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APPLICATION OF THE FUNCTIONAL CALCULUS TO SOLVING OF INFINITE DIMENSIONAL HEAT EQUATION

In this paper we study infinite dimensional heat equation associated with the Gross Laplacian. Using the functional calculus method, we obtain the solution of appropriate Cauchy problem in the space of polynomial ultradifferentiable functions. The semigroup approach is considered as well.

Key words and phrases: infinite dimensional heat equation, Gross Laplacian, space of polynomial ultradifferentiable functions, space of polynomial ultradistributions.

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INTRODUCTION

The mathematical framework of white noise analysis, which was founded in works of Yu. Berezansky, Yu. Samoilenko [1] and T. Hida [5], is based on an infinite dimensional analogue of the Schwartz distribution theory.

In 1967 L. Gross [4] introduced Laplacian Δ_G on an abstract Wiener space as a natural infinite dimensional analogue of the finite dimensional Laplacian and studied potential theory associated with Δ_G. Within the white noise framework, the Gross Laplacian has been formulated by Kuo in [8] as a continuous linear operator acting on test white noise functions. The Gross Laplacian and appropriate Cauchy problem have been studied for example in [2, 9].

The aim of this work is to use the functional calculus constructed in [12] in order to solve the infinite dimensional heat equation associated with the Gross Laplacian.

1 PRELIMINARIES

1.1 Spaces of functions

Denote $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\partial^k := \partial^k/\partial t^k$. Fix any real $\beta > 1$. An infinitely differentiable function φ is called an ultradifferentiable function of the Gevrey class (see [7]) if for each segment $[\mu, \nu] \subset \mathbb{R}$ there exist constants $h > 0$ and $C > 0$ such that the inequality $\sup_{t\in[\mu,\nu]} |\partial^k \varphi(t)| \le C h^k k^{k\beta}$ holds for all $k \in \mathbb{Z}_+$. For a fixed $h > 0$ let us consider the subspace

$$\mathcal{G}^h_\beta[\mu,\nu] := \Big\{\varphi \in C^\infty : \operatorname{supp}\varphi \subset [\mu,\nu], \|\varphi\|_{\mathcal{G}^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+} \sup_{t\in[\mu,\nu]} \frac{|\partial^k\varphi(t)|}{h^k k^{k\beta}} < \infty\Big\}.$$

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Each subspace $\mathcal{G}^h_\beta[\mu,\nu]$ is a Banach space (see [7]) and all maps $\mathcal{G}^h_\beta[\mu,\nu] \hookrightarrow \mathcal{G}^l_\beta[\mu,\nu]$ with $h<l$ are compact inclusions. Consider the space

$$\mathcal{G}_\beta := \bigcup_{\mu<\nu,\, h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad \mathcal{G}_\beta \simeq \liminf_{\mu<\nu,\, h>0} \mathcal{G}^h_\beta[\mu,\nu],$$

of Gevrey ultradifferentiable functions with compact supports and endow it with topology of inductive limit with respect to above mentioned compact inclusions. Let $\mathcal{G}'_\beta$ be its dual space of Roumieu ultradistributions.

Let $h>0$ be any positive real and $\mu,\nu\in\mathbb{R}$ be any reals such that $\mu<\nu$. In the space of entire functions of exponential type we consider the subspace $E^h_\beta[\mu,\nu]$ of functions with the finite norm

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+}\sup_{z\in\mathbb{C}} \frac{|z^k\psi(z)e^{-H_{[\mu,\nu]}(\eta)}|}{h^k k^{k\beta}}, \qquad \text{where} \quad H_{[\mu,\nu]}(\eta) := \sup_{t\in[\mu,\nu]} t\eta.$$

Each space $E^h_\beta[\mu,\nu]$ is a Banach one, and all maps $E^h_\beta[\mu,\nu] \hookrightarrow E^{h'}_\beta[\mu',\nu']$ with $[\mu,\nu]\subset[\mu',\nu']$, $h<h'$, are compact inclusions. Consider the space

$$E_\beta := \bigcup_{\mu<\nu,\, h>0} E^h_\beta[\mu,\nu], \qquad E_\beta \simeq \liminf_{\mu<\nu,\, h>0} E^h_\beta[\mu,\nu],$$

and endow it with the topology of inductive limit with respect to above mentioned compact inclusions.

