x) \cdot h(y)$ for all $x, y \in X$. The uniqueness of the inverse element in an inverse semigroup implies that each homomorphism $h : X \to Y$ between inverse semigroups preserves the inversion in the sense that $h(x^{-1}) = h(x)^{-1}$ for all $x \in X$. More information on inverse semigroups can be found in [5].

A *topological semigroup* is a semigroup $S$ endowed with a topology making the semigroup operation $\cdot : S \times S \to S$ continuous. A *topological inverse (Clifford) semigroup* is an inverse (Clifford) semigroup $S$ endowed with a topology making the multiplication $\cdot : S \times S \to S$ and the inversion $(\ )^{-1} : S \to S$ continuous.

A topological semilattice $E$ is *Lawson* if open subsemilattices form a base of the topology of $E$.

**1.2. Unosemigroups and unomorphisms.** By a *left unit operation* on a semigroup $S$ we understand a unary operation $\lambda_S : S \to S$ such that $\lambda_S(x) \cdot x = x$ for all $x \in S$. A *left unosemigroup* is a semigroup $S$ endowed with a left unit operation $\lambda_S : S \to S$. A left unosemigroup $S$ is called *$\lambda$-regular* if for each $x \in S$ there is $x^* \in S$ such that $\lambda_S(x) = xx^*$. In this case the element $\lambda_S(x) = xx^*$ is an idempotent because $\lambda_S(x) \cdot \lambda_S(x) = \lambda_S(x)xx^* = xx^* = \lambda_S(x)$. So, for each $\lambda$-regular left unosemigroup $S$ we get $\lambda_S(S) \subset E_S$.

By an *unomorphism* between left unosemigroups $(X, \lambda_X)$ and $(Y, \lambda_Y)$ we understand a semigroup homomorphism $h : X \to Y$ preserving the left unit operation in the sense that $h \circ \lambda_X = \lambda_Y \circ h$.

Left unosemigroups were introduced in [1]. By analogy we can define right unosemigroups, see [1].

Each inverse semigroup $S$ endowed with the left unit operation $\lambda_S : S \to S$, $\lambda_S : x \mapsto xx^{-1}$, carries a canonical structure of a $\lambda$-regular left unosemigroup. If $S$ is Clifford, then the left unit operation $\lambda_S$ is a homomorphism coinciding with the projection $\pi : S \to E_S, \pi : x \mapsto xx^{-1} = x^{-1}x$. If $S$ is a semilattice, then $\lambda_S$ coincides with the identity map of $S$.

The uniqueness of the inverse element in an inverse semigroup implies that each homomorphism between inverse semigroups is a unomorphism of the corresponding left unosemigroups.

By a *topological left unosemigroup* we understand a topological semigroup $S$ endowed with a continuous left unit operation $\lambda_S : S \to S$.

**Proposition 1.** *If a topological left unosemigroup $(S, \lambda_S)$ is $\lambda$-regular, then for any idempotent $e \in S$ and any point $x \in S$ with $e \cdot \lambda_S(x) = e$ the right shift $s_x : H_e \to H_e x$, $s_x : z \mapsto zx$, is a homeomorphism.*

*Proof.* Since $(S, \lambda_S)$ is $\lambda$-regular, $\lambda_S(x) = xx^*$ for some element $x^* \in S$. Consider the right shift $s_{x^*} : S \to S$, $s_{x^*} : z \mapsto zx^*$, and observe that for every element $z$ of the maximal subgroup $H_e$, we get $s_{x^*} \circ s_x(z) = zxx^* = z \cdot \lambda_S(x) = ze \cdot \lambda_S(x) = ze = z$. This implies that the restriction $s_{x^*}|H_e x : H_e x \to H_e$ is a continuous map, inverse to $s_x$. So, $s_x : H_e \to H_e x$ is a homeomorphism. □

**1.3. Ditopological unosemigroups.** For two subsets $A, B$ of a semigroup $S$ consider the subsets

$$B \leftthreetimes A = \{y \in S : \exists b \in B\ \exists a \in A\ \ by = a\} \text{ and } A \rightthreetimes B = \{x \in S : \exists a \in A\ \exists b \in B\ \ a = xb\}$$

which can be thought as the results of left and right division of $A$ by $B$ in the semigroup $S$.

A topological left unosemigroup $(S, \lambda_S)$ is called *a ditopological left unosemigroup* if for each $x \in X$ and neighborhood $O_x \subset S$ there are neighborhoods $W_{\lambda_S(x)} \subset \lambda_S(S)$ and $U_x \subset S$ of the points $\lambda_S(x)$ and $x$, respectively, such that

$$(W_{\lambda_S(x)} \leftthreetimes U_x) \cap \lambda_S^{-1}(W_{\lambda_S(x)}) \subset O_x.$$

Ditopological left unosemigroups were introduced and studied in [1]. By analogy, ditopological right unosemigroups can be introduced; see [1]. By Theorem 4 of [1], each compact topological left unosemigroup is ditopological.

