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POINTWISE STABILIZATION OF THE POISSON INTEGRAL FOR THE DIFFUSION TYPE EQUATIONS WITH INERTIA

In this paper we consider the pointwise stabilization of the Poisson integral for the diffusion type equations with inertia in the case of finite number of parabolic degeneracy groups. We establish necessary and sufficient conditions of this stabilization for a class of bounded measurable initial functions.

Key words and phrases: Poisson integral, Kolmogorov equation, diffusion type equation with inertia, stabilization, degenerate parabolic equation, surface level, average on border.

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INTRODUCTION

In this paper we consider pointwise stabilization of the Poisson integral for diffusion type equations with inertia which have finite number groups of variables with diffusion degeneration.

Stabilization problems for solutions of the Cauchy problem for parabolic equations were studied by S.D. Eidelman and V.P. Repnikov [1, 2]. Necessary and sufficient conditions of pointwise stabilization of the Poisson integral for the Kolmogorov equation were obtained by S.D. Eidelman, V.P. Repnikov and G.P. Malytska [3, 4]. Generalization of these results in the case of three degeneration groups can be found in the work [5].

1 NOTATIONS AND PROBLEM STATEMENT

Let $x := (x_{11}, x_{12}, \dots, x_{1n_1}; \dots; x_{k1}, x_{k2}, \dots, x_{kn_k}; \dots; x_{p1}, x_{p2}, \dots, x_{pn_p}; x_{p+1,1}, \dots, x_{m1})$, $n_1 \geq n_2 \geq \dots \geq n_p > 1$, $n_k \in \mathbb{N}$, $k = \overline{1,p}$, $p \in \mathbb{N}$, $m \geq p$, $\sum\limits_{k=1}^{p} n_k + m - p = n$, $x \in \mathbb{R}^n$. Consider the Cauchy problem

$$\partial_t u\,(t,x) - \sum_{k=1}^{p} \sum_{j=1}^{n_k} x_{kj} \partial_{x_{k\,j+1}} u\,(t,x) = \sum_{v=1}^{m} \partial^2_{x_v^2} u\,(t,x)\,, \tag{1}$$

$$u\,(t,x)\,|_{t=\tau} = u_0\,(x)\,, \;\; 0 \leq \tau < t \leq T < +\infty, \;\; x \in \mathbb{R}^n, \tag{2}$$

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where $u_0(x)$ is a Lebesgue measurable and bounded function in $\mathbb{R}^n$. The fundamental matrix of solutions $G(t-\tau,\, x,\, \xi)$ with $t>\tau, x\in\mathbb{R}^n, \xi\in\mathbb{R}^n$ of the Cauchy problem (1), (2) was found in [6]. Hence,

$$G(t-\tau,\, x,\, \xi) = (2\sqrt{\pi})^{-n}(t-\tau)^{-\mu}\prod_{\nu=1}^{p}\prod_{k=1}^{n_\nu} k(k+1)\dots(2k-2)(2k-1)^{-\frac{1}{2}}\, e^{-\rho(t,x;\tau,\xi)}, \qquad (3)$$

where

$$\rho(t,x;\tau,\xi) = \sum_{\nu=1}^{m}|x_{\nu 1}-\xi_{\nu 1}|^2 4^{-1}(t-\tau)^{-1}\sum_{\nu=1}^{p}\sum_{k=2}^{n_\nu}(k-1)^2k^2\dots(2k-3)^2(2k-1)$$
$$(t-\tau)^{-(2k-1)}\Big|\sum_{j=0}^{k-1}\frac{x_{\nu k-j}(t-\tau)^j}{j!}-\xi_{\nu k}-\Big(\sum_{j=0}^{k-2}\frac{x_{\nu k-1-j}(t-\tau)^j}{j!}-\xi_{\nu k-1}\Big)(t-\tau)2^{-1}+\dots$$
$$+(-1)^{k-l}\frac{2l(2l+1)\dots(2l+(k-l)-2)(2l+2(k-l)-1)}{k\dots(2k-1)}\frac{(t-\tau)^{(k-l)}}{(k-l)!}\Big(\sum_{j=0}^{l-1}\frac{x_{\nu l-j}(t-\tau)^j}{j!}-\xi_{\nu l}\Big)+\dots$$
$$+\frac{(-1)^{k-1}(t-\tau)^{(k-1)}}{k\dots(2k-2)}(x_{\nu 1}-\xi_{\nu 1})\Big|^2,\ \mu = \frac{m}{2}+\frac{\sum_{k=1}^{p}(n_k-1)^2}{2}.$$

Here $\rho(t,x;0,\xi)=r^2$ is the family of surfaces of the fundamental solutions of the problem (1), (2). Let us denote by $F^{x,0}_{r,t}$ a figure which is bounded by the ellipsoid

$$\rho(t,x;0,\xi) = r^2, \qquad (4)$$

where ξ is a variable. Let v_n be the volume of the figure which is bounded by the surface $\rho_1(\alpha)\equiv 1$, where

$$\rho_1(\alpha) = \sum_{\nu=1}^{m}\alpha_{\nu 1}^2+\sum_{\nu=1}^{m}\sum_{k=2}^{n_\nu}(\alpha_{\nu k}-(2k-3)^{1/2}(2k-1)^{1/2}(k-1)^{-1}\alpha_{\nu k-1}).$$

Let $M^x_t(r)$ is the average of $u_0(x)$ with respect to $F^x_{r,t}$ which is bounded by surfaces (4).

