ISSN 2075-9827
Carpathian Math. Publ. 2015, **7** (1), 78–82
doi:10.15330/cmp.7.1.78-82

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.78–82

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TWO-SIDED INEQUALITIES WITH NONMONOTONE SUBLINEAR OPERATORS

Theorems on existence of solutions and their two-sided estimates for one class of nonlinear operator equations $x = Fx$ with nonmonotone operators are proved.

Key words and phrases: two-sided inequalities, extreme solutions, norm.

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PROBLEM STATEMENT

While we studying in Banach space E the equation

$$x = Fx \tag{1}$$

with the nonlinear operator $F : E \to E$, in general, the following condition is often used

$$\lim_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|} = \alpha. \tag{2}$$

For example, condition (2) for $\alpha = 0$ satisfies the integral equation

$$x(t) = f(t) + \int_D K(t,s)x^{\gamma}(s)ds, \quad 0 < \gamma < 1, \tag{3}$$

which was studied by M.A. Krasnoselsky [1] (see also references in [1]), B.Z. Vulykh [2], C.A. Stuart [3] and others. In particular, C.A. Stuart [3] uses the results obtained for equation (3) while investigating some boundary values problems for equations with partial derivatives.

In the present paper, we apply methods suggested in [4, §8] to the study of equation (1) with the operator F satisfying condition (2) under the assumption $0 < \alpha \leqslant 1$, which makes it possible to get some specification and generalization of respective results from [4, §8]. It also allows of applying the obtained results to the equation

$$x(t) = f(t) + \int_D K(t,s)x^{\gamma}(s)ds + \int_D K_1(t,s)x(s)ds, \quad 0 < \gamma \leqslant 1.$$

1 MAIN RESULTS AND THEIR EXPLANATION

Definition 1. *The norm of the pair of elements from E is $\|y,z\|$, satisfying the following conditions: a) if $y, z \in E$, the inequalities are $\|y\| \leqslant \|y,z\|$, $\|z\| \leqslant \|y,z\|$; b) the norm $\|y,z\|$ is monotone when introducing into $E \times E$ semiorderedness of pairs (y,z) elements from E, generated in this or that way of semiorderedness in E.*

УДК 517.968.4
2010 *Mathematics Subject Classification:* 41A65, 30B70.

For example, if the norm $\|y, z\|$ is introduced with the help of one of the formulas

$$\|y, z\| = \max\{\|y\|, \|z\|\}, \quad \|y, z\| = (\|y\|^p + \|z\|^p)^{\frac{1}{p}}, \quad p \geqslant 1, \tag{4}$$

and the semiorderedness of pairs (y, z) is defined as $(y, z) \leqslant (u, v)$ for $y \leqslant u, \quad z \geqslant v$ or $(y, z) \leqslant (u, v)$ for $y \leqslant u, \quad z \leqslant v$, then conditions a), b) are satisfied. With this semiorderedness, the space $E \times E$ is a fully regular space if E has this property.

Theorem 1. *Suppose that: 1) there are nondecreasing with respect to y and nonincreasing with respect to z continuous operators $T_1(y, z), T_2(y, z) : E \times E \to E$, such that*

$$T_1(x, x) = T_2(x, x) = Fx, \quad x \in E; \tag{5}$$

2) there exists $M > 0$, and from the inequality $\|y, z\| > M$ it follows that $\|T_1(y, z), T_2(z, y)\| \leqslant \|y, z\|$; 3) if $y, z \in E$, then $T_1(y, z) \leqslant T_2(y, z)$; 4) there are elements $u, v \in E$, for which $u \leqslant T_1(u, v), v \geqslant T_2(v, u)$; 5) simultaneous equations

$$y = T_1(y, z), \quad z = T_2(z, y) \tag{6}$$

have no more than one solution; 6) equation (1) *has at least one solution. Then the unique solution $x^* \in E$ of equation* (1) *satisfies the estimates*

$$y_n \leqslant y_{n+1} \leqslant x^* \leqslant z_{n+1} \leqslant z_n, \quad n = 0, 1, \ldots, \tag{7}$$

where the sequences $\{y_n\}$, $\{z_n\}$ are built with the help of

$$y_0 = u, \quad z_0 = v, \quad y_{n+1} = T_1(y_n, z_n), \quad z_{n+1} = T_2(z_n, y_n), \quad n = 0, 1, \ldots \tag{8}$$

At that the sequences $\{y_n\}$, $\{z_n\}$ converge in E to x^* by the norm.