A topological Clifford semigroup $S$ is *ditopological* if it is ditopological as a topological left unosemigroup (endowed with the canonical left unit operation $\lambda_S : x \mapsto xx^{-1}$). By [1], the class of ditopological Clifford semigroups contains all compact topological Clifford semigroups, all topological groups, all topological semilattices and is closed under many operations over topological Clifford semigroups (in particular, taking Clifford subsemigroups, Tychonoff products, reduced products, semidirect products).

## 2 TOPOLOGICAL LEFT $U$-UNOSEMIGROUPS

In this section we introduce the notion of a left $U$-unosemigroup, which is crucial in the proof of our main results.

**Definition 1.** *A topological left unosemigroup $(X, \lambda_X)$ is called a* left $U$-unosemigroup *if for each point $x \in X$ and each neighborhood $O_{\lambda_X(x)} \subset X$ of the element $\lambda_X(x)$ there is an open neighborhood $U_x \subset X$ of $x$ and an idempotent $e \in O_{\lambda_X(x)}$ such that $e\lambda_X(x) = e$ and $eU_x \subset H_e x$.*

In case $S$ is a topological semilattice the notion of a left $U$-unosemigroup agrees with the notion of a $U$-semilattice.

A topological semilattice $S$ is called a *$U$-semilattice* if for each point $x \in S$ and its neighborhood $U \subset S$ there is an idempotent $y \in U$ such that $x \in \Uparrow y$. We recall that by $\Uparrow y$ we denote the interior of the upper cone $\uparrow y$ in $S$.

The definitions of a left $U$-unosemigroup and a $U$-semilattice imply the following characterization:

**Proposition 2.** *A topological semilattice $E$ is a left $U$-unosemigroup if and only if it is a $U$-semilattice.*

The interplay between topological $U$-semilattices and other classes of topological semilattices was studied in [2]. In particular, let us recall for future references that each locally compact Lawson semilattice is a $U$-semilattice. The same is true for locally compact zero-dimensional semilattices, as they are Lawson. Let us recall that a regular topological space $X$ is *locally compact* if every point has a compact neighborhood and *zero-dimensional* if closed-and-open sets form a base of the topology of $X$.

**2.1. Topological Clifford $U$-semigroups.** Topological Clifford semigroups which are left $U$-unosemigroups can be characterized as follows.

**Proposition 3.** *A topological Clifford semigroup $S$ is a left $U$-unosemigroup if and only if its band $E_S = \{x \in S : xx = x\}$ is a $U$-semilattice.*

*Proof.* Assume first that $S$ is a left $U$-unosemigroup. Given any idempotent $e \in E_S$ and its neighborhood $U \subset E_S$, we need to find an idempotent $e' \in U$ such that $e \in \Uparrow e'$. The set $U$ is open in $E_S$ and so $U = W \cap E_S$ for some open neighborhood $W \subset S$ of $e$. Since $S$ is a $U$-unosemigroup, for the element $e$ and the neighborhood $W$ of the point $ee^{-1} = e$ we can find an open neighborhood $W_e \subset S$ of $e$ and an idempotent $e' \in W$ such that $e'e = e'$ and $e'W_e \subset H_{e'}e$. Without loss of generality we can assume that $W_e \subset W$ and therefore $U_e = W_e \cap E_S$ is an open neighborhood of $e$ in $E_S$.

It remains to check that $e \in \Uparrow e'$. For this observe that the inclusion $e'W_e \subset H_{e'}e$ implies that $e'U_e \subset (H_{e'}e) \cap E_S = \{e'\}e = e'$ and consequently $e \in U_e \subset \uparrow e'$. Thus $e \in \Uparrow e'$, which means that $E_S$ is a $U$-semilattice.

Now assume that the maximal semilattice $E_S$ of $S$ is a $U$-semilattice. To show that $S$ is a topological left $U$-unosemigroup, take any point $x \in S$ and neighborhood $O_{xx^{-1}} \subset S$ of the idempotent $\pi(x) = xx^{-1}$. Since $E_S$ is a $U$-semilattice, we can find an idempotent $e \in O_{xx^{-1}}$ such that $xx^{-1} \in \Uparrow e$. Then $U_x = \pi^{-1}(\Uparrow e)$ is an open neighborhood of $x$.

It remains to show that $eU_x \subset H_e x$. First observe that for any element $z \in H_e$ we have $z = ze = zex^{-1}x$. It follows from

$$(zex^{-1})(zex^{-1})^{-1} = zex^{-1}xez^{-1} = zez^{-1} = e$$

that $zex^{-1} \in H_e$ and $z = (zex^{-1})x \in H_e x$. Hence, $H_e \subset H_e x$.

