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A WORPITZKY BOUNDARY THEOREM FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

For a branched continued fraction of a special form we propose the limit value set for the Worpitzky-like theorem when the element set of the branched continued fraction is replaced by its boundary.

Key words and phrases: element set, value set, branched continued fraction of special form.

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INTRODUCTION

A lot of convergence criteria for continued fractions are characterized by convergence domains. Such domains are indicated in the complex plane, that if elements a_k, b_k of a continued fraction belong to these domains then the continued fraction

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ddots}}} = \prod_{k=1}^{\infty} \frac{a_k}{b_k}$$

converges. At first convergence domains for continued fractions we can find in papers of Worpitzky (1865), Pringsheim (1899) and Van Vleck (1901) [8].

Despite of the fact that a well known convergence theorem for continued fractions was proposed by J. Worpitzky in 1865, its new proofs, generalizations and applications are actual even at present [3, 6, 8].

H. Waadeland [10] formulated the Worpitzky theorem in a slightly more general form than classical one [8], using conditions on the coefficients of the continued fraction proposed by F. Paydon and H. Wall [9].

Theorem 1. Let $\rho \in (0, 1/2]$ be any positive number, and let all elements of a continued fraction

$$\frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \ddots}}} = \prod_{i=1}^{\infty} \frac{a_i}{1}, \quad (1)$$

$a_i, i = 1, 2, \dots$, be complex numbers, bounded by

$$|a_i| \leq \rho(1 - \rho), \quad i = 1, 2, \dots \quad (2)$$

Then the continued fraction (1) converges and its values are contained in the disk $|w| \leq \rho$.

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For the continued fraction (1) Haakon Waadeland raised the question: What happens to the set of values of the continued fraction (1) when the condition (2) in the Worpitzky theorem would be replaced by $|a_i| = \rho(1 - \rho)$, $i = 1, 2, \dots$? Answering on his question H.Waadeland proved [10], that the set of all possible values of the continued fraction (1) is the annulus

$$\rho \cdot \frac{1 - \rho}{1 + \rho} \leq |w| \leq \rho.$$

In the classical case of the theorem ($\rho = 1/2$), i.e. $|a_i| = 1/4$, $i = 1, 2, \dots$, the annulus is $1/6 \leq |w| \leq 1/2$.

The same question one can put for multidimensional generalizations of the continued fraction, such as for example,

a branched continued fraction (BCF) [3]

$$1 + \sum_{i_1=1}^N \frac{a_{i_1} z_{i_1}}{1 + \sum_{i_2=1}^N \frac{a_{i_1 i_2} z_{i_2}}{1 + \sum_{i_3=1}^N \frac{a_{i_1 i_2 i_3} z_{i_3}}{1 + \dots}}} = 1 + \prod_{k=1}^{\infty} \sum_{i_k=1}^N \frac{a_{i(k)} z_{i_k}}{1}, \tag{3}$$

where $a_{i_1 i_2 \dots i_k}$ be complex numbers, z_{i_k} be complex variables, $i(k) = i_1 i_2 \dots i_k$ be multiindex; a branched continued fraction with independent variables [1]

$$\frac{a_{00}}{1 + \sum_{i_1=1}^N \frac{a_{i_1} z_{i_1}}{1 + \sum_{i_2=1}^{i_1} \frac{a_{i_1 i_2} z_{i_2}}{1 + \sum_{i_3=1}^{i_2} \frac{a_{i_1 i_2 i_3} z_{i_3}}{1 + \dots}}}} = \frac{a_{00}}{1 + \prod_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)} z_{i_k}}{1}}, \tag{4}$$

where $a_{i_1 i_2 \dots i_k}$ be complex numbers, z_{i_k} be complex variables, $i(k) = i_1 i_2 \dots i_k$ be multiindex $1 \leq i_k \leq i_{k-1}$, $k = 1, 2, \dots$, $i_0 = N$;

or a two-dimensional continued fraction (TDCF) [6]

$$\prod_{i=0}^{\infty} \frac{a_{i,i} z_1 z_2}{\Phi_i}, \quad \Phi_i = 1 + \prod_{j=1}^{\infty} \frac{a_{i+j,i} z_1}{1} + \prod_{j=1}^{\infty} \frac{a_{i,i+j} z_2}{1}, \tag{5}$$

where $a_{i,j}$, $i = 0, 1, \dots$, $j = 1, 2, \dots$, be complex numbers, z_1, z_2 be complex variables.