Consider the Fourier-Laplace transformation

$$\widehat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}} e^{-itz}\varphi(t)\,dt, \qquad \varphi\in\mathcal{G}_\beta,\ z\in\mathbb{C}.$$

Let $F' : E'_\beta \longrightarrow \mathcal{G}'_\beta$ be the adjoint mapping. It is known [13], that $F(\mathcal{G}_\beta) = E_\beta$ and $F'(E'_\beta) = \mathcal{G}'_\beta$.

1.2 Polynomial ultradifferentiable functions and polynomial ultradistributions

For any locally convex space $\mathcal{X}$, let $\mathcal{X}^{\widehat{\otimes} n}$, $n\in\mathbb{N}$, be the symmetric nth tensor degree of $\mathcal{X}$, completed in the projective tensor topology. For any $x\in\mathcal{X}$ we denote $x^{\otimes n} := \underbrace{x\otimes\cdots\otimes x}_{n} \in \mathcal{X}^{\widehat{\otimes} n}$, $n\in\mathbb{N}$. Set $\mathcal{X}^{\widehat{\otimes} 0} := \mathbb{C}$, $x^{\otimes 0} := 1\in\mathbb{C}$.

To define the locally convex space $\mathcal{P}({}^n\mathcal{G}'_\beta)$ of n-homogeneous polynomials on $\mathcal{G}'_\beta$ we use the canonical topological linear isomorphism $\mathcal{P}({}^n\mathcal{G}'_\beta) \simeq (\mathcal{G}'^{\widehat{\otimes} n}_\beta)'$, described in [3]. We equip $\mathcal{P}({}^n\mathcal{G}'_\beta)$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{G}'_\beta$. Set $\mathcal{P}({}^0\mathcal{G}'_\beta) := \mathbb{C}$. The space $\mathcal{P}(\mathcal{G}'_\beta)$ of all continuous polynomials on $\mathcal{G}'_\beta$ is defined to be the complex linear span of all $\mathcal{P}({}^n\mathcal{G}'_\beta)$, $n\in\mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Let $\mathcal{P}'(\mathcal{G}'_\beta)$ mean the strong dual of $\mathcal{P}(\mathcal{G}'_\beta)$. Elements of the spaces $\mathcal{P}(\mathcal{G}'_\beta)$ and $\mathcal{P}'(\mathcal{G}'_\beta)$ we call the polynomial test ultradifferentiable functions and polynomial ultradistributions, respectively.

Denote

$$\Gamma(\mathcal{G}_\beta) := \bigoplus_{n\in\mathbb{Z}_+}{}_{fin}\, \mathcal{G}^{\widehat{\otimes} n}_\beta \subset \bigoplus_{n\in\mathbb{Z}_+} \mathcal{G}^{\widehat{\otimes} n}_\beta \qquad \text{and} \qquad \Gamma(\mathcal{G}'_\beta) := \bigtimes_{n\in\mathbb{Z}_+} \mathcal{G}'^{\widehat{\otimes} n}_\beta.$$

