ISSN 2075-9827 e-ISSN 2313-0210 **http://www.journals.pu.if.ua/index.php/cmp**
Carpathian Math. Publ. 2016, **8** (2), 251–262 Карпатські матем. публ. 2016, Т.8, №2, С.251–262
doi:10.15330/cmp.8.2.251-262

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COINCIDENCE POINT THEOREMS FOR $\varphi - \psi-$CONTRACTION MAPPINGS IN METRIC SPACES INVOLVING A GRAPH

Some new coupled coincidence and coupled common fixed point theorems for $\varphi - \psi-$contraction mappings are established. We have also an application to some integral system to support the results.

Key words and phrases: coupled coincidence point, coupled fixed point, edge preserving, directed graph.

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INTRODUCTION AND PRELIMINARIES

In 2009, *Lakshmikantham* and *Ćirić* [2] introduced a generalization of monotonicity that called mixed g-monotone property. The authors established some coupled coincidence and coupled fixed point results related the mappings have mixed g-monotone property in the partially ordered metric space.

Definition 1 ([2]). *An element $(x, y) \in X^2$ is said to be a coupled coincidence point of a mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 2 ([2]). *An element $(x, y) \in X^2$ is said to be a coupled common fixed point of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.*

Definition 3 ([2]). *Let X be a nonempty set and $F : X^2 \to X$ and $g : X \to X$. We say F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.*

Now, we furnish the following class of auxiliary functions which will be used densely in the sequel.

Definition 4 ([11]). *Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$, which satisfy following:*

(φ_1) *φ is continuous and non-decreasing;*

(φ_2) *$\varphi(t) = 0$ iff $t = 0$;*

(φ_3) *$\varphi(t + s) \le \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$ and Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$, which satisfy (ψ_1);*

УДК 517.98, 515.14
2010 *Mathematics Subject Classification:* 47H10, 54H25.

(ψ_1) *ψ is continuous function with the condition $\varphi(t) > \psi(t)$ for all $t > 0$.*

By (φ_1), (φ_2) and (ψ_1) we have that $\psi(0) = 0$.

Next, we give the following coupled fixed point theorems as the main results of Işık and Türkoğlu [11].

Theorem 1 ([11]). *Let $(X, \leq, d)$ be a complete partially ordered metric space. Suppose that $F : X^2 \to X$ is a mapping having the mixed monotone property on X. Assume there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi\left(d\left(F\left(x,y\right),F\left(u,v\right)\right)\right) \leq 2^{-1} \times \psi\left(d\left(x,u\right) + d\left(y,v\right)\right) \tag{1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.
Suppose that either

(a) F is continuous or;

(b) X has the following properties:

1) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,

2) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n.

If there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Then F has a coupled fixed point.

The existence of fixed points of contraction mappings in metric space endowed with graph has been initiated by Jachymski [4]. Fixed point theorems for single valued and multivalued operators in such metric spaces have been studied by some authors since 2007 (see [5]— [10] and so on).

Let (X, d) be a metric space, Δ be a diagonal of X^2, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $(V(G), E(G))$. Furthermore, denote by G^{-1} the graph obtained from G by reversing the direction of the edges in G. Hence, $E(G^{-1}) = \{(x,y) \in X^2 : (y,x) \in E(G)\}$.

Definition 5 ([4]). *A function $g : X \to X$ is $G-$continuous if*

(a) for all x, $x_* \in X$ and any sequence $(n_i)_{i\in N}$ of positive integers, $(x_{n_i}) \to x_*$ and $(x_{n_i}, x_{n_i+1}) \in E(G)$, for $n \in N$, implies $g(x_{n_i}) \to gx_*$;

(b) for all y, $y_* \in X$ and any sequence $(n_i)_{i\in N}$ of positive integers, $(y_{n_i}) \to y_*$ and $(y_{n_i}, y_{n_i+1}) \in E(G^{-1})$, for $n \in N$, implies $g(y_{n_i}) \to gy_*$.

