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PERIODIC WORDS CONNECTED WITH THE FIBONACCI WORDS

In this paper we introduce two families of periodic words (FLP-words of type 1 and FLP-words of type 2), that are connected with the Fibonacci words. The properties of the families are investigated.

Key words and phrases: Fibonacci number, Fibonacci word.

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INTRODUCTION

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all integer $n > 1$, and with initial values $F_0 = 0$ and $F_1 = 1$. These numbers and their generalizations have interesting properties. Different kinds of the Fibonacci sequence and their properties have been presented in the literature, see, e.g., [1, 6, 11].

Many properties of Fibonacci numbers require the full ring structure of the integers. However, generalizations to the ring $\mathbb{Z}_m$ and groups have been considered, see, e.g., [3, 5, 14, 16]. The sequence $F_n \pmod{m}$ is periodic and it repeats by returning to its starting values because there are only a finite number m^2 of pairs of possible terms. Therefore, we obtain the repeating of all the sequence elements.

In analogy to the definition of the Fibonacci numbers, one defines the Fibonacci finite words as the concatenation of the two previous terms $f_n = f_{n-1}f_{n-2}$, $n > 1$, with initial values $f_0 = 1$ and $f_1 = 0$ and defines the infinite Fibonacci word f, $f = \lim f_n$ [2]. It is the archetype of a Sturmian word [7]. The properties of the Fibonacci infinite word have been studied extensively by many authors, see, e.g., [7, 8, 9, 10, 12, 15].

Using Fibonacci words, in the present article we shall introduce some new kinds of the infinite words, namely FLP-words, and investigate some of their properties.

For any notations not explicitly defined in this article we refer to [4, 6, 7].

1 FIBONACCI SEQUENCE MODULO m

The letter p, $p > 2$, is reserved to designate a prime, m may be arbitrary integer, $m > 2$.

Let $F_n^*(m)$ denote the n-th member of the sequence of integers $F_n \equiv F_{n-1} + F_{n-2} \pmod{m}$, for all integer $n > 1$, and with initial values $F_0 = 0$ and $F_1 = 1$. We reduce F_n modulo m taking the least nonnegative residues, and let $k(m)$ denote the length of the period of the repeating sequence $F_n^*(m)$.

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The problem of determining the length of the period of the recurring sequence arose in connection with a method for generating random numbers. A few properties of the function $k(m)$ are in the following theorem [14].

Theorem 1. *In $\mathbb{Z}_m$ the following statements hold.*

1. *Any Fibonacci sequence modulo m is periodic.*
2. *If $p \equiv \pm 1 \pmod{10}$, then $k(p)|(p-1)$. If $p \equiv \pm 3 \pmod{10}$, then $k(p)|2(p+1)$.*
3. *If m has prime factorization $m = \prod_{i=1}^{n} p_i^{e_i}$, then $k(m) = \mathrm{lcm}(k(p_1^{e_1}), \dots, k(p_n^{e_n}))$.*
4. *If $k(p^2) \neq k(p)$, then $k(p^i) = p^{i-1}k(p)$ for $i > 1$.*

The results in Theorem 1 give upper bounds for $k(p)$ but there are primes for which $k(p)$ is less than the given upper bound.

Let $h(m)$ denote the length of the period of the repeating sequence $2^{F_n} \pmod{m}$ and $\varphi(m)$ be Euler's totient function.

Theorem 2. *Let m be odd and $m > 1$. Then $h(m)|k(\varphi(m))$.*

Proof. This follows from Euler's theorem: if m and a are coprime positive integers, then $a^{\varphi(m)} \equiv 1 \pmod{m}$. When reducing the power of a a modulo m, one needs to work modulo $\varphi(m)$ in the exponent of a: if $x \equiv y \pmod{\varphi(m)}$ then $a^x \equiv a^y \pmod{m}$. $\square$

Corollary 1. *Let $p \geq 3$. Then $h(p)|k(p-1)$.*

2 FIBONACCI WORDS

Let $f_0 = 1$ and $f_1 = 0$. Now $f_n = f_{n-1}f_{n-2}$, $n > 1$, the concatenation of the two previous terms. The successive initial finite Fibonacci words are:

$$\begin{aligned}
&f_0 = 1, \quad f_1 = 0, \quad f_2 = 01, \quad f_3 = 010, \\
&f_4 = 01001, \quad f_5 = 01001010, \quad f_6 = 0100101001001, \\
&f_7 = 010010100100101001010, \quad f_8 = 0100101001001010010100100101001001, \dots
\end{aligned} \tag{1}$$

