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REPRESENTATION OF SPECTRA OF ALGEBRAS OF BLOCK-SYMMETRIC ANALYTIC FUNCTIONS OF BOUNDED TYPE

The paper contains a description of a symmetric convolution of the algebra of block-symmetric analytic functions of bounded type on ℓ_1-sum of the space $\mathbb{C}^2$. We show that the specrum of such algebra does not coincide of point evaluation functionals and we describe characters of the algebra as functions of exponential type with plane zeros.

Key words and phrases: algebraic basis, block-symmetric polynomials, block-symmetric analytic functions, spectrum, symmetric intertwining, symmetric convolution.

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INTRODUCTION

In resent years there is an increasing interest to investigations of invariants of the permutation group S_∞ of positive integers. This group can be represented on a Banach space X with symmetric basis as a group of operators of perturbation of basis vectors. The action of this group has a natural extension to the action on the algebra $H_b(X)$ of analytic functions of bounded type on X. Invariants of this representation of S_∞ are so-called symmetric analytic functions of bounded type on X. The algebras of symmetric analytic functions $H_{bs}(X)$ were investigated by many authors ([1, 2, 9]). In particular, it is known that $H_{bs}(\ell_p)$ admits an algebraic basis for $1 \leq p < \infty$.

On the other hand, there are more representations of S_∞ in Banach spaces. For example, if $\mathcal{X}$ is a directs sum of infinite many of "blocks" which consists of linear subspaces isomorphic each to other, then S_∞ may to act as a group of permutations of the "blocks". For this case we have invariants — the algebra of block-symmetric analytic functions. Note that this algebra is much more complicated and in the general case has no algebraic basis (see e.g. [6, 12]). In the case $\dim \mathcal{X} < \infty$, block-invariant polynomials were investigated in the classical theory of invariants [5, 11].

1 MAIN RESULTS

Let

$$\mathcal{X}^2 = \oplus_{\ell_1}\mathbb{C}^2 = \ell_1 \otimes \mathbb{C}^2$$

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be an infinite ℓ_1-sum of copies of Banach space $\mathbb{C}^2$. So any element $u = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X}^2$ can be represented as a sequence $u = \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \ldots \right)$, where $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathbb{C}^2$, with the norm $\|u\| = \sum\limits_{k=1}^{\infty} (|x_k| + |y_k|)$. Also, we will use notation $u(x,y)$, where $x, y \in \ell_1$, $x = \sum\limits_{k=1}^{\infty} x_k e_k$, $y = \sum\limits_{k=1}^{\infty} y_k e_k$. Here e_k is the standard symmetric basis in ℓ_1.

A polynomial P on the space $\mathcal{X}^2$ is called *block-symmetric* (or *vector-symmetric*) if:

$$P\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right) = P\left(\begin{pmatrix} x_{\sigma(1)} \\ y_{\sigma(1)} \end{pmatrix}, \ldots, \begin{pmatrix} x_{\sigma(m)} \\ y_{\sigma(m)} \end{pmatrix}, \ldots \right),$$

for every permutation σ on the set of natural numbers $\mathbb{N}$, where $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{C}^2$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In [7] it was shown that the following vectors form an algebraic bases of "power" block-symmetric polynomials of $\mathcal{P}_{vs}(\mathcal{X}^2)$:

$$H^{p,n-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{n-p}, \tag{1}$$

where $0 \le p \le n$, $(x_i, y_i) \in \mathbb{C}^2$, $i \ge 1$. Also, there is a basis of "elementary" block-symmetric polynomials:

$$R^{p,n-p}(x,y) = \sum_{\substack{i_1 < \ldots < i_p \\ j_1 < \ldots < j_{n-p} \\ i_k \ne j_l}}^{\infty} x_{i_1} \ldots x_{i_p} y_{j_1} \ldots y_{j_{n-p}}, \tag{2}$$

where $0 \le p \le n$, $n \ge 1$ and $(x_i, y_i) \in \mathbb{C}^2$.