Finally, the inclusion $\pi(eU_x) = \pi(e)\pi(U_x) \subset \{e\}\Uparrow e = \{e\}$ implies that $eU_x \subset \pi^{-1}(e) = H_e \subset H_e x$, which is the desired conclusion. □

Having in mind the previous proposition we define a topological Clifford semigroup $S$ to be *a topological Clifford $U$-semigroup* if its maximal semilattice $E_S$ is a $U$-semilattice. This happens if and only if $S$ is a topological left $U$-unosemigroup.

## 3 THE CONTINUITY OF EH-CONTINUOUS UNOMORPHISMS BETWEEN TOPOLOGICAL LEFT UNOSEMIGROUPS

The following theorem is a key ingredient in the proof of Theorem 3, which is our main result. This theorem can be considered as a generalization of Bowman's result [3] to topological left unosemigroups.

**Theorem 2.** *Any EH-continuous unomorphism $h : X \to Y$ from a $\lambda$-regular topological left $U$-unosemigroup $(X, \lambda_X)$ into a ditopological left unosemigroup $(Y, \lambda_Y)$ is continuous.*

*Proof.* Given any point $x \in X$ and an open neighborhood $O_y \subset Y$ of the point $y = h(x)$ we need to find a neighborhood $V_x \subset X$ of $x$ such that $h(V_x) \subset O_y$.

Since the left unosemigroup $(Y, \lambda_Y)$ is ditopological, there are open neighborhoods $W_{\lambda_Y(y)} \subset \lambda_Y(Y)$ and $U_y \subset Y$ of the elements $\lambda_Y(y)$ and $y$, respectively, such that $(W_{\lambda_Y(y)} \leftthreetimes U_y) \cap \lambda_Y^{-1}(W_{\lambda_Y(y)}) \subset O_y$. Taking into account that $\lambda_Y(y) \cdot y \in U_y$, we can replace $W_{\lambda_Y(y)}$ by a smaller neighborhood and additionally assume that $W_{\lambda_Y(y)} \cdot y \subset U_y$.

Since the unomorphism $h$ preserves the left unit operation, we have $h(\lambda_X(x)) = \lambda_Y(y)$. The $\lambda$-regularity of the left unit operation $\lambda_X$ implies that $\lambda_X(X) \subset E_X$. By the continuity of the restriction $h|\lambda_X(X)$, there is an open neighborhood $W_{\lambda_X(x)} \subset \lambda_X(X)$ such that $h(W_{\lambda_X(x)}) \subset W_{\lambda_Y(y)}$.

Since $X$ is a left $U$-unosemigroup, for the point $x$ and the neighborhood $W_{\lambda_X(x)}$ of $\lambda_X(x)$ we can find an idempotent $e \in W_{\lambda_X(x)}$ and an open neighborhood $V_x \subset X$ of $x$ such that $e\lambda_X(x) = e$ and $eV_x \subset H_e x$. Replacing $V_x$ by a smaller neighborhood, if necessary, we can additionally assume that $\lambda_X(V_x) \subset W_{\lambda_X(x)}$. In this case

$$\lambda_Y \circ h(V_x) = h \circ \lambda_X(V_x) \subset h(W_{\lambda_X(x)}) \subset W_{\lambda_Y(y)}$$

and $h(ex) = h(e) \cdot h(x) \in h(W_{\lambda_X(x)}) \cdot y \subset W_{\lambda_Y(y)} \cdot y \subset U_y$.

We claim that the restriction $h|H_e x$ is continuous. Indeed, by the $\lambda$-regularity of the left unit operation $\lambda_X$, there is an element $x^* \in X$ such that $\lambda_X(x) = xx^*$. By Proposition 1 the right shift $s_x : H_e \to H_e x$, $s_x : z \mapsto zx$, is a homeomorphism with inverse $s_{x^*} : H_e x \to H_e$, $s_{x^*} : z \mapsto zx^*$. The EH-continuity of $h$ guarantees that the restriction $h|H_e$ is continuous and so is the composition $h \circ s_{x^*} : H_e x \to Y$. For every point $z \in H_e x$ we can find an element $g \in H_e$ with $z = gx$ and observe that $zx^*x = gxx^*x = g\lambda_X(x)x = gx = z$. So, $h(z) = h(zx^*x) = h(zx^*) \cdot h(x) = h(zx^*) \cdot y$, which implies that the restriction $h|H_e x$ is continuous as the composition of the continuous map $h \circ s_{x^*}$ and the continuous right shift $s_y : Y \to Y$, $s_y : u \mapsto uy$.