Definition 6 ([9]). *Let (X, d) be a complete metric space, G be a directed graph and $F : X \times X \to X$ be a mapping. Then*

(i) F is called $G-$continuous if for all (x, y), $(x_*, y_*) \in X^2$ and for any sequence $(n_i)_{i\in N}$ of positive integers such that $(x_{n_i}) \to x_*$, $(y_{n_i}) \to y_*$ as $i \to \infty$ and $(x_{n_i}, x_{n_i+1}) \in E(G)$, $(y_{n_i}, y_{n_i+1}) \in E(G^{-1})$, for $n \in N$, implies $F(x_{n_i}, y_{n_i}) \to F(x_*, y_*)$ and $F(y_{n_i}, x_{n_i}) \to F(y_*, x_*)$ as $i \to \infty$;

(ii) (X, d, G) *has property A if (a) for any sequence* $(x_n)_{n \in N}$ *in* X *with* $(x_n) \to x_*$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$*, then* $(x_n, x_*) \in E(G)$*; (b) for any sequence* $(y_n)_{n \in N}$ *in* X *with* $(y_n) \to y_*$ *as* $n \to \infty$ *and* $(y_n, y_{n+1}) \in E(G^{-1})$ *for* $n \in \mathbb{N}$*, then* $(y_n, y_*) \in E(G^{-1})$*.*

Consider the set $CCoinFix(Fg)$ of all coupled coincidence points of mappings $F : X^2 \to X$, $g : X \to X$ and the set $(X^2)_{Fg}$ as follows:

$$CCoinFix(Fg) = \left\{(x, y) \in X^2 : gx = F(x, y) \text{ and } gy = F(y, x)\right\} \text{ and}$$
$$\left(X^2\right)_{Fg} = \left\{(x, y) \in X^2 : (gx, F(x, y)) \in E(G) \text{ and } (gy, F(y, x)) \in E\left(G^{-1}\right)\right\}.$$

In 2016, Eshi et al. [12] introduced the concept of $G - g-$contraction mapping as follows.

Definition 7 ([12]). *$F : X^2 \to X$ is called $G - g-$contraction if:*

(i) g *is edge preserving, i.e.,* $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E(G^{-1}) \Rightarrow (g(gx), g(gu)) \in E(G)$ *and* $(g(gy), g(gv)) \in E(G^{-1})$*;*

(ii) F *is* $g-$*edge preserving, i.e.,* $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E(G^{-1}) \Rightarrow (F(x, y), F(u, v)) \in E(G)$ *and* $(F(y, x), F(v, u)) \in E(G^{-1})$*;*

(iii) for all x, y, u, $v \in X$ *such that,* $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E(G^{-1})$, $d(F(x, y), F(u, v)) \leq \frac{k}{2}[(gx, gu) + (gy, gv)]$*, where* $k \in \left[0, \frac{1}{2}\right)$ *is called the contraction constant of* F*.*

Proposition 1 ([12]). *If* $F : X^2 \to X$ *is* $g-$*edge preserving and* $F(X^2) \subseteq g(X)$*. Also, let* $(x_n)_{n \in N}$, $(y_n)_{n \in N}$, $(u_n)_{n \in N}$ *and* $(v_n)_{n \in N}$ *be sequences in metric space* (X, d) *endowed with a directed graph* G*. Then*

(a) $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E(G^{-1}) \Rightarrow (F(x_n, y_n), F(u_n, v_n)) \in E(G)$ *and* $(F(y_n, x_n), F(v_n, u_n)) \in E(G^{-1})$ *for all* $n \in N$*;*

(b) $(x, y) \in (X^2)_{Fg} \Rightarrow (F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \in E(G)$ *and* $(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \in E(G^{-1})$ *for all* $n \in N$*;*

(c) $(x, y) \in (X^2)_{Fg} \Rightarrow (F(x_n, y_n), F(y_n, x_n)) \in (X^2)_{Fg}$ *for all* $n \in N$*.*

In this paper, we prove coupled coincidence and coupled common fixed point theorems for contaction mappings in metric spaces endowed with a directed graph. Our results extend and improve the results obtained by Eshi et al. in [12], Işık and Türkoğlu in [11], Chifu and Petrusel in [9] so on. Moreover, we have an application to some integral system to support the results.