In the finite case, generating elements of algebra of block-symmetric polynomials on the space $\mathcal{X}_m^2 = \oplus_{\ell_1}^m \mathbb{C}^2$ are algebraic dependent. In [12] was proved the following theorem.

Theorem 1. *For every nonsymmertic polynomial ξ of a system of generating elements of $\mathcal{P}_{vs}(\mathcal{X}_m^2)$ there exist symmetric polynomials a_k in this system such that*

$$\xi^{m!} - a_1 \xi^{m!-1} + \cdots + (-1)^{m!-1} a_{m!-1} \xi^1 + (-1)^{m!} a_{m!} = 0.$$

Let σ be some permutation on the set of natural numbers $\mathbb{N}$. We denote by T_σ the linear operator on $\mathcal{X}^2$ associated with σ by the formula

$$T_\sigma \Big(\sum_{k=1}^{\infty} x_k e_k, \sum_{k=1}^{\infty} y_k e_k \Big) = \Big(\sum_{k=1}^{\infty} x_{\sigma(k)} e_k, \sum_{k=1}^{\infty} y_{\sigma(k)} e_k \Big).$$

For any $(x,y), (z,t) \in \mathcal{X}^2$ we denote $(x,y) \sim (z,t)$ if there exists a permutation σ on $\mathbb{N}$ such that $(x,y) = T_\sigma(z,t)$.

Theorem 2. *Let $(x,y), (z,t) \in \mathcal{X}^2$ and $H^{p,i-p}(x,y) = H^{p,i-p}(z,t)$, where $0 \le p \le i$ for every $i \ge 1$. Then $(x,y) \sim (z,t)$.*

Proof. Let $G(x)$ be a symmetric polynomial of degree n in the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ on ℓ_1. We set $P(x,y) = G(x+jy)$, where $0 \le j \le n$, $(x,y) \in \mathcal{X}^2$. Obviously, $P(x,y)$ is a block-symmetric polynomial. In [13] it was proved that the block-symmetric polynomial $P(x,y)$ will be represented as an algebraic combination of $F_k(x+jy)$, where $F_n(x) = \sum_{k=1}^{\infty} x_k^n$. So for the polynomial $P(x,y)$ according to [1, Theorem 1.3] we obtain that $x + jy = T_\sigma(z+jt)$. On the other hand, we can denote by $T_\sigma(x) = T_\sigma(x,0)$, $T_\sigma(y) = T_\sigma(0,y)$ and we obtain that $x + jy = T_\sigma((z,0) + j(0,t)) = T_\sigma(z) + jT_\sigma(t)$.

For us it is enough to consider $j = 1,2$. We obtain two equalities

$$x + y = T_\sigma(z) + T_\sigma(t), \quad x + 2y = T_\sigma(z) + 2T_\sigma(t),$$

which imply $x = T_\sigma(z)$, $y = T_\sigma(t)$. That is, $(x, jy) = T_\sigma(z,t)$.

Since $H^{p,i-p}(x,y) = H^{p,i-p}(z,t)$, $0 \le p \le i$ for every $i \ge 1$ it follows that $F_i(x+jy) = F_i(z+jt)$ and so $(x,y) \sim (z,t)$. □

Let $H_{bvs}(\mathcal{X}^2)$ be the algebra of block-symmetric analytic functions of bounded type (that is, bounded on bounded subsets) on $\mathcal{X}^2$. This algebra is generated by polynomials $H^{1,0}, \ldots, H^{p,n-p}, \ldots, H^{0,n}, \ldots$, where $n \ge 1, 0 \le p \le n$. Let us denote by $M_{bvs}(\mathcal{X}^2)$ the specrum of algebra $H_{bvs}(\mathcal{X}^2)$.