By the continuity of the map $h|H_e x$, the set $h^{-1}(U_y) \cap H_e x$ is an open neighborhood of the point $ex$. Replacing the neighborhood $V_x$ by a smaller one, if necessary, we can assume that $eV_x \subset h^{-1}(U_y) \cap H_e x$. Then $h(eV_x) \subset h(h^{-1}(U_y)) \subset U_y$.

To finish the proof of the continuity of $h$ at $x$, it remains to check that $h(V_x) \subset O_y$. For this observe that for every $v \in V_x$ we get $h(e) \cdot h(v) = h(ev) \in U_y$ and $h(e) \in h(W_{\lambda_X(x)}) \subset W_{\lambda_Y(y)}$.

Combined with the inclusion $\lambda_Y \circ h(v) \in \lambda_Y \circ h(V_x) \subset W_{\lambda_Y(y)}$ proved above, this yields

$$h(v) \in (W_{\lambda_Y(y)} \setminus U_y) \cap \lambda_Y^{-1}(W_{\lambda_Y(y)}) \subset O_y$$

according to the choice of the neighborhoods $W_{\lambda_Y(y)}$ and $U_y$. □

## 4 THE CONTINUITY OF EH-CONTINUOUS HOMOMORPHISMS BETWEEN CLIFFORD $U$-SEMIGROUPS

Now we are in a position to prove the main result of the paper and state some its corollaries. Let us recall that a topological Clifford semigroup $X$ is called topological Clifford $U$-semigroup if its band $E_X$ is a $U$-semilattice.

**Theorem 3.** *Each EH-continuous homomorphism $h : X \to Y$ from a topological Clifford $U$-semigroup $X$ to a ditopological Clifford semigroup $Y$ is continuous.*

*Proof.* By Proposition 3, the topological Clifford $U$-semigroup $X$ endowed with a canonical left unit operation $\lambda : x \mapsto xx^{-1}$ is a $\lambda$-regular topological left $U$-unosemigroup. The homomorphism $h$, being a homomorphism between Clifford semigroups, preserves the operation of inversion. It follows that $h$ preserves the canonical unit operation on $X$ and so is a unomorphism. Thus, $h : X \to Y$ is a EH-continuous unomorphism and by Theorem 2, it is continuous. □

Since each locally compact Lawson semilattice is a $U$-semilattice (see Proposition 2.4(3) of [2]), this Theorem implies

**Corollary 1.** *For any topological Clifford semigroup $X$ with locally compact Lawson maximal semilattice $E_X$, every EH-continuous homomorphism $h : X \to Y$ to a ditopological Clifford semigroup $Y$ is continuous.*

Since each locally compact zero-dimensional semilattice is Lawson (see Theorem 2.6 in [4]), we obtain

**Corollary 2.** *For any topological Clifford semigroup $X$ with locally compact zero-dimensional maximal semilattice $E_X$, every EH-continuous homomorphism $h : X \to Y$ to a ditopological Clifford semigroup $Y$ is continuous.*

Since each compact Hausdorff topological Clifford semigroup is ditopological (see Theorem 4 in [1]), Corollary 1 implies the following result of Bowman [3].

**Corollary 3** (Bowman)**.** *Each EH-continuous homomorphism $h : X \to Y$ from a compact Hausdorff topological Clifford semigroup $X$ with Lawson maximal semilattice $X$ into a compact Hausdorff topological Clifford semigroup $Y$ is continuous.*

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Узагальнюється результат, отриманий у статті [3], і доводиться неперервність гомоморфізму $f : X \to Y$ між топологічними кліфордовими напівгрупами за умов:

- множина $E_X = \{x \in X : xx = x\} \subset X$ ідемпотентів є $U$-напівграткою;
- топологічна кліфордова напівгрупа $Y$ дітопологічна;
- звуження $f|E_X$ неперервне;
- звуження $f|H$ неперервне для кожної підгрупи $H \subset X$.

*Ключові слова і фрази:* дітопологічна унонапівгрупа, кліфордова напівгрупа, топологічна напівгратка.

Пастухова И. *О непрерывности гомоморфизмов между топологическими клиффордовыми полугруппами* // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 123–129.

Обобщается результат, полученный в работе [3], и доказывается непрерывность гомоморфизма $f : X \to Y$ между топологическими клиффордовыми полугруппами при условиях:

- множество $E_X = \{x \in X : xx = x\} \subset X$ идемпотентов является $U$-полурешеткой;
- топологическая клиффордова полугруппа $Y$ дитопологическая;
- сужение $f|E_X$ непрерывно;
- сужение $f|H$ непрерывно для каждой подгруппы $H \subset X$.

*Ключевые слова и фразы:* дитопологическая унополугруппа, клиффордова полугруппа, топологическая полурешетка.