1 MAIN RESULTS

Definition 8. *Let* (X, d) *be a complete metric space endowed with a directed graph* G*. The mappings* $F : X^2 \to X$, $g : X \to X$ *are called a* $\varphi - \psi-$*contraction if:*

1) g *is edge preserving,* F *is* $g-$*edge preserving;*

2) *there exists* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that for all* $x, y, u, v \in X$ *satisfying* $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E(G^{-1})$,

$$\varphi(d(F(x,y), F(u,v))) \leq 2^{-1} \times \psi(d(gx,gu) + d(gy,gv)). \tag{2}$$

Lemma 1. *Let* (X,d) *be complete metric space endowed with a directed graph* G, *and let* $F : X^2 \to X$, $g : X \to X$ *be a* $\varphi - \psi-$*contraction and* $F(X^2) \subseteq g(X)$. *Also, let* (x_n), (y_n) *be sequences in* X. *If for each* $(x,y) \in (X^2)_{Fg}$, *then*

$$\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0 \text{ as } n \to \infty.$$

Proof. Let x_0, $y_0 \in X$. Since $F(X^2) \subseteq g(X)$, we can constitute x_1, $y_1 \in X$ such that $F(x_0,y_0) = gx_1$ and $F(y_0,x_0) = gy_1$. Again, we can constitute x_2, $y_2 \in X$ such that $F(x_1,y_1) = gx_2$ and $F(y_1,x_1) = gy_2$. Continuing this procedure above we obtain sequences (x_n) and (y_n) in X such that

$$gx_n = F(x_{n-1}, y_{n-1}) \text{ and } gy_n = F(y_{n-1}, x_{n-1}) \tag{3}$$

for all $n \geq 1$, $x = x_0$ and $y = y_0$. Let $(x_0,y_0) \in (X^2)_{Fg}$ such that $(gx_0, F(x_0,y_0)) = (gx_0, gx_1) \in E(G)$ and $(gy_0, F(y_0,x_0)) = (gy_0, gy_1) \in E(G^{-1})$. Then, by Proposition 1 (b), we get $(F(x_{n-1},y_{n-1}), F(x_n,y_n)) \in E(G)$ and $(F(y_{n-1},x_{n-1}), F(y_n,x_n)) \in E(G^{-1})$. Thus we have that $(gx_n, gx_{n+1}) \in E(G)$ and $(gy_n, gy_{n+1}) \in E(G^{-1})$ for all $n \in N$. Using the $\varphi - \psi-$contaction (2) and (3), we have that

$$\begin{aligned} \varphi(d(gx_{n+1}, gx_n)) &= \varphi(d(F(x_n,y_n), F(x_{n-1},y_{n-1}))) \\ &\leq 2^{-1} \times \psi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) \text{ and} \end{aligned} \tag{4}$$

$$\begin{aligned} \varphi(d(gy_{n+1}, gy_n)) &= \varphi(d(F(y_n,x_n), F(y_{n-1},x_{n-1}))) \\ &\leq 2^{-1} \times \psi(d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})) \end{aligned} \tag{5}$$

for all $n \in N$. From (4) and (5) we get

$$\varphi(d(gx_{n+1}, gx_n)) + \varphi(d(gy_{n+1}, gy_n)) \leq \psi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})). \tag{6}$$

From (φ_3), we obtain that

$$\varphi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) \leq \psi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})).$$

Regarding the properties φ and ψ, we conclude that

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}).$$

It follows that $\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)$ is decreasing. Then $\lim_{n\to\infty} \rho_n = \rho$ for some $\rho \geq 0$. Taking the limit as $n \to \infty$ in (6), we have $\varphi(\rho) \leq \psi(\rho)$. From the properties φ and ψ, we obtain that $\rho = 0$, and thus

$$\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0 \text{ as } n \to \infty.$$

□

Theorem 2. *Let (X, d) be complete metric space endowed with a directed graph G, and let $F : X^2 \to X$, $g : X \to X$ be a $\varphi - \psi-$contraction and $F\left(X^2\right) \subseteq g(X)$. Let g be $G-$continuous and commutes with F. Suppose that:*

(i) F is $G-$continuous, or

(ii) the tripled (X, d, G) has a property A.

Then $CCoinFix(Fg) \neq \varnothing$ iff $\left(X^2\right)_{Fg} \neq \varnothing$.

Proof. Let $CCoinFix(Fg) \neq \varnothing$. Then there exists $(x_*, y_*) \in CCoinFix(Fg)$ such that $(gx_*, F(x_*, y_*)) = (gx_*, gx_*) \in \Delta \subset E(G)$ and $(gy_*, F(y_*, x_*)) = (gy_*, gy_*) \in \Delta \subset E\left(G^{-1}\right)$. It follows that $(x_*, y_*) \in \left(X^2\right)_{Fg}$, so that $\left(X^2\right)_{Fg} \neq \varnothing$.