Note, that we consider only the case when the elements of direct sum consist of finite but not fixed number of addends. It is well known [11, 4.4], that $\langle\, \Gamma(\mathcal{G}'_\beta), \Gamma(\mathcal{G}_\beta)\,\rangle$ is a dual pair with respect to the bilinear form

$$\langle\, \boldsymbol{f}, \boldsymbol{p}\,\rangle = \Big\langle \bigtimes_{n\in\mathbb{Z}_+} f_n, \bigoplus_{n\in\mathbb{Z}_+} p_n \Big\rangle = \sum_{n\in\mathbb{Z}_+} \langle\, f_n, p_n\,\rangle, \quad \boldsymbol{p} \in \Gamma(\mathcal{G}_\beta), \quad \boldsymbol{f} \in \Gamma(\mathcal{G}'_\beta), \tag{1}$$

where $p_n \in \mathcal{G}_\beta^{\widehat{\otimes} n}$ and $f_n \in \mathcal{G}_\beta'^{\widehat{\otimes} n} \simeq (\mathcal{G}_\beta^{\widehat{\otimes} n})'$.

By analogy we can construct the dual pairs $\langle\, \Gamma(E'_\beta), \Gamma(E_\beta)\,\rangle$ and $\langle\, \mathcal{P}'(E'_\beta), \mathcal{P}(E'_\beta)\,\rangle$.

We have the following assertion (see also [10, Proposition 2.1]).

Proposition 1.1. *There exist the linear topological isomorphisms*

$$\Upsilon : \mathcal{P}'(\mathcal{G}'_\beta) \longrightarrow \Gamma(\mathcal{G}'_\beta), \qquad \Psi : \mathcal{P}'(E'_\beta) \longrightarrow \Gamma(E'_\beta).$$

Using the Proposition 1.1 and tensor structure of the space $\Gamma(\mathcal{G}'_\beta)$, we extend the map F'^{-1} onto $\Gamma(\mathcal{G}'_\beta)$. First, for elements of total subset of the space $\mathcal{G}_\beta'^{\widehat{\otimes} n}$ we define the operator $\mathcal{F}'^{\otimes n}$: $f^{\otimes n} \longmapsto \widehat{f}^{\otimes n}$, $\mathcal{F}'^{\otimes 0} := I_{\mathbb{C}}$, where $\widehat{f}^{\otimes n} := \left(F'^{-1} f\right)^{\otimes n}$. Next, we extend the map $\mathcal{F}'^{\otimes n}$ onto whole space $\mathcal{G}_\beta'^{\widehat{\otimes} n}$ by linearity and continuity. As a result we obtain the map $\mathcal{F}'^{\otimes n} \in \mathscr{L}\big(\mathcal{G}_\beta'^{\widehat{\otimes} n}, E_\beta'^{\widehat{\otimes} n}\big)$ And finally, we define the mapping $\mathcal{F}'^{\otimes}$ by the formula

$$\mathcal{F}'^{\otimes} := \big(\mathcal{F}'^{\otimes n}\big) : \Gamma(\mathcal{G}'_\beta) \ni \boldsymbol{f} = \big(f_n\big) \quad \longmapsto \quad \widehat{\boldsymbol{f}} := \big(\widehat{f}_n\big) \in \Gamma(E'_\beta),$$

where $f_n \in \mathcal{G}_\beta'^{\widehat{\otimes} n}$, $\widehat{f}_n := \mathcal{F}'^{\otimes n} f_n \in E_\beta'^{\widehat{\otimes} n}$.

The following commutative diagram

$$\begin{array}{ccc} \mathcal{P}'(\mathcal{G}'_\beta) & \xrightarrow{\;\mathcal{F}'^{\otimes}_{\mathcal{P}}\;} & \mathcal{P}'(E'_\beta) \\ {\scriptstyle\Upsilon}\big\downarrow & & \big\uparrow{\scriptstyle\Psi^{-1}} \\ \Gamma(\mathcal{G}'_\beta) & \xrightarrow{\;\mathcal{F}'^{\otimes}\;} & \Gamma(E'_\beta) \end{array} \tag{2}$$

uniquely defines the operator $\mathcal{F}'^{\otimes}_{\mathcal{P}} \in \mathscr{L}\big(\mathcal{P}'(\mathcal{G}'_\beta), \mathcal{P}'(E'_\beta)\big)$.