Definition 1. *Function $u_0(x)$ has threshold average $M^x(r)$ on bodies $F^x_{r,t}$ if there exists the following limit $\lim\limits_{t\to\infty} M^x_t(r) = M^x(r)$.*

2 POINTWISE STABILIZATION OF THE POISSON INTEGRAL OF THE CAUCHY PROBLEM (1), (2)

Theorem 1. *If $u_0(x)$ has a threshold average on ellipsoids $F^{x,0}_{r,t}$, which almost for all r is equal to $M^x(r)$, then the Poisson integral of the equation* (1) *stabilizes (as $t\to\infty$) to the number*

$$\iota = (2\pi)^{-n/2}v_n\int_0^{+\infty} r^{n+1}e^{-r^2}M^x(r)dr.$$

Proof. Consider the Poisson integral of the equation (1)

$$u(t,x) = \int_{\mathbb{R}^n} G(t,x;0,\xi)u_0(\xi)d\xi. \qquad (5)$$

Let us make the following change of variables

$$\begin{cases} x_{\nu 1}-\xi_{\nu 1}=-2t^{1/2}\alpha_{\nu 1},\ \nu=\overline{1,m}, \\ (k-1)k\dots(2k-3)(2k-1)^{1/2}t^{-\frac{2k-1}{2}}\Big[\sum\limits_{j=0}^{k-1}\frac{x_{\nu k-j}(t-\tau)^j}{j!}-\xi_{\nu k} \\ -\Big(\sum\limits_{j=0}^{k-2}\frac{x_{\nu k-1-j}(t-\tau)^j}{j!}-\xi_{\nu k-1}\Big)\frac{t-\tau}{2}+\dots \\ +\frac{(-1)^{k-l}(t-\tau)^{(k-l)}}{(k-l)!}\frac{2l(2l+1)\dots(2l+(k-l)-2)(2l+2(k-l)-1)}{k\dots(2k-1)}\Big(\sum\limits_{j=0}^{l-1}\frac{x_{\nu l-j}(t-\tau)^j}{j!}-\xi_{\nu l}\Big) \\ +\dots+(-1)^{k-2}(t-\tau)^{(k-2)}\frac{x_{\nu 2}-\xi_{\nu 2}+(t-\tau)x_{\nu 1}}{2(k+1)\dots(2k-3)}+\frac{(-1)^{k-1}(t-\tau)^{(k-1)}}{k\dots(2k-2)}(x_{\nu 1}-\xi_{\nu 1})\Big] \\ =-(\alpha_{\nu k}-(2k-3)^{1/2}(2k-1)^{1/2}(k-1)^{-1}\alpha_{\nu k-1}+\dots+(-1)^l\alpha_{k-l}l \\ (2l+1)\dots(2l+(k-l)-2)(2k+2(k-l)-1)(2k-1)^{-1/2}((2(k-l)-1)!)^{-1} \\ (k-l)^{-1}(2(k-l)-1)^{1/2}+\dots+(-1)^{k-1}2\alpha_{\nu 1}(2k-1)^{1/2}),\quad \nu=\overline{1,p},\ k=\overline{1,n_\nu}. \end{cases} \tag{6}$$

Then equation (5) takes the form

$$u(t,x)=\pi^{-x/2}\int\limits_{R^x}exp\{-\rho_1(\alpha)\}u_0(\xi(\alpha,x,t))d\alpha, \tag{7}$$

where $u_0(\xi(\alpha,x,t))$ is the value of $u_0(\xi)$, and $\xi(\alpha,x,t)$ is determined by the system (6). Let us consider positively defined quadratic form

$$\rho_1(\alpha)=\sum_{\nu=1}^{m}\sum_{k,j=1}^{n_\nu}c_{\nu kj}\alpha_{\nu k}\alpha_{\nu j},$$

and respective family of disjoint ellipsoids

$$\sum_{\nu=1}^{m}\sum_{k,j=1}^{n_\nu}c_{\nu kj}\alpha_{\nu k}\alpha_{\nu j}=r^2.$$