Now, suppose that $\left(X^2\right)_{Fg} \neq \varnothing$. Then there exists $(x_0, y_0) \in \left(X^2\right)_{Fg}$, e.g., $(gx_0, F(x_0, y_0)) \in E(G)$, $(gy_0, F(y_0, x_0)) \in E\left(G^{-1}\right)$. Then, by Proposition 1 (b), we get $(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \in E(G)$ and $(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \in E\left(G^{-1}\right)$. Thus we have that

$$(gx_n, gx_{n+1}) \in E(G) \text{ and } (gy_n, gy_{n+1}) \in E\left(G^{-1}\right) \tag{7}$$

for all $n \in N$. By Lemma 1, we have

$$\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0 \text{ as } n \to \infty. \tag{8}$$

Next, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. If possible, assume that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\left\{gx_{n(k)}\right\}$, $\left\{gx_{m(k)}\right\}$ of $\{gx_n\}$ and $\left\{gy_{n(k)}\right\}$, $\left\{gy_{m(k)}\right\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\gamma_k := d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \geq \varepsilon. \tag{9}$$

Farther, corresponding to $m(k)$, we can choose $n(k)$ in the manner that it is the smallest integer for which (9) holds. Then,

$$d\left(gx_{n(k)-1}, gx_{m(k)}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right) < \varepsilon. \tag{10}$$

Using (9), (10), and triangular inequality, we obtain

$$\varepsilon \leq \gamma_k < \varepsilon + d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gy_{n(k)}, gy_{n(k)-1}\right). \tag{11}$$

Letting $k \to \infty$ in (11) and by (8), we have

$$\gamma_k := d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \to \varepsilon \text{ as } k \to \infty. \tag{12}$$

From the triangle inequality, we get

$$\begin{aligned}\gamma_k &= d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \\ &\leq d\left(gx_{n(k)+1}, gx_{m(k)+1}\right) + d\left(gy_{n(k)+1}, gy_{m(k)+1}\right) + \rho_{n(k)} + \rho_{m(k)}.\end{aligned}$$

From property φ, we have

$$\begin{aligned}
\varphi(\gamma_k) &\leq \varphi\left(d\left(gx_{n(k)+1}, gx_{m(k)+1}\right)\right) + \varphi\left(d\left(gy_{n(k)+1}, gy_{m(k)+1}\right)\right) + \varphi\left(\rho_{n(k)} + \rho_{m(k)}\right) \\
&\leq \varphi\left(d\left(F\left(x_{n(k)}, y_{n(k)}\right), F\left(x_{m(k)}, y_{m(k)}\right)\right)\right) \\
&+ \varphi\left(d\left(F\left(y_{n(k)}, x_{n(k)}\right), F\left(y_{m(k)}, x_{m(k)}\right)\right)\right) + \varphi\left(\rho_{n(k)} + \rho_{m(k)}\right) \\
&\leq 2^{-1} \times \psi\left(d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)\right) \\
&+ 2^{-1} \times \psi\left(d\left(gy_{n(k)}, gy_{m(k)}\right) + d\left(gx_{n(k)}, gx_{m(k)}\right)\right) + \varphi\left(\rho_{n(k)} + \rho_{m(k)}\right) \\
&\leq \psi(\gamma_k) + \varphi\left(\rho_{n(k)} + \rho_{m(k)}\right).
\end{aligned} \tag{13}$$

Taking $k \to \infty$ in (13) and from (8) and (12), we obtain a following contradiction:

$$\varphi(\varepsilon) \leq \psi(\varepsilon) + \varphi(0) = \psi(\varepsilon).$$

Thus, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. As (X, d) is complete, there exists $x_*, y_* \in X$ such that

$$gx_n \to x_* \text{ and } gy_n \to y_* \text{ as } n \to \infty. \tag{14}$$

Since g be $G-$continuous, we have

$$g(gx_n) \to gx_* \text{ and } g(gy_n) \to gy_* \text{ as } n \to \infty.$$

Moreover as F and g are commutative

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n), \tag{15}$$

$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n). \tag{16}$$

We now prove that

$$F(x_*, y_*) = gx_* \text{ and } F(y_*, x_*) = gy_*.$$

Suppose assumption (i) holds. From (15) and (16), we have

$$\begin{aligned}
gx_* &= \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} F(gx_n, gy_n) = F(x_*, y_*), \\
gy_* &= \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} F(gy_n, gx_n) = F(y_*, x_*);
\end{aligned}$$

that is, (x_*, y_*) is a coincidence point of F and g.