2 CONVOLUTION OF POLYNOMIAL ULTRADISTRIBUTIONS

Let $g \in \mathcal{G}'_\beta$. Define the shift operator on the space $\mathcal{P}(\mathcal{G}'_\beta)$ with the formula

$$\mathscr{T}_g P(f) := P(f+g), \qquad P \in \mathcal{P}(\mathcal{G}'_\beta), \quad f \in \mathcal{G}'_\beta.$$

It is easy to see, that $\mathscr{T}_g$ is a linear continuous operator from the space $\mathcal{P}(\mathcal{G}'_\beta)$ into itself.

Let the symbol $\circledcirc_k$ denotes the (right) k-contraction [6] of symmetric tensor product, i.e., $g^{\otimes k} \circledcirc_k \varphi^{\otimes s} := \langle g, \varphi\rangle^k \varphi^{\otimes(s-k)}$, $k \le s$, $g \in \mathcal{G}'_\beta$, $\varphi \in \mathcal{G}_\beta$.

Let us show, that for any $g \in \mathcal{G}'_\beta$ the shift operator $\mathscr{T}_g$ acts as follows $P = \sum_n \langle \cdot^{\otimes n}, p_n\rangle \mapsto \mathscr{T}_g P = \sum_n \langle \cdot^{\otimes n}, q_n\rangle$, where $p_n, q_n \in \mathcal{G}_\beta^{\widehat{\otimes} n}$, $n = 0, 1, \ldots, m$, $m = \deg P$, and the elements q_n can be obtained by the formula

$$q_n = \sum_{k=0}^{m-n} \frac{(n+k)!}{n!k!} g^{\otimes k} \circledcirc_k p_{n+k}.$$

Without loss of generality we can prove this for polynomials of view $P_{\varphi,m} = \sum_{k=0}^{m} \langle \,\cdot\, ^{\otimes k}, \varphi^{\otimes k}\rangle$, where $(1, \varphi, \varphi^{\otimes 2}, \dots, \varphi^{\otimes m}, 0, \dots) \in \Gamma(\mathcal{G}_\beta)$, $\varphi \in \mathcal{G}_\beta$, $m \in \mathbb{Z}_+$.

Indeed,

$$
\begin{aligned}
\mathcal{T}_g P_{\varphi,m}(f) = P_{\varphi,m}(f+g) &= \sum_{k=0}^{m} \langle (f+g)^{\otimes k}, \varphi^{\otimes k}\rangle = \sum_{k=0}^{m}\sum_{n=0}^{k} C_k^n \langle f^{\otimes n}\widehat{\otimes} g^{\otimes (k-n)}, \varphi^{\otimes k}\rangle \\
&= \sum_{n=0}^{m}\sum_{k=n}^{m} C_k^n \langle f^{\otimes n}\widehat{\otimes} g^{\otimes (k-n)}, \varphi^{\otimes k}\rangle = \sum_{n=0}^{m}\sum_{k=0}^{m-n} C_{n+k}^n \langle f^{\otimes n}\widehat{\otimes} g^{\otimes k}, \varphi^{\otimes (n+k)}\rangle \\
&= \sum_{n=0}^{m}\sum_{k=0}^{m-n} C_{n+k}^n \big\langle f^{\otimes n}, \langle g, \varphi\rangle^k \varphi^{\otimes n}\big\rangle = \sum_{n=0}^{m} \Big\langle f^{\otimes n}, \sum_{k=0}^{m-n} C_{n+k}^n \langle g, \varphi\rangle^k \varphi^{\otimes n}\Big\rangle \\
&= \sum_{n=0}^{m} \Big\langle f^{\otimes n}, \sum_{k=0}^{m-n} C_{n+k}^n g^{\otimes k} \circledcirc_k \varphi^{\otimes (n+k)}\Big\rangle.
\end{aligned}
$$

Let us define the convolution of a polynomial ultradistribution $U \in \mathcal{P}'(\mathcal{G}'_\beta)$ and a test function $P \in \mathcal{P}(\mathcal{G}'_\beta)$ with the formula $(U * P)(g) := \langle U, \mathcal{T}_g P\rangle$, $g \in \mathcal{G}'_\beta$, where in the right side there is the pairing of the dual pair $\big\langle \mathcal{P}'(\mathcal{G}'_\beta), \mathcal{P}(\mathcal{G}'_\beta)\big\rangle$ (see Proposition 1.1 and formula (1)).