For given $(x,y), (z,t) \in \mathcal{X}^2$,

$$(x,y) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right)$$

and

$$(z,t) = \left(\begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots \right),$$

where $(x_i, y_i), (z_i, t_i) \in \mathbb{C}^2$, we put

$$(x,y) \bullet (z,t) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots \right)$$

and define

$$\mathcal{T}_{(z,t)}(f)(x,y) := f((x,y) \bullet (z,t)). \tag{3}$$

We will say that $(x,y) \to (x,y) \bullet (z,t)$ is the *intertwining* and the operator $\mathcal{T}_{(z,t)}$ is the *intertwining operator*. Some elementary properties of $\mathcal{T}_{(z,t)}$ was proved in [6].

Let $\mathbb{C}\{t_1, t_2\}$ be the space of all pover series over $\mathbb{C}^2$. We denote by $\mathcal{R}$ and $\mathcal{H}$ the following maps from $M_{bvs}(\mathcal{X}^2)$ into $\mathbb{C}\{t_1, t_2\}$

$$\mathcal{R}(\varphi) = \sum_{\substack{n=0 \\ 0 \le p \le n}}^{\infty} t_1^p t_2^{n-p} \varphi(R^{p,n-p}),$$

and

$$\mathcal{H}(\varphi) = \sum_{\substack{n=1 \\ 0 \le p \le n}}^{\infty} t_1^p t_2^{n-p} \varphi(H^{p,n-p}).$$

Note

$$\mathcal{R}((x,y)\bullet(z,t)) = \mathcal{R}(x,y)\mathcal{R}(z,t),$$

and

$$\mathcal{H}((x,y)\bullet(z,t)) = \mathcal{H}(x,y) + \mathcal{H}(z,t),$$

where $(x,y),(z,t) \in \mathcal{X}^2$. We will prove these equalities in Theorem 4 for more general situation.

Following [3] we define the symmetric convolution.

Definition 1. *For any $f \in H_{bvs}(\mathcal{X}^2)$ and $\theta \in H_{bvs}(\mathcal{X}^2)'$, symmetric convolution $\theta \star f$ is defined by*

$$(\theta \star f)(x,y) = \theta[\mathcal{T}_{(x,y)}(f)].$$

Definition 2. *For any $\varphi, \theta \in H_{bvs}(\mathcal{X}^2)'$, symmetric convolution $\varphi \star \theta$ is defined by*

$$(\varphi \star \theta)(f) = \varphi(\theta \star f) = \varphi((z,t) \mapsto \theta(\mathcal{T}_{(z,t)}f)).$$

Theorem 3. *For any $\varphi, \theta \in M_{bvs}(\mathcal{X}^2)$ the symmetric convolution is commutative, associative and*

$$(\varphi \star \theta)(H^{p,n-p}) = \varphi(H^{p,n-p}) + \theta(H^{p,n-p}), \tag{4}$$

where $0 \le p \le n$.

Proof. First we will prove the equality (4). Indead, for polynomials $H^{p,n-p}$ we have

$$\begin{aligned}(\theta \star H^{p,n-p})(x,y) &= \theta(\mathcal{T}_{(x,y)}(H^{p,n-p}))\\ &= \theta(H^{p,n-p}(x,y) + H^{p,n-p}) = H^{p,n-p}(x,y) + \theta(H^{p,n-p}).\end{aligned}$$

Therefore,

$$\begin{aligned}(\varphi \star \theta)(H^{p,n-p}) &= \varphi(H^{p,n-p}(x,y) + \theta(H^{p,n-p}))\\ &= \varphi(H^{p,n-p}) + \theta(H^{p,n-p}).\end{aligned}$$

From this equality it follows the associativity and commutativity of $\varphi \star \theta \in M_{bvs}(\mathcal{X}^2)$. □

Similarly to Lemma 3.1 and Proposition 8.2 in [4] (see also [12]) it is possible to show that

$$||R^{p,n-p}|| \le \frac{2}{p!(n-p)!}$$

and $\mathcal{R}(\varphi)(t)$ is a function of exponential type for every fixed $\varphi \in M_{bvs}(\mathcal{X}^2)$.

Theorem 4. *The following identities hold*

1. $\mathcal{H}(\varphi \star \theta) = \mathcal{H}(\varphi) + \mathcal{H}(\theta)$,

2. $\mathcal{R}(\varphi \star \theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$.