Suppose now assumption (ii) holds. From (7) and (14), using property A, we get $(gx_n, x_*) \in E(G)$ and $(gy_n, y_*) \in E\left(G^{-1}\right)$ for each $n \in \mathbb{N}$. By (2), we get

$$\begin{aligned}
&\varphi(d(gx_*, F(x_*, y_*)) + d(gy_*, F(y_*, x_*))) \\
&\leq \varphi(d(gx_*, gx_{n+1}) + d(gx_{n+1}, F(x_*, y_*)) + d(gy_*, gy_{n+1}) + d(gy_{n+1}, F(y_*, x_*))) \\
&\leq \varphi(d(gx_*, gx_{n+1})) + \varphi(d(F(x_n, y_n), F(x_*, y_*))) \\
&+ \varphi(d(gy_*, gy_{n+1})) + \varphi(d(F(y_n, x_n), F(y_*, x_*))) \\
&\leq \psi(d(gx_n, gx_*) + d(gy_n, gy_*)) + \varphi(d(gx_*, gx_{n+1})) + \varphi(d(gy_*, gy_{n+1})).
\end{aligned}$$

Letting $n \to \infty$, we obtain $\varphi(d(gx_*, F(x_*, y_*)) + d(gy_*, F(y_*, x_*))) = 0$. From properties φ, we have $d(gx_*, F(x_*, y_*)) + d(gy_*, F(y_*, x_*)) = 0$. Hence, $gx_* = F(x_*, y_*)$ and $gy_* = F(y_*, x_*)$. □

Definition 9. *Let (X,d) be a complete metric space endowed with a directed graph G. The mappings $F: X^2 \to X$, $g: X \to X$ are called a $\psi-$contraction if:*

(i) g is edge preserving, F is $g-$edge preserving;

(ii) there exists $\psi \in \Psi$ such that for all $x, y, u, v \in X$ satisfying $(gx, gu) \in E(G)$ and $(gy, gv) \in E(G^{-1})$,

$$d(F(x,y), F(u,v)) \leq 2^{-1} \times \psi(d(gx, gu) + d(gy, gv)).$$

Theorem 3. *Let (X,d) be complete metric space endowed with a directed graph G, and let $F: X^2 \to X$, $g: X \to X$ be a $\psi-$contraction and $F(X^2) \subseteq g(X)$. Let g be $G-$continuous and commutes with F. Suppose that:*

(i) F is $G-$continuous, or

(ii) the tripled (X,d,G) has a property A.

Then $CCoinFix(Fg) \neq \varnothing$ iff $(X^2)_{Fg} \neq \varnothing$.

Proof. Taking $\varphi(t) = t$, along the lines of the proof of Theorem 2, we have the requested results. By virtue of the analogy, we skip the details of the proof. □

If we choose the functions $\varphi(t) = t$ and $\psi(t) = kt$, for $t \in [0,\infty)$ and $k \in \left[0, \frac{1}{2}\right)$ in Theorem 2, we have the following corollary.

Corollary 1 ([12]). *Let (X,d) be complete metric space endowed with a directed graph G, and let $F: X^2 \to X$ be a $G-g-$contraction with contraction constant $k \in \left[0, \frac{1}{2}\right)$ and $F(X^2) \subseteq g(X)$. Let g be $G-$continuous and commutes with F. Suppose that (i) F is $G-$continuous, or (ii) the tripled (X,d,G) has a property A. Then $CCoinFix(Fg) \neq \varnothing$ iff $(X^2)_{Fg} \neq \varnothing$.*

Remark 1. *In the case where $(X, \preccurlyeq)$ is partially ordered complete metric space, taking $E(G) = \{(x,y) \in X \times X : x \preccurlyeq y\}$, the functions $\varphi(t) = t$ and $\psi(t) = kt$, for $t \in [0,\infty)$ and $k \in [0,1)$, Theorem 2 generalize and improve the results obtained by Bhaskar and Lakshmikantham ([1], Theorem 2.1) and Chifu and Petrusel ([9], Theorem 2.1). If we take the function $\psi(t) = \varphi(t) - \psi_1(t)$, for $t \in [0,\infty)$, where $\psi_1 \in \Psi$, Theorem 2 generalize the results given by Luong and Thuan ([3], Theorem 2.1). In Theorem 2, let g be the identity mapping. Then it is easy to see that our conclusions enhance the results achieved by Işık and Türkoğlu [11].*

Theorem 4. *In addition to Theorem 2, suppose that for any two elements $(x,y), (x_*, y_*) \in X^2$, there exists $(p,r) \in X^2$ such that*

$$(F(x,y), F(p,r)) \in E(G), \ (F(y,x), F(r,p)) \in E\left(G^{-1}\right) \text{ and}$$
$$(F(x_*,y_*), F(p,r)) \in E(G), \ (F(y_*,x_*), F(r,p)) \in E\left(G^{-1}\right).$$

Then, F and g have a unique coupled common fixed point.