The infinite Fibonacci word f is the limit $f = \lim f_n$. It is referenced A003849 in the On-line Encyclopedia of Integer Sequences [13] and is certainly one of the most studied examples in the combinatorial theory of infinite words. The combinatorial properties of the Fibonacci infinite word are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, cryptography, formal language, computational complexity, quasicrystals etc. (see [7]).

We denote as usual by $|f_n|$ the length (the number of symbols) of f_n (see [7]). The following proposition summarizes basic properties of the Fibonacci words [7, 10].

Theorem 3. *The infinite Fibonacci word and the finite Fibonacci words satisfy the following properties.*

1. *The words 11 and 000 are not subwords of the infinite Fibonacci word.*

2. *For all $n > 1$ let ab be the last two symbols of f_n, then we have $ab = 01$ if n is even and $ab = 10$ if n is odd.*

3. *The concatenation of two successive Fibonacci words is "almost commutative", i.e., $f_n f_{n-1}$ and $f_{n-1} f_n$ differ only by their last two symbols for all $n > 1$.*

4. *For all n $|f_n| = F_{n+1}$.*

5. *The number of 0 and 1 in f_n equals F_n and F_{n-1}, respectively.*

3 PERIODIC FLP-WORDS

Let us start with the classical definition of periodicity on words over arbitrary alphabet $\{a_0, a_1, a_2, \dots\}$ (see [4]).

Definition 1. *Let $w = a_0 a_1 a_2 \dots$ be an infinite word. We say that w is*

1) *a periodic word if there exists a positive integer t such that $a_i = a_{i+t}$ for all $i \geq 0$. The smallest t satisfying the previous condition is called the period of w;*

2) *an eventually periodic word if there exist two positive integers k, p such that $a_i = a_{i+p}$, for all $i > k$;*

3) *an aperiodic word if it is not eventually periodic.*

Theorem 4. *The infinite Fibonacci word is aperiodic.*

This statement is proved in [10]. We consider the finite Fibonacci words f_n (1) as numbers written in the binary system and denote them by b_n. Denote by d_n the value of the number b_n in usual decimal numeration system. We write $b_n = d_n$ meaning that b_n and d_n are writing of the same number in different numeration systems.

Example.

$$f_0 = 1,\ f_1 = 0,\ f_2 = 01,\ f_3 = 010,\ f_4 = 01001,\ f_5 = 01001010,\ f_6 = 0100101001001,\ \dots,$$
$$b_0 = 1,\ b_1 = 0,\ b_2 = 1,\ b_3 = 10,\ b_4 = 1001,\ b_5 = 1001010,\ b_6 = 100101001001,\ \dots,$$
$$d_0 = 1,\ d_1 = 0,\ d_2 = 1,\ d_3 = 2,\ d_4 = 9,\ d_5 = 74,\ d_6 = 2377, \dots.$$

Formally, for arbitrary $n > 1$ f_n coincide with the b_n, taken with prefix 0: $f_n = 0b_n$.

Theorem 5. *For any finite Fibonacci word f_n, $n > 1$, in decimal numeration system we have*

$$d_n = d_{n-1} 2^{F_{n-1}} + d_{n-2}, \text{ where } d_0 = 1 \text{ and } d_1 = 0. \tag{2}$$

Proof. One can easily verify (2) for the first few n : $d_2 = b_2 = 1 = 0 + 1 = d_1 + d_0$, $d_3 = b_3 = 10 = 10 + 0 = d_2 2^1 + d_1$, $d_4 = b_4 = 1001 = 1000 + 01 = d_3 2^2 + d_2$, $d_5 = b_5 = 1001010 = 1001000 + 010 = d_4 2^3 + d_3$. Statement (2) follows from Theorem 3 (statement 4) and the equality $d_n = b_n = b_{n-1} \underbrace{0 \dots 0}_{F_{n-1}} + b_{n-2} = d_{n-1} 2^{F_{n-1}} + d_{n-2}$. □