Proof. The first statement it follows from Theorem 3. To prove the second statement we observe that

$$R^{p,n-p}((x,y)\bullet(z,t)) = \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)R^{p-r,n-p-(i-r)}(z,t).$$

Thus

$$\begin{aligned}(\theta\star R^{p,n-p})(x,y) &= \theta(\mathcal{T}_{(x^1,x^2)}(R^{p,n-p}))\\ &= \theta\Big(\sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)R^{p-r,n-p-(i-r)}\Big)\\ &= \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)\theta\Big(R^{p-r,n-p-(i-r)}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}(\varphi\star\theta)\Big(R^{p,n-p}\Big) &= \varphi\Big(\sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x^1,x^2)\theta\Big(R^{p-r,n-p-(i-r)}\Big)\Big)\\ &= \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} \varphi\Big(R^{r,i-r}\Big)\theta\Big(R^{p-r,n-p-(i-r)}\Big).\end{aligned}$$

On the other hand

$$\begin{aligned}\mathcal{R}(\varphi)\mathcal{R}(\theta) &= \sum_{\substack{i=0\\0\le k\le i}}^{\infty} t_1^k t_2^{i-k}\varphi(R^{k,i-k}) \sum_{\substack{m=0\\0\le r\le m}}^{\infty} t_1^r t_2^{m-r}\theta(R^{r,m-r})\\ &= \sum_{\substack{n=0\\0\le p\le n}}^{\infty} \sum_{\substack{k+r=p\\i+m=n}} t_1^p t_2^{n-p}\varphi(R^{k,i-k})\theta(R^{r,m-r})\\ &= \sum_{\substack{n=0\\0\le p\le n}}^{\infty} t_1^p t_2^{n-p} \sum_{\substack{k+r=p\\i+m=n}} \varphi(R^{k,i-k})\theta(R^{r,m-r}) = \sum_{\substack{n=0\\0\le p\le n}}^{\infty} t_1^p t_2^{n-p}(\varphi\star\theta)\Big(R^{p,n-p}\Big)\\ &= \mathcal{R}(\varphi\star\theta).\end{aligned}$$

□

Lemma 1. *If $\varphi = \delta_{(x,y)}$, then for every $(x,y)\in\mathcal{X}^2$:*

$$\mathcal{R}(\delta_{(x,y)})(t_1,t_2) = \prod_{i=1}^{\infty}(1+x_it_1+y_it_2) = \sum_{n=0}^{\infty} G_n(xt_1+yt_2),$$

where $(x_i,y_i)\in\mathbb{C}^2$, $i\ge 1$ and $G_n(xt_1+yt_2) = \sum\limits_{k_1<k_2<\ldots<k_n}^{\infty}(x_{k_1}t_1+y_{k_1}t_2)\ldots(x_{k_n}t_1+y_{k_n}t_2)$ and $G_0=1$.

Proof. For every $(x,y) \in \mathcal{X}^2$, the product

$$\prod_{i=1}^{\infty}(1+x_it_1+y_it_2)$$

is absolutely convergent if the series $\sum\limits_{i=1}^{\infty}(x_it_1+y_it_2)$ is absolutely convergent. Since

$$\begin{aligned}\sum_{i=1}^{\infty}|x_it_1+y_it_2| &\le \sum_{i=1}^{\infty}(|x_i||t_1|+|y_i||t_2|) = |t_1|\sum_{i=1}^{\infty}|x_i|+|t_2|\sum_{i=1}^{\infty}|y_i| \\ &\le \max\{|t_1|,|t_2|\}\Big(\sum_{i=1}^{\infty}|x_i|+\sum_{i=1}^{\infty}|y_i|\Big) \\ &\le \max\{|t_1|,|t_2|\}\sqrt{2}\Big(\sum_{i=1}^{\infty}\big(|x_i|^2+|y_i|^2\big)^{1/2}\Big) < \infty,\end{aligned}$$

we obtain that $\prod\limits_{i=1}^{\infty}(1+x_it_1+y_it_2)$ is absolutely convergent, and so the product is convergent as well. Since for every $1 \le m < \infty$ will be performed the equality

$$\sum_{\substack{n=0 \\ 0\le p\le n}}^{m} t_1^p t_2^{n-p}\delta_{(x,y)}(R^{p,n-p}) = \prod_{i=1}^{m}(1+x_it_1+y_it_2)$$

and series and product are convergent, we obtain that

$$\mathcal{R}(\delta_{(x,y)})(t_1,t_2) = \prod_{i=1}^{\infty}(1+x_it_1+y_it_2).$$