Proof. By Theorem 2, we have $CCoinFix\,(Fg) \neq \varnothing$. Suppose $(x, y), (x_*, y_*)$ are coupled fixed points of F, e.g.,

$$gx = F(x, y),\ gy = F(y, x) \text{ and } gx_* = F(x_*, y_*),\ gy_* = F(y_*, x_*). \tag{17}$$

Consider sequences $\{p_n\}$ and $\{r_n\}$ as follows

$$p_0 = p, r_0 = r, p_{n+1} = F(p_n, r_n) \text{ and } r_{n+1} = F(r_n, p_n) \text{ for all } n \geq 0.$$

From assumption, we get

$$(F(x, y), F(p, r)) = (gx, gp_1) \in E(G),\ (F(y, x), F(r, p)) = (gy, gr_1) \in E\left(G^{-1}\right) \text{ and}$$

$$(F(x_*, y_*), F(p, r)) = (gx_*, gp_1) \in E(G),$$
$$(F(y_*, x_*), F(r, p)) = (gy_*, gr_1) \in E\left(G^{-1}\right).$$

Since F is $g-$edge preserving, we have

$$(F(x, y), F(p_1, r_1)) = (gx, gp_2) \in E(G),\ (F(y, x), F(r_1, p_1)) = (gy, gr_2) \in E\left(G^{-1}\right),$$

$$(F(x_*, y_*), F(p_1, r_1)) = (gx_*, gp_2) \in E(G),$$
$$(F(y_*, x_*), F(r_1, p_1)) = (gy_*, gr_2) \in E\left(G^{-1}\right).$$

Continuing this procedure above, we obtain

$$(gx, gp_n) \in E(G),\ (gy, gr_n) \in E\left(G^{-1}\right) \text{ and}$$

$$(gx_*, gp_n) \in E(G),\ (gy_*, gr_n) \in E\left(G^{-1}\right).$$

By (2), we have

$$\begin{aligned}
&\varphi(d(gx_*, p_{n+1})) + \varphi(d(r_{n+1}, gy_*)) \\
&\quad = \varphi(d(F(x_*, y_*), F(p_n, r_n))) + \varphi(d(F(r_n, p_n), F(y_*, x_*))) \\
&\quad \leq 2^{-1} \times \psi(d(gx_*, gp_n) + d(gy_*, gr_n)) + 2^{-1} \times \psi(d(gr_n, gy_*) + d(gp_n, gx_*)).
\end{aligned}$$

By the property of φ, we have

$$\varphi(d(gx_*, gp_{n+1}) + d(gr_{n+1}, gy_*)) \leq \psi(d(gx_*, gp_n) + d(gy_*, gr_n)). \tag{18}$$

By (φ_1) and (ψ_1), we have

$$d(gx_*, gp_{n+1}) + d(gr_{n+1}, gy_*) \leq d(gx_*, gp_n) + d(gy_*, gr_n).$$

Therefore, the sequence $\{f_n\}$ defined by $f_n = d(gx_*, gp_n) + d(gy_*, gr_n)$, is a nonnegative decreasing sequence, and consequently, there exists some $f \geq 0$ such that

$$d(gx_*, gp_n) + d(gy_*, gr_n) \to f \text{ as } n \to \infty.$$

Suppose that $f > 0$. Then taking limit as $n \to \infty$ in (18) and using the continuity of φ and ψ, we get

$$\varphi(f) \leq \psi(f)$$

which implies, from the properties of φ and ψ, that $\psi(f) = 0$ and eventually, $f = 0$. Hence

$$d(gx_*, gp_n) + d(gy_*, gr_n) \to 0 \text{ as } n \to \infty,$$

which implies

$$\lim_{n\to\infty} d(gx_*, gp_n) = 0 = \lim_{n\to\infty} d(gy_*, gr_n).$$

Similarly

$$\lim_{n\to\infty} d(gx, gp_n) = 0 = \lim_{n\to\infty} d(gy, gr_n).$$

By the triangular inequality we obtain

$$d(gx_*, gx) \le d(gx_*, gp_n) + d(gp_n, gx), \quad d(gy_*, gy) \le d(gy_*, gr_n) + d(gr_n, gy), \tag{19}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (19), we obtain that $d(gx_*, gx) = 0 = d(gy_*, gy)$. Hence, we get

$$gx_* = gx \text{ and } gy_* = gy. \tag{20}$$

Let $gx_* = gx = t$ and $gy_* = gy = s$.