Proof. Let us prove the inequalities

$$y_0 \leqslant y_1 \leqslant \ldots \leqslant y_n \leqslant y_{n+1} \leqslant \ldots, \quad z_0 \geqslant z_1 \geqslant \ldots \geqslant z_n \geqslant z_{n+1} \geqslant \ldots \tag{9}$$

If $n = 0$ from conditions 4) and (8) we obtain $y_0 = u \leqslant T_1(y_0, z_0) = y_1, z_0 = v \geqslant T_2(z_0, y_0) = z_1$. Assuming that $y_{n-1} \leqslant y_n, \quad z_{n-1} \geqslant z_n$, based on (8) and 1) we get

$$y_{n+1} = T_1(y_n, z_n) \geqslant T_1(y_{n-1}, z_{n-1}) = y_n, \quad z_{n+1} = T_2(z_n, y_n) \leqslant T_2(z_{n-1}, y_{n-1}) = z_n.$$

By induction, we come to a conclusion that the inequalities (9) are valid for any $n \in \mathbb{N}$.

Let us make sure that the sequences $\{y_n\}$, $\{z_n\}$ are limited by the norm. If starting from some number $n = N$, all the members of the sequence $\{(y_n, z_n)\}$ satisfied the inequality

$$\|(y_n, z_n)\| \leqslant M, \tag{10}$$

then the sequence $\{(y_n, z_n)\}$ would be limited by the norm, and the sequences $\{y_n\}$, $\{z_n\}$ would be limited by the norm. Assuming that for any $N > 0$ we have $n > N$ so that

$$\|y_n, z_n\| > M, \tag{11}$$

let us consider two mutually exclusive cases. Let in the first one exist no more than a finite number of the members of the sequence $\{(y_n, z_n)\}$, for which $\|y_n, z_n\| \leqslant M$. Then starting from some number $n = N$ inequality (11) holds. In virtue of (8) and condition 2) we obtain

$$\|y_{N+1}, z_{N+1}\| = \|T_1(y_N, z_N), T_2(z_N, y_N)\| \leqslant \|y_n, z_n\|.$$

Assuming that $\|y_{N+k}, z_{N+k}\| \leqslant \|y_{N+k-1}, z_{N+k-1}\|$, we shall similarly find that

$$\|y_{N+k+1}, z_{N+k+1}\| = \|T_1(y_{N+k}, z_{N+k}), T_2(z_{N+k}, y_{N+k})\| \leqslant \|y_{N+k}, z_{N+k}\|.$$

By induction, we come to a conclusion that the sequence $\{(y_n, z_n)\}$ and the sequences $\{y_n\}$, $\{z_n\}$ are limited by the norm. Suppose that the sequence $\{(y_n, z_n)\}$ has an infinite number of members for which inequality (10) holds as well as an infinite number of members for which inequality (11) holds. It means that this property pertains to the sequence $\{y_n\}$, $\{z_n\}$. Let us choose arbitrary n_1 and n_2, $n_1 < n_2$, for which, for example, $\|y_{n_1}\| \leqslant M, \|y_{n_2}\| \leqslant M$. Let us have n_3 so that $n_1 < n_3 < n_2$ and $\|y_{n_3}\| > M$, from (9) we obtain $y_{n_1} \leqslant y_{n_3} \leqslant y_{n_2}$. Based on Lemma 8.1 [4, p. 37] we get $\|y_{n_3}\| \leqslant \|y_{n_1}\| + \|y_{n_2}\| \leqslant 2M$. This proves that the sequence $\{y_n\}$ is limited by the norm. It is similarly proved that the sequence $\{z_n\}$ is also limited by the norm. For the fully regular ordered space E, the monotonely nondecreasing sequence $\{y_n\}$ and the monotonely nonincreasing sequence $\{z_n\}$, which are limited by the norm, have limits y^* and z^*, $y^*, z^* \in E$, which are components of the solution of system (6). The solution $x^* \in E$ of equation (1) and equality (5) mean that (x^*, x^*) is the solution of system (6). Since system (6) has a unique solution, then $y^* = z^* = x^*$. The proof of the theorem is complete. □