It is known from Combinatorics [8] that $\sum\limits_{n=0}^{\infty} t^nG_n(x) = \prod\limits_{i=1}^{\infty}(1+x_it_1)$ for every $x \in c_{00}$, where $G_n(x) = \sum\limits_{k_1<\ldots<k_n}^{\infty} x_{k_1}\ldots x_{k_n}$ is the basis of elementary symmetric polynomials of algebra $\mathcal{H}_{bs}(\ell_1)$. Since it is true for every $x \in \ell_1$,

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(xt_1+yt_2) &= \sum_{n=0}^{\infty}(t_1t_2)^nG_n\Big(\frac{x}{t_2}+\frac{y}{t_1}\Big) = \prod_{i=1}^{\infty}\Big(1+\Big(\frac{x_i}{t_2}+\frac{y_i}{t_1}\Big)t_1t_2\Big) \\ &= \prod_{i=1}^{\infty}(1+x_it_1+y_it_2).\end{aligned}$$

□

Now we show that the spectrum of the algebra of block-symmetric analytic functions of bounded type on $\mathcal{X}^2$ does not coincide of point evaluation functionals.

Example 1. *Let k, l are same fixed nonzero complex numbers. Now we consider the sequence of elements*

$$\begin{aligned} e_1(k,l) &= \left(\begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right), \\ e_2(k,l) &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ l \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right), \\ &\ldots\ldots\ldots\ldots \\ e_n(k,l) &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} k \\ l \end{pmatrix}, \ldots \right), \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

in $\mathcal{X}^2$ and for every n put

$$v_n(k,l) = \frac{1}{n}(e_1(k,l) + e_2(k,l) + \ldots + e_n(k,l)) \in \mathcal{X}^2.$$

Then $\delta_{v_n(k,l)}(H^{0,1}) \to l$, $\delta_{v_n(k,l)}(H^{1,0}) \to k$, $\delta_{v_n(k,l)}(H^{p,i-p}) \to 0$ as $n \to \infty$ for every $1 \le k \le i$, where $1 \le p \le i$. By the reletive compactness of bounded subset of $M_{bvs}(\mathcal{X}^2)$ there is an accumulation point $\varphi_{(k,l)}$ of the sequence $\delta_{v_n(k,l)}$, such that $\varphi_{(k,l)}(H^{0,1}) = l$, $\varphi_{(k,l)}(H^{1,0}) = k$, $\varphi_{(k,l)}(H^{p,i-p}) = 0$ for all $1 \le i \le m$, where $1 \le p \le i$. From Theorem 2 it follows that there is no poit $(x,y) \in \mathcal{X}^2$, such that $\delta_{(x,y)} = \varphi_{(k,l)}$. Indeed, if such a point exists, then $(x,y) \sim (0,0)$. Therefore $\delta_{v_n(k,l)}(H^{0,1}) = \delta_{v_n(k,l)}(H^{1,0}) = 0$, but we have that $\delta_{v_n(k,l)}(H^{0,1}) = l$, $\delta_{v_n(k,l)}(H^{1,0}) = k$.