Theorem 6. *Let $p > 3$. The sequence $d_n \pmod{p}$ has period $T(p) = p \cdot h(p)$.*

Proof. By Theorem 1 we have $\gcd(k(p-1), p) = 1$. By Corollary 1 we have $h(p)|k(p-1)$. Therefore $\gcd(h(p), p) = 1$. From (2) it follows that for arbitrary integer i, $0 \le i < h(p)$, if j runs from 0 to $p-1$ then numbers $d_{i+jh(p)} \pmod p$ runs all residues $\mod p$ or stationary. Then sequence $d_n \pmod p$ has period $p \cdot h(p)$. □

Let $d_0(m) = 1$, $w_0(m) = 1$ and for arbitrary integer n, $n \ge 1$, $d_n(m) = d_n \pmod m$ in binary numeration system, $w_n(m) = w_{n-1}(m)d_n(m)$. Denote by $w(m)$ the limit $w(m) = \lim_{n\to\infty} w_n(m)$.

Definition 2. *We say that*

1. $w_n(m)$ *is a finite FLP-word of type 1 by modulo* m*;*
2. $w(m)$ *is a infinite FLP-word of type 1 by modulo* m*.*

Theorem 7. *The infinite FLP-word of type 1* $w(m)$ *is periodic.*

Proof. The statement follows from (2) and Theorem 2 because there are only a finite number of $d_n \pmod m$ and $2^{F_{n-1}} \pmod{\varphi(m)}$ possible, and the recurrence of the first few terms of sequence $d_n \pmod m$ gives recurrence of all subsequent terms. □

Theorem 8. *Let* $p > 3$*. The sequence subwords* $d_n(p)$ *of the infinite FLP-word* $w(p)$ *of type 1 has period* $T(p) = p \cdot h(p)$*.*

Proof. The proof is a direct corollary of Theorem 6. □

Using Fibonacci words (1) we define periodic FLP-word $w^*(m)$ (infinite FLP-word of type 2 by modulo m). We denote as usual by ε the empty word [7]. First we define words $w_n^*(m)$. Let $w_n^*(m)$ be the last $F_{n+1}^*(m)$ symbols of the word f_n. If $F_{n+1}^*(m) = 0$ for some n, then $w_n^*(m) = \varepsilon$. Since $F_n^*(m)$ is periodic sequence with period $k(m)$, the sequence $|w_n^*(m)|$ is periodic with the same period.

Theorem 9. *The word length* $|w_n^*(m)|$ *coincides with* $F_{n+1}^*(m)$*.*

Proof. This is clear by construction of $w^*(m)$. □

Theorem 10. *The word* $w_n^*(m)$ *coincides with the word* $w_{n+k(m)}^*(m)$*.*

Proof. Since $f_n = f_{n-1}f_{n-2}$, the last F_{n-1} symbols of the word f_n coincide with the word f_{n-2}, and therefore the last F_n elements of the word f_{n+2k} coincide with the word f_{n-2} for any natural number k. The period $k(m)$ is an even number [14], so the last $F_{n+1}^*(m)$ elements of the words f_n and $f_{n+k(m)}$ are equivalent. □

Let $f_0^*(m) = 1$ and for arbitrary integer n, $n \ge 1$, $f_n^*(m) = f_{n-1}^*(m)w_n^*(m)$. Denote by $w^*(m)$ the limit $w^*(m) = \lim_{n\to\infty} f_n^*(m)$.

Definition 3. *We say that*

1) $f_n^*(m)$ *is a finite FLP-word of type 2 by modulo* m*;*
2) $w^*(m)$ *is a infinite FLP-word of type 2 by modulo* m*.*

Theorem 11. *The infinite FLP-word* $w^*(m)$ *of type 2 is a periodic word and sequence subwords* $w_n^*(m)$ *of* $w^*(m)$ *has period* $k(m)$*.*

Proof. The proof is a direct corollary of Theorem 10. □

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У цій статті означено два види періодичних слів (FLP-слова типу 1 та FLP-слова типу 2), які пов'язані зі словами Фібоначчі, та досліджено їх властивості.

Ключові слова і фрази: число Фібоначчі, слово Фібоначчі.