Theorem 2. *Suppose that: 1) there are nondecreasing with respect to y, nonincreasing with respect to z continuous operators $T_1(y,z), T_2(y,z) : E \times E \to E$, such that*

$$T_1(x,x) \leqslant Fx \leqslant T_2(x,x), \quad x \in E; \tag{12}$$

2) conditions 2)–6) of Theorem 1 are satisfied. Then for any solution $x^* \in E$ of equality (1) *we have inequalities* (7)*, where sequences $\{y_n\}$, $\{z_n\}$ are built with the help of formulae* (8)*. Besides, the sequences $\{y_n\}$, $\{z_n\}$ converge to the components y^*, z^* of the solution (y^*, z^*) of system* (6) *and the estimates $u \leqslant y^* \leqslant x^* \leqslant z^* \leqslant v$ are valid.*

Proof. Let us build an iteration process with the help of

$$\varphi_0 = \psi_0 = x^*, \quad \varphi_{n+1} = T_1(\varphi_n, \psi_n), \quad \psi_{n+1} = T_2(\psi_n, \varphi_n), \quad n = 0, 1, \ldots,$$

where x^* is the solution of equation (1). From inequality (12), nondecreasing with respect to y and nonincreasing with respect to z properties of operators $T_1(y,z)$, $T_2(y,z)$ are observed by

$$\varphi_0 \geqslant \varphi_1 \geqslant \ldots \geqslant \varphi_n \geqslant \varphi_{n+1} \geqslant \ldots, \quad \psi_0 \leqslant \psi_1 \leqslant \ldots \leqslant \psi_n \leqslant \psi_{n+1} \leqslant \ldots, \quad n = 0, 1, \ldots$$

As in the proof of Theorem 1, we find that the sequence $\{\varphi_n\}$ converges to its limit φ^* without its monotone increase, and the sequence $\{\psi_n\}$ converges to its limit ψ^* without its monotone decrease. At that (φ^*, ψ^*) is the solution of system (6) and $\varphi^* \leqslant x^* \leqslant \psi^*$. Besides, for the sequences $\{y_n\}$, $\{z_n\}$, built with the help of formulae (8), we can fully repeat relevant considerations in the proof of Theorem 1 and reach the same conclusions concerning y^*, z^* as a component of the solution (y^*, z^*) of system (6) and about inequalities (9). The solution of system (6) being unique, it makes us possible to state that $\varphi^* = y^*$, $\psi^* = z^*$.

The proof of the theorem is complete. □

Theorem 3. *Suppose that: 1) condition 1) of Theorem 2 is satisfied; 2) there are linear positive relative to $w \in E$, nondecreasing with respect y, nonincreasing with respect to z operators $A_1(y,z)w$, $A_2(y,z)w$, for which if $x, y, z \in E$ the following inequalities hold*

$$-A_1(z,y)(z-y) \leqslant T_1(z,x) - T_1(y,x), \quad T_2(x,z) - T_2(x,y) \leqslant A_2(z,y)(z-y);$$

3) there is $M>0$, so that from the inequality $\|z,y\|\geqslant M$ follows

$$\|T_1(y,z)-(A_1(z,y)+A_2(z,y))(z-y),T_2(y,z)+(A_1(z,y)+A_2(z,y))(z-y)\|\leqslant\|z,y\|;$$

4) there are $u,v\in E$, such that

$$u\leqslant-(A_1(v,u)+A_2(v,u))(v-u)+T_1(u,v),\quad v\geqslant(A_1(v,u)+A_2(v,u))(v-u)+T_2(v,u);$$

5) simultaneous equations

$$\begin{aligned} y&=-(A_1(z,y)+A_2(z,y))(z-y)+T_1(y,z),\\ z&=(A_1(z,y)+A_2(z,y))(z-y)+T_2(z,y)\end{aligned}\tag{13}$$

have in $E\times E$ no more than one solution. Then if there is a solution of equation (1)*, it is unique and the sequences $\{y_n\}$, $\{z_n\}$ converge to it without increasing and decreasing respectively. These sequences are built with the help of formulae*

$$\begin{aligned} y_{n+1}&=-(A_1(z_n,y_n)+A_2(z_n,y_n))(z_n-y_n)+T_1(y_n,z_n),\\ z_{n+1}&=(A_1(z_n,y_n)+A_2(z_n,y_n))(z_n-y_n)+T_2(z_n,y_n),\quad n=0,1,\ldots\end{aligned}$$

if $y_0=u$, $z_0=v$. Besides, there are estimates (7)*.*

Proof of the sequences $\{y_n\}$, $\{z_n\}$ being monotone and limited by the norm in fact follows along the lines of respective considerations from the proof of Theorem 1. That's why $y_n\uparrow y^*$, $z_n\downarrow z^*$ $(y^*,z^*\in E)$, and (y^*,z^*) is solution of system (13). If x^* is the solution of equation (1), then (x^*,x^*) is the solution of system (13), and this system can have no more than one solution. The proof of the theorem is complete. □