Owing to commutativity of F and g, by (17), we have

$$g(gx_*) = g(F(x_*, y_*)) = F(gx_*, gy_*) \Rightarrow gt = F(t, s) \text{ and}$$

$$g(gy_*) = g(F(y_*, x_*)) = F(gy_*, gx_*) \Rightarrow gs = F(s, t).$$

Hence, (t, s) is a coupled coincidence point. Thus, by repeating previous argument for (x_*, y_*) and (t, s),

$$gx_* = gt \Rightarrow t = gt \text{ and } gy_* = gs \Rightarrow s = gs.$$

Therefore, $t = gt = F(t, s)$ and $s = gs = F(s, t)$. Hence, (t, s) is a coupled common fixed point of F and g.

To show the uniqueness, suppose that (k, l) is another coupled common fixed point of F and g. Hence,

$$k = gk = F(k, l) \text{ and } l = gl = F(l, k). \tag{21}$$

By (20), we have

$$gk = gt = t \text{ and } gl = gs = s. \tag{22}$$

Thus, from (21) and (22), we get $k = t$ and $l = s$. Then, $k = gk = gt = t$ and $l = gl = gs = s$. □

2 APPLICATION

We consider the following integral system:

$$\begin{aligned} x(t) &= h(t) + \lambda \int_{-t}^{t} A(t, s, x(s), y(s))\, ds, \\ y(t) &= h(t) + \lambda \int_{-t}^{t} A(t, s, y(s), x(s))\, ds, \end{aligned} \tag{23}$$

for $t \in [-T, T]$, $T > 0$, $\lambda \in R$.

Recall that the Bielecki-type norm on $X := C([-T,T], R^n)$,

$$\|x\|_B = \max_{t\in[-T,T]} \left| x(t) e^{-\tau(t-T)} \right| \text{ for all } x \in X,$$

where $\tau > 0$, is arbitrarily chosen. Consider $\|x-y\|_B = \max_{t\in[-T,T]} |x(t) - y(t)| e^{-\tau(t-T)}$ for all $x, y \in X$.

Define the graph G with partial order relation by

$$x, y \in X, x \leq y \Leftrightarrow x(t) \leq y(t) \quad \text{for any } t \in I.$$

Thus $(X, \|x\|_B)$ is complete metric space endowed with a directed graph G.

If we take into consideration $E(G) := \{(x,y) \in X^2 : x \leq y\}$, then $\Delta(X^2) \subseteq E(G)$. On the other hand $E(G^{-1}) := \{(x,y) \in X^2 : y \leq x\}$. Furthermore, $(X, \|x\|_B, G)$ has property A.

Then $(X^2)_{Fg} = \{(x,y) \in X^2 : gx \leq F(x,y) \text{ and } F(y,x) \leq gy\}$. We consider the following conditions:

1. $A : [-T,T] \times [-T,T] \times R^n \times R^n \to R^n$ and $h : [-T,T] \to R^n$ are continuous;
2. for all $x, y, u, v \in R^n$ with $x \leq u$, $v \leq y$ we have $A(t,s,x,y) \leq A(t,s,u,v)$ for all t, $s \in [-T,T]$;
3. for all $t, s \in [-T,T]$ and for all $x, y, u, v \in R^n$

   $$|A(t,s,x,y) - A(t,s,u,v)| \leq \psi(|x-u| + |y-v|),$$

   where $\psi \in \Psi$ such that $\psi(\alpha t) \leq \alpha\psi(t)$ for all $t \in [-T,T]$ and for all $\alpha \geq 0$;
4. there exists $(x_0, y_0) \in X^2$ such that

   $$\begin{aligned} x_0(t) &\leq h(t) + \lambda \int_{-t}^{t} A(t,s,x_0(s), y_0(s))\, ds, \\ y_0(t) &\geq h(t) + \lambda \int_{-t}^{t} A(t,s,y_0(s), x_0(s))\, ds, \end{aligned}$$

   where $t \in [-T,T]$.