Example 2. *Let $\varphi_{(k,l)}$ be as in Example 1. We know that $\mathcal{H}(\varphi_{(k,l)}) = k + l$. To find $\mathcal{R}(\varphi_{(k,l)})$ note that*

$$R^{p,s-p}(v_n(k,l)) = \frac{k^p l^{s-p}}{n^p n^{s-p}} \binom{n}{s} \binom{s}{p},$$

hence

$$\varphi(R^{p,s-p}) = \lim_{n\to\infty} R^{p,s-p}(v_n(k,l)) = \frac{k^p l^{s-p}}{p!(s-p)!}$$

and so

$$\begin{aligned} \mathcal{R}(\varphi_{(k,l)})(t_1,t_2) &= \lim_{n\to\infty} \sum_{\substack{s=0\\0\le p\le s}}^{n} t_1^p t_2^{s-p} \varphi(R^{p,s-p}) \\ &= \lim_{n\to\infty} \sum_{\substack{s=0\\0\le p\le s}}^{n} \frac{(kt_1)^p (lt_2)^{s-p}}{p!(s-p)!} = e^{kt_1+lt_2}. \end{aligned}$$

Theorem 5. *The invertible elements of semigroup $\left(M_{bvs}(\mathcal{X}^2), \star\right)$ are functionals only of the form $\varphi_{(k,l)} = \mathcal{R}(\varphi_{(k,l)})(t_1,t_2) = e^{kt_1+lt_2}$.*

Proof. Since by Theorem 4 $\mathcal{R}(\varphi \star \theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$, $\varphi_{(-k,-l)}$ is inverse to $\varphi_{(k,l)}$. In the other hand, if φ is invertible and $\psi = \varphi^{-1}$, then $\mathcal{R}(\psi) = \frac{1}{\mathcal{R}(\varphi)(t_1,t_2)}$ is an entire function of exponential type and so has no zeros. So we have that $\mathcal{R}(\varphi)(t_1,t_2) = e^{kt_1+lt_2}$ for some complex numbers k,l. □

Corollary 1. *Let Φ be a homomorphism on the subspace of block-symmetric polynomials in $H_{bvs}(\mathcal{X}^2)$ to itself such that $\Phi(H^{p,k-p}) = -H^{p,k-p}$ for every p,k. Then Φ is discontinuous.*

Proof. If Φ is continuous it may be extended to continuous homomorphism $\tilde{\Phi}$ of $H_{bvs}(\mathcal{X}^2)$. Then for $(x,y) \in \mathcal{X}^2$

$$H^{p,k-p}(x,y) + \Phi(H^{p,k-p})(x,y) = 0 \tag{5}$$

for all p,k. Note that this equality is true for

$$(x_0,y_0) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right).$$

Let us denote $\psi = \delta_{(x_0,y_0)} \circ \tilde{\Phi}$. From the continuity of homomorphism $\tilde{\Phi}$ we have, that $\psi \in M_{bvs}(\mathcal{X}^2)$. From equality (5) we have, that $\delta_{(x,y)} \star \psi = \delta_{(0,0)}$, $\psi = \delta^{-1}_{(x_0,y_0)}$. According to the Theorem 5 $\delta_{(x_0,y_0)}$ is not invertible. □

Let $f(z)$ be an entire function of many variable. We will say that $f(z)$, where $z \in \mathbb{C}^n$, has "plane" zeros if the set of zeros is

$$Z_f = \Big\{ z \in \mathbb{C}^n : f(z) = 0 \Big\} = \bigcup_{k=1}^{\infty} H_k,$$

where $H_k = \{z : \langle z, a^k |a^k|^{-2} \rangle = 1\}$ is hyperplane in $\mathbb{C}^n$. Here $a^k \in \mathbb{C}^n$ are feets of perpendiculars dropped from the origin onto zeros hyperplanes H_k of the function $f(z)$ (see [10]).