2 APPLYING LIMITED ELEMENTS TO EQUATIONS IN KN-SPACES

Theorem 4. *Suppose that: 1) E is KN-space of limited elements and in $E\times E$ the norm is defined with the help of the first formula from* (4)*; 2) condition 2) of Theorem 1 and condition 1) of Theorem 2 are satisfied; 3) if $y\leqslant z$ $(y,z\in E)$, then $T_1(y,z)\leqslant T_2(z,y)$. Then there is extreme (see, e.g., [4, p. 22]) in $E\times E$ solution (y^*,z^*) of system* (6)*, the components of which belong to some segment $[-a,a]\subset E$, and for any solution $x^*\in E$ of equation* (1) *we have*

$$-a\leqslant x^*\leqslant a,\quad y^*\leqslant x^*\leqslant z^*.\tag{14}$$

Proof. If we replace condition 2) of Theorem 1 by the condition: if $\|y,z\|\geqslant M$ we have

$$\|T_1(y,z),T_2(z,y)\|<\|y,z\|,\tag{15}$$

then any solution (y,z), $y,z\in E$ of system (6) is within $D=\{(y,z)|\|y,z\|<M,\quad y,z\in E\}$. If for some solution (y,z) $(y,z\in E)$ of system (6) we have $\|y,z\|\geqslant M$, then from (6) and (15) we obtain $\|y,z\|=\|T_1(y,z),T_2(z,y)\|<\|y,z\|$, which is impossible. It allows us to draw a conclusion that any solution of system (6) belongs to the segment $[-a;a]$. If e is a unit of the space E of limited elements, it follows from what has been said that

$$|y|\leqslant\|y\|e\leqslant\|y,z\|e\leqslant Me,\quad |z|\leqslant\|z\|e\leqslant\|y,z\|e\leqslant Me.$$

Let us denote $Me=a$. Considering obvious inequality $-a\leqslant a$, inequality (15) and determination of domain D, we shall have

$$|T_1(-a,a)|\leqslant\|T_1(-a,a),T_2(a,-a)\|e\leqslant\|-a,a\|e=Me=a,$$

$$|T_2(a,-a)| \leqslant \|T_1(-a,a), T_2(a,-a)\|e \leqslant \|-a,a\|e = Me = a.$$

This implies that $-a \leqslant T_1(-a,a)$, $a \geqslant T_2(a,-a)$. To prove existence of extreme solution (y^*, z^*) of system (6) on the segment $[-a,a]$, it is enough to use iterations (8) setting $u = -a$, $v = a$ in them. As any solution of system (6) has components belonging to the segment $[-a,a]$, we draw a conclusion that (y^*, z^*) is extreme in $E \times E$ solution of system (6). The proof of the theorem is complete. □

Theorem 5. *Suppose that condition 1) of Theorem 4, condition 1) of Theorem 2 and condition 2) of Theorem 3 are satisfied. Then there is an extreme in $E \times E$ solution (y^*, z^*) of system* (13), *the components of which belong to some segment $[-a,a] \subset E$, and for any solution $x^* \in E$ of equation* (1), *there are estimates* (14).

The proof differs from the proof of Theorem 4 unessentially.

REMARK

If $T_1(y,z)$, $T_2(y,z)$ are fully continuous operators, then for the solution of equation (1) to exist, it is enough to satisfy condition 2) of Theorem 1. In this case, the operator generated by the right member of (6) will turn some sphere S of the radius M from $E \times E$ into compact in $E \times E$ set D_1. Let us choose the number $M_1 > M$ so high that the sphere $S_1 \subset E \times E$ contains the sphere S, as well as the compact, and therefore limited, set D_1. Thus, it turns out that the operator generated by the right member of (6), turns the sphere S into itself. Therefore, let us apply the Schauder principle.

Obtaining results supplement and specify results [5, §21] (see also references in [5]).

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Received 22.12.2014

Revised 18.05.2015

Копач М.І., Обшта А.Ф., Шувар Б.А. *Двосторонні нерівності з немонотонними підлінійними операторами* // Карпатські матем. публ. — 2015. — Т.7, №1. — С. 78–82.

Встановлено теореми про існування розв'язків та їх двосторонніх оцінок для одного класу нелінійних операторних рівнянь вигляду $x = Fx$ з немонотонними операторами.

Ключові слова і фрази: двосторонні нерівності, крайні розв'язки, норма.