It was found this question for the branched continued fraction (3) with $z_1 = z_2 = \dots = z_N = 1$ is answered by the following theorem [11].

Theorem 2. *Let $\rho \in (0, 1/2]$ and $N \geq 2$ be an integer. In the family of branched continued fractions*

$$1 + \sum_{i_1=1}^N \frac{a_{i_1}}{1 + \sum_{i_2=1}^N \frac{a_{i_1 i_2}}{1 + \sum_{i_3=1}^N \frac{a_{i_1 i_2 i_3}}{1 + \dots}}} = 1 + \prod_{k=1}^{\infty} \sum_{i_k=1}^N \frac{a_{i(k)}}{1}, \tag{6}$$

where $a_{i_1 i_2 \dots i_k}$ be complex numbers, $i(k) = i_1 i_2 \dots i_k$ be multiindex, $a_{i(k)}$ satisfy the conditions $|a_{i(k)}| = \frac{\rho(1 - \rho)}{N}$, then the set of possible branched continued fraction values is the closed disk $|w| \leq \rho$.

Thus, in this case the set of possible BCF values is unchanged when all elements of (6) are restricted to the boundary of the disk.

For TDCF (5) with $z_1 = z_2 = 1$ the answer is proposed by the following theorem [7].

Theorem 3. *Let ρ be a real number in $(0, 1/2]$, and let F_ρ be the family of two-dimensional continued fractions*

$$\underset{i=0}{\overset{\infty}{D}} \frac{a_{i,i}}{\Phi_i}, \quad \Phi_i = 1 + \underset{j=1}{\overset{\infty}{D}} \frac{a_{i+j,i}}{1} + \underset{j=1}{\overset{\infty}{D}} \frac{a_{i,i+j}}{1}, \tag{7}$$

where $a_{i,j}$, $i = 0, 1, \dots, j = 1, 2, \dots$, be complex numbers that satisfy conditions $|a_{i,j}| = \frac{1}{2}\rho(1 - \rho)$, $i, j \geq 1$.

Then the set of all possible values f of the TDCF (7) in F_ρ is the annulus A_ρ , given by

$$R \cdot \frac{\rho(1 - \rho)}{4R - \rho(1 - \rho)} \leq |f| \leq R, \quad R = \frac{1}{2}(\sqrt{1 - 2\rho(1 - \rho)} + \sqrt{1 - 4\rho(1 - \rho)}).$$

In the case $\rho = 1/2$ the annulus is $(8 + \sqrt{2}) / 124 \leq |f| \leq 1/2\sqrt{2}$.

In the present paper the answer will be done for the branched continued fraction with independent variables (4) with $z_1 = z_2 = \dots = z_N = 1$ (named the branched continued fraction of the special form [2, 5, 4]).

1 THE WORPITZKY-LIKE THEOREMS FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

Since the beginning we prove the Worpitsky-like theorem in a slightly more general form than it was done in [1].

Theorem 4. *Let $\rho \in (0, 1/2]$ and $N \geq 2$ be an integer. In the BCF of the special form*

$$\frac{a_{00}}{1 + \underset{k=1}{\overset{\infty}{D}} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1}}, \tag{8}$$

where $a_{i_1 i_2 \dots i_k}$ be complex numbers, $i(k) = i_1 i_2 \dots i_k$ be multiindex $1 \leq i_k \leq i_{k-1}$, $k = 1, 2, \dots$, $i_0 = N$, $a_{i(k)}$ satisfy the conditions $|a_{i(k)}| \leq \alpha_{i_{k-1}} = \frac{\rho(1 - \rho)}{i_{k-1}}$, $|a_{00}| \leq \rho(1 - \rho)$.

Then the BCF of the special form (8) converges, and its values are contained in the disk $|w| \leq \rho$.

Proof. It is not difficult to show that a periodic continued fraction

$$1 - \frac{\rho(1 - \rho)}{1 - \frac{\rho(1 - \rho)}{1 - \frac{\rho(1 - \rho)}{\vdots}}}} \tag{9}$$

is the majorant fraction for the BCF of special form (8).