In the integral (7) we consider new integration variables

$$\begin{cases} \alpha_{11}=r\Phi(\Psi)\cos\Psi_1, \\ \alpha_{21}=r\Phi(\Psi)\sin\Psi_1\cos\Psi_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{m1}=r\Phi(\Psi)\sin\Psi_1\sin\Psi_2\dots\sin\Psi_{n-1}, \end{cases} \tag{8}$$

where $0\le r<+\infty, \Psi=(\Psi_1\dots\Psi_{n-1}), 0\le\Psi_j\le\pi, j=\overline{1,n-2}, 0\le\Psi_{n-1}\le 2\pi$. The function $\Phi(\Psi)$ is defined by the equality

$$\Phi^2(\Psi)\sum_{\nu=1}^{m}\sum_{k,j=1}^{n_\nu}c_{\nu kj}\alpha'_{\nu k}\alpha'_{\nu j}=1,$$

where $\alpha_1^{'}1=\cos\Psi_1$, $\alpha_1^{'}2=\sin\Psi_1\cos\Psi_2,\dots,\alpha_{m1}^{'}=\sin\Psi_1\sin\Psi_2\dots\sin\Psi_{n-2}\cos\Psi_{n-1}$. Note that $J=r^{n-1}J_1$ is the Jacobian of the transformation (8), where

$$J_1=\Phi^n(\Psi)\sin^{n-2}\Psi_1\sin^{n-3}\Psi_2\dots\sin\Psi_{n-1}.$$

Let us denote $u_0(t, r, \Psi, x) := u_0(\xi(\alpha, x, t))$, where α is defined by (8). Then we obtain

$$u(t,x) = \pi^{-n/2}\int_0^{+\infty} r^{n-1}e^{-r^2}dr\int_{\Sigma_1} u_0(t,r,\Psi,x)Jd\Psi = \pi^{-n/2}\int_0^{+\infty} e^{-r^2}\frac{\partial}{\partial r}\int_0^r \rho^{n-1}d\rho\int_{\Sigma_1} u_0(t,r,\Psi,x)J_1 d\Psi dr$$
$$= 2\pi^{-n/2}\int_0^{+\infty} re^{-r^2}\int_0^r \rho^{n-1}d\rho\int_{\Sigma_1} u_0(t,r,\Psi,x)Jd\Psi dr,$$

where Σ_1 is the unit sphere in $\mathbb{R}^n$, J is the Jacobian of the transformation (8). Therefore for $M_t^x(r)$ we have

$$u(t,x) = 2\pi^{-n/2}v_n\int_0^{+\infty} r^{n+1}e^{-r^2}(r^n v_n)^{-1}\int_0^r \rho^{n-1}d\rho\int_{\Sigma_1} u_0(t,r,\Psi,n)Jd\Psi dr$$
$$= 2\pi^{-n/2}v_n\int_0^{+\infty} r^{n+1}e^{-r^2}M_t^x(r)dr.$$

It remains to pass to the limit in the above integral as $t \to \infty$. It can be done according to the Lebesgue theorem because there exists a threshold average. From boundedness of $u_0(x)$ immediately follows uniform boundedness of $M_t^x(r)$ by t.

Note that it is sufficient to show the existence of threshold average in some fixed point x_1 that leads to existence of threshold average in any point x and to stabilization at every compact. □

Theorem 2. *Let $u_0(x) \geq 0$. For stabilization of the Poisson integral* (5) *to zero it is necessary and sufficient that $u_0(x)$ has a threshold average $M^x(r)$, which almost everywhere is equal to zero.*

Proof. The sufficiency follows from Theorem 1. Let us show that from stabilization of the integral (5) it follows the existence of a zero threshold average on $F_{r,t}^x$:

$$M_t^x(r) = \frac{1}{mesF_{r,t}^x}\int_{F_{r,t}^x} u_0(\xi)d\xi \leq ct^{-N_1/3}\int_{\mathbb{R}^N} exp\{-\rho(t^{1/3},x,0,\xi)\}u_0(\xi)d\xi = c_1 u(t^{1/3},x), \quad (9)$$

where $N_1 = \frac{m-p}{2} + \sum\limits_{k=1}^{p} n_k^2$. In the inequality (9) $mesF_{r,t}^x$ replaced by volume of the parallelepiped

$$\begin{cases} |\xi_{\nu 1} - x_{\nu 1}| \leq t^{1/6},\ \nu = \overline{1,m}, \\ |\xi_{\nu k} - x_{\nu k}| \leq t^{\frac{2k-1}{6}}, \nu = \overline{1,p},\ k = \overline{2,n_p}. \end{cases}$$

Since $u(t,x) \to 0$ as $t \to \infty$, then from (9) it follows that $M_{t,r}^x \to 0$ as $t \to \infty$ for any r. □

3 CONCLUSION

If there exists a threshold average of a measurable bounded initial function, then theorems about pointwise stabilization of the Poisson integral for diffusion type equations with inertia also take place for systems of Kolmogorov equations with constant coefficients [7, 8]. Stabilization of the Poisson integral of the equation (1) is related to the stability problem of derivative prices on financial markets [9, 10, 11].

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Малицька Г.П., Буртняк І.В. *Поточкова стабілізація інтеграла Пуассона для рівнянь типу дифузії з інерцією* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 279–283.

В роботі розглянуто поточкову стабілізацію інтеграла Пуассона для рівнянь типу дифузії з інерцією у випадку скінченної кількості груп виродження параболічності, встановлено необхідні і достатні умови такої стабілізації у класі обмежених вимірних початкових функцій.

Ключові слова і фрази: інтеграл Пуассона, рівняння Колмогорова, рівняння типу дифузії з інерцією, стабілізація, вироджене параболічне рівняння, поверхні рівня, граничне середнє.