Theorem 5. *Suppose that conditions (1)—(4) are satisfied. Then there exists at least one solution of (23).*

Proof. Let $F : X^2 \to X$ and $g : X \to X$ be defined as

$$\begin{aligned} &F(x,y)(t) = h(t) + \lambda \int_{-t}^{t} A(t,s,x(s), y(s))\, ds, \ t \in [-T,T], \\ &g(x)(t) = x(t). \end{aligned}$$

Then (23) can be indicated as

$$gx = F(x,y) \text{ and } gy = F(y,x). \tag{24}$$

By (24), the solution of this system is a coupled coincidence point of the mappings F and g, if we prove the assumptions in Theorem 3.

Let $x, y, u, v \in X$ be such that $gx \leq gu$ and $gv \leq gy$,

$$\begin{aligned} F(x,y)(t) &= h(t) + \lambda \int_{-t}^{t} A(t,s,x(s),y(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,g(x)(s),g(y)(s))\,ds \\ &\leq h(t) + \lambda \int_{-t}^{t} A(t,s,g(u)(s),g(v)(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,u(s),v(s))\,ds = F(u,v)(t) \end{aligned}$$

for all $t \in [-T,T]$. Therefore $(F(x,y), F(u,v)) \in E(G)$.

$$\begin{aligned} F(v,u)(t) &= h(t) + \lambda \int_{-t}^{t} A(t,s,v(s),u(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,g(v)(s),g(u)(s))\,ds \\ &\leq h(t) + \lambda \int_{-t}^{t} A(t,s,g(y)(s),g(x)(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,y(s),x(s))\,ds = F(y,x)(t) \end{aligned}$$

for all $t \in [-T,T]$. Therefore $(F(y,x), F(v,u)) \in E(G^{-1})$. Then, F is $g-$edge preserving.

We shall show that F is $\psi-$contraction. We have

$$\begin{aligned} &|F(x,y)(t) - F(u,v)(t)| \\ &\leq |\lambda| \int_{-t}^{t} |A(t,s,x(s),y(s)) - A(t,s,u(s),v(s))|\,ds \\ &\leq |\lambda| \int_{-t}^{t} \psi(|x(s)-u(s)| + |y(s)-v(s)|) \left(e^{-\tau(t-T)} e^{\tau(t-T)}\right) \\ &\leq \frac{|\lambda|}{\tau} \psi(\|x-u\|_B + \|y-v\|_B)\, e^{\tau(t-T)} \end{aligned}$$

for all $t \in [-T,T]$; therefore,

$$|F(x,y)(t) - F(u,v)(t)|\, e^{-\tau(t-T)} \leq \frac{|\lambda|}{\tau} \psi(\|x-u\|_B + \|y-v\|_B). \tag{25}$$

Applying maximum in (25), we have

$$\|F(x,y) - F(u,v)\|_B \leq \frac{|\lambda|}{\tau} \psi(\|x-u\|_B + \|y-v\|_B).$$

If we take τ such that $\frac{|\lambda|}{\tau} = \frac{1}{2} \Leftrightarrow |\lambda| = \frac{\tau}{2}$, then F is $\psi-$contraction.

From assumption (4) show that there exists $(x_0, y_0) \in X^2$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \leq F(y_0, x_0)$, which implies that $(X^2)_{Fg} \neq \varnothing$. Also, F and g are commutative.

On the other hand, by virtue of (1) and of the fact that $(X, \|x\|_B, G)$ has property A we get that (i) or (ii) from Theorem 3 is fulfilled. Hence, there exists a coupled coincidence point $(x_*, y_*) \in X^2$ of the mapping F and g, which is the solution of the integral system (23). □

Competing interests. The authors declare that they have no competing interests.

Acknowledgement. The authors would like to express their sincere appreciation to the referees for their very helpful suggestions and many kind comments.

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Received 02.08.2016

Revised 28.11.2016

Йолакан Е., Кізілтанк Г., Кір М. *Теореми про точки співпадіння для $\varphi - \psi -$скоротних відображень в метричних просторах еволюції графів* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 251–262.

У статті отримано деякі нові теореми про зв'язні точки співпадання та зв'язні фіксовані точки для $\varphi - \psi -$скоротних відображень. Також були отримані застосування отриманих результатів у дослідженні інтегральних систем.

Ключові слова і фрази: зв'язна точка співпадання, зв'язна фіксована точка, вершина збереження, напрямлений граф.