It means that approximants of these fractions satisfy the relation:

$$|f_n - f_m| \leq M \cdot |g_n - g_m|,$$

where f_n, g_n are the n th approximants of the BCF of the special form (8) and continued fraction (9) respectively, M is a certain constant, m, n are natural numbers.

For the difference between the n th and m th approximants of the BCF of the special form (8) the following relation is true [1]:

$$f_n - f_m = (-1)^m \sum_{i_1=1}^N \sum_{i_2=1}^{i_1} \dots \sum_{i_m=1}^{i_{m-1}} \frac{a_{00} \cdot \prod_{k=1}^m a_{i(k)}}{\prod_{k=0}^m Q_{i(k)}^{(n-1)} \prod_{k=0}^{m-1} Q_{i(k)}^{(m-1)}}, \quad n > m \geq 1, \tag{10}$$

where

$$Q_{i(s)}^{(s)} = 1, \quad Q_{i(k)}^{(s)} = 1 + \sum_{i_{k+1}=1}^{i(k)} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(s)}}, \quad k = \overline{1, s-1}, \quad s \geq 2,$$

$$Q^{(s)} = Q_{i(0)}^{(s)} = 1 + \sum_{i_1=1}^N \frac{a_{i(1)}}{Q_{i(1)}^{(s)}}, \quad s \geq 1, \quad f_n = \frac{a_{00}}{Q_{i(0)}^{(n-1)}}.$$

Using the method of complete mathematical induction it is easy to prove that

$$\left| Q_{i(k)}^{(s)} \right| \geq h_{s-k}, \tag{11}$$

where h_m is the m th approximant of the continued fraction

$$1 - \frac{\rho(1-\rho)}{1 - \frac{\rho(1-\rho)}{1 - \dots}}$$

for all possible index sets.

Let us write the difference formula for approximants of the continued fraction (9)

$$g_n - g_m = \frac{\rho^{m+1}(1-\rho)^{m+1}}{\prod_{i=0}^m h_{n-i-1} \prod_{i=0}^{m-1} h_{m-i-1}}. \tag{12}$$

From (11) follows that all $Q_{i(k)}^{(s)} \neq 0$. Hence, taking into account (10) and (12) we have

$$\begin{aligned} |f_n - f_m| &\leq \sum_{i_1=1}^N \sum_{i_2=1}^{i_1} \dots \sum_{i_m=1}^{i_{m-1}} \frac{|a_{00}| \cdot \prod_{k=1}^m |a_{i(k)}|}{\prod_{k=0}^m |Q_{i(k)}^{(n-1)}| \prod_{k=0}^{m-1} |Q_{i(k)}^{(m-1)}|} \\ &\leq \frac{\rho^{m+1}(1-\rho)^{m+1}}{\prod_{k=0}^m h_{n-k-1} \prod_{k=0}^{m-1} h_{m-k-1}} = g_n - g_m. \end{aligned}$$

The continued fraction (9) converges, and therefore the BCF of the special form (8) is also convergent.

Let us write the m th approximant of (8) in the form

$$z = \frac{a_{00}}{1 + \sum_{i_1=1}^N \frac{a_{i(1)}}{Q_{i(1)}^{(m-1)}}} = \frac{a_{00}}{(1+w)}.$$

From the conditions of the theorem on the fraction coefficients and inequalities (11) one can write

$$|w| = \left| \sum_{i_1=1}^N \frac{a_{i(1)}}{Q_{i(1)}^{(m-1)}} \right| \leq \frac{\rho(1-\rho)}{h_{m-2}} = g_{m-1}.$$

Putting $g_n = P_n/Q_n$, where P_n is the n th numerator and Q_n is the n th denominator of the approximant g_n it is easy to find by induction that

$$Q_n = \sum_{i=0}^n \rho^i (1-\rho)^{n-i}.$$

If Q is the value of the infinite fraction (9), and $Q_n > 0, n = 1, 2, \dots$, then we get

$$g_n - g_{n-1} = \frac{(\rho(1-\rho))^n}{Q_n Q_{n-1}} \geq 0,$$

i.e., the sequence $\{g_n\}$ grows monotonically. Hence, $|w| \leq Q$. Since $Q = \rho(1-\rho) \cdot (1-Q)^{-1}$, and taking into account that $Q = 0$, if $\rho = 0$, the solution of this quadratic equation with respect to Q gives $Q = \rho$.