Theorem 6. *Let φ be a character such that $\mathcal{R}(\varphi)$ is a polynomial. Then $R(\varphi)$ have a plane zeros, that is $KerR(\varphi)$ consists of one-codimensional linear subspaces.*

Proof. Let us denote $\Lambda_{t_1 t_2}(G_n) = G_n(xt_1 + yt_2)$. Now we consider the equation $\sum\limits_{n=0}^{m} \lambda^n \varphi(\Lambda_{t_1 t_2}(G_n)) = 0$ with m solutions z_k, $1 \le k \le m$. Hence $\prod\limits_{i=1}^{m}(1 + z_k\lambda) = 0$. Obviously, every solution z_k can be represented as $z_k = x_k t_1 + y_k t_1$, where x_k, y_k are indeterminants and t_1, t_2 are some complex numbers. If we take $t_1 = 1, t_2 = 0$ and $t_1 = 2, t_2 = 1$, then can fined x_k, y_k. So we have the system of $2m$ equation and $2m$ indeterminants x_k, y_k, $1 \le k \le m$. The solutions of that system are $x_k = z_k$, $y_k = -z_k$, $1 \le k \le m$. Hence x_k, y_k can be clearly define. If $\lambda = 1$, then we obtain the equality

$$\mathcal{R}(\varphi)(t_1, t_2) = \sum_{n=0}^{m} \varphi(\Lambda_{t_1 t_2}(G_n)) = \prod_{i=1}^{m}(1 + x_i t_1 + y_i t_2) = 0.$$

Hence φ has plane zeros. ☐

According to the analog of Hadamard's Theorem [10] the function $\mathcal{R}(\varphi)(t_1, t_2)$ with plane zeros is of the form

$$\mathcal{R}(\varphi)(t_1, t_2) = \exp(P(t_1, t_2)) \prod_{i=1}^{n} \Big(1 - \big(t_1 \tfrac{\overline{a}_1^k}{|a^k|^2} + t_2 \tfrac{\overline{a}_2^k}{|a^k|^2}\big)\Big),$$

where $\{(a_1^k, a_2^k)\}$ are the zeros of $\mathcal{R}(\varphi)(t_1, t_2)$, $P(t_1, t_2)$ is analytic polynomial and we have

$$\sum_{k=1}^{n} \tfrac{1}{|a_k|} < \infty.$$

According to the Lemma 1

$$\mathcal{R}(\delta_{(x,y)})(t_1, t_2) = \prod_{i=1}^{m}(1 + x_i t_1 + y_i t_2),$$

and so the zeros of $\mathcal{R}(\delta_{(x,y)})(t_1, t_2)$ are

$$a_1^k = -\frac{\overline{x}_k}{|x_k|^2 + |y_k|^2}, \qquad a_2^k = -\frac{\overline{y}_k}{|x_k|^2 + |y_k|^2}.$$

On the other hand, if $f(t_1, t_2)$ is the function of the exponential type with plane zeros, then it can be represented as

$$\mathcal{R}(\varphi)(t_1, t_2) = \exp(P(t_1, t_2)) \prod_{i=1}^{\infty} \Big(1 - \big(t_1 \tfrac{\overline{a}_1^k}{|a^k|^2} + t_2 \tfrac{\overline{a}_2^k}{|a^k|^2}\big)\Big),$$

if

$$\sum_{k=1}^{\infty} \tfrac{1}{|a_k|} < \infty.$$

So for $\varphi \in M_{bvs}(\mathcal{X}^2)$, which we can represanted as $\varphi = \varphi_{(k,l)} \star \delta_{(x,y)}$, where $(x, y) \in \mathcal{X}^2$, $(x_k, y_k) = -\big(\frac{\overline{a}_1^k}{|a_k|^2}, \frac{\overline{a}_2^k}{|a_k|^2}\big)$ and $\varphi_{(k,l)}$ was defined in Example 1, we have that

$$\mathcal{R}(\varphi)(t_1, t_2) = f(t_1, t_2).$$

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У статті описано симетричну згортку характерів алгебри блочно-симетричних аналітичних фнкцій обмеженого типу на ℓ_1-сумі простору $\mathbb{C}^2$. Авторами показано, що спектр такої алгебри не не збігається з множиною класів еквівалентності функціоналів значенні в точках, описано характери такої алгебри, як функції експоненціального типу з "плоскими" нулями.

Ключові слова і фрази: алгебраїчний базис, блочно-симетичні поліноми, блочно-симетричні аналітичні фукції, спектр, симетричний зсув, симетрична згортка.