Therefore, $|w| \leq \rho$, and $|z| \leq \rho$. □

Now we obtain the boundary version of this theorem.

Theorem 5. Let $\rho \in (0, 1/2]$ and $N \geq 2$ be an integer. In the family of branched continued fractions of the special form F_ρ

$$\frac{a_{00}}{1 + \underset{D}{\overset{\infty}{\sum}}_{k=1} \frac{i_{k-1}}{i_k} \frac{a_{i(k)}}{1}}, \tag{13}$$

where $a_{i_1 i_2 \dots i_k}$ be complex numbers, $i(k) = i_1 i_2 \dots i_k$ be multiindex $1 \leq i_k \leq i_{k-1}, k = 1, 2, \dots, i_0 = N, a_{i(k)}$ satisfy the conditions $|a_{i(k)}| = \frac{\rho(1-\rho)}{i_{k-1}}, |a_{00}| = \rho(1-\rho)$, the set of all possible branched continued fractions of the special form values is the annulus A_ρ , given by

$$\rho \cdot \frac{1-\rho}{1+\rho} \leq |w| \leq \rho.$$

Proof. Let f_0 be a possible value of the BCF of the special form (13). Then all values f with $|f| = |f_0|$ are possible BCF of the special form values in F_ρ . Hence the set of values of such fraction must be a disk or an annulus, in both cases centered at the origin. From the Worpitzky-like theorem (Theorem 4) follows that this disk or annulus must be contained in the disk $|f| \leq \rho$.

We shall first prove that the set of all values must be contained in A_ρ . Any BCF of the special form in F_ρ can be written in the form

$$f = \frac{\rho(1-\rho)e^{i\theta}}{1+\omega}, \theta \in [0, 2\pi), \omega = \sum_{i_1=1}^N \frac{a_{i(1)}}{1 + \underset{D}{\overset{\infty}{\sum}}_{k=1} \frac{i_k}{i_{k+1}} \frac{a_{i(k+1)}}{1}}.$$

Since $\frac{a_{i(1)} \cdot N}{1 + \overset{\infty}{D} \sum_{k=1}^{i_k} \frac{a_{i(k+1)}}{1}} \in F_\rho$ we have, using the previous Theorem 4

$$\left| a_{i(1)} \cdot N \cdot \left(1 + \overset{\infty}{D} \sum_{k=1}^{i_k} \frac{a_{i(k+1)}}{1} \right)^{-1} \right| \leq \rho.$$

It means that

$$\left| a_{i(1)} \cdot \left(1 + \overset{\infty}{D} \sum_{k=1}^{i_k} \frac{a_{i(k+1)}}{1} \right)^{-1} \right| \leq \frac{\rho}{N},$$

and $|\omega| \leq \rho$. Since $|\omega| \leq \rho$ it follows that for any value f of a BCF of the special form in F_ρ we have $|f| \geq \rho \cdot \frac{1-\rho}{1+\rho}$.

That is sharp, follows from the fact that

$$\rho = \frac{\rho(1-\rho)}{1 - \frac{\rho(1-\rho)}{1 - \dots}}$$

and that the right-hand side is in F_ρ .

We next prove that A_ρ is contained in the set of values of BCFs of the special form in F_ρ with independent variables $|\omega| \leq \rho$.

By the mapping $\xi = 1/1 + \omega$ the circle $\omega = \rho$ is mapped onto the circle

$$\left| \xi - \frac{1}{1-\rho^2} \right| = \frac{\rho}{1-\rho^2}.$$

Then, by $\xi \rightarrow \rho(1-\rho)e^{i\theta}\xi$, for all $\theta \in [0, 2\pi)$ we get all points in the annulus A_ρ .

Hence, A_ρ is contained in the set of BCF with independent variables values for F_ρ . \square

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Для гіллястого ланцюгового дроби спеціального вигляду запропоновано межову множину значень у теоремі типу Ворпіцького, коли множина елементів гіллястого ланцюгового дроби замінена її межею.

Ключові слова і фрази: множина елементів, множина значень, гіллястий ланцюговий дріб спеціального вигляду.