If $U \in \mathcal{P}'(\mathcal{G}'_\beta)$ and $P \in \mathcal{P}(\mathcal{G}'_\beta)$ are represented in the form $U = \bigtimes_{n\in\mathbb{Z}_+} \langle u_n, \cdot\,^{\otimes n}\rangle$ and $P = \sum_{n=0}^{m} \langle \cdot\,^{\otimes n}, p_n\rangle$ respectively, then the convolution may be written in the explicit form

$$
\begin{aligned}
(U * P)(g) &= \sum_{n=0}^{m} \Big\langle u_n, \sum_{k=0}^{m-n} C_{n+k}^n g^{\otimes k} \circledcirc_k p_{n+k}\Big\rangle = \sum_{n=0}^{m}\sum_{k=0}^{m-n} C_{n+k}^n \langle u_n \widehat{\otimes} g^{\otimes k}, p_{n+k}\rangle \\
&= \sum_{k=0}^{m}\sum_{n=0}^{m-k} C_{n+k}^n \langle g^{\otimes k}, u_n \circledcirc_n p_{n+k}\rangle = \sum_{k=0}^{m} \Big\langle g^{\otimes k}, \sum_{n=0}^{m-k} C_{n+k}^n u_n \circledcirc_n p_{n+k}\Big\rangle.
\end{aligned} \tag{3}
$$

It is clear, that $q_k = \sum_{n=0}^{m-k} C_{n+k}^n u_n \circledcirc_n p_{n+k}$ belongs to the space $\mathcal{G}_\beta^{\widehat{\otimes} k}$ for each $k = 0, 1, \dots, m$. It follows, that the convolution $U * P$ is a polynomial from the space $\mathcal{P}(\mathcal{G}'_\beta)$.

For any polynomial ultradistribution $U \in \mathcal{P}'(\mathcal{G}'_\beta)$ the mapping C_U, defined with the formula $C_U : \mathcal{P}(\mathcal{G}'_\beta) \ni P \mapsto U * P \in \mathcal{P}(\mathcal{G}'_\beta)$, is said to be the convolution operator, associated with U.

Let us show, that the composition of two convolution operators C_V and C_U, associated with any $V, U \in \mathcal{P}'(\mathcal{G}'_\beta)$, is a convolution operator, associated with some polynomial ultradistribution, which we denote by $V * U$. Let $V, U \in \mathcal{P}'(\mathcal{G}'_\beta)$ and $P \in \mathcal{P}(\mathcal{G}'_\beta)$ are represented in the form $V = \bigtimes_{n\in\mathbb{Z}_+} \langle g^{\otimes n}, \cdot\,^{\otimes n}\rangle$, $U = \bigtimes_{n\in\mathbb{Z}_+} \langle f^{\otimes n}, \cdot\,^{\otimes n}\rangle$ and $P = \sum_{n=0}^{m} \langle \cdot\,^{\otimes n}, \varphi^{\otimes n}\rangle$ respectively, where $f, g \in \mathcal{G}'_\beta$, $\varphi \in \mathcal{G}_\beta$.

Using formula (3), we obtain the following equalities.

$$
\begin{aligned}
(C_V \circ C_U)(P) = V * (U * P) &= \sum_{n=0}^{m} \Big\langle \,\cdot\, ^{\otimes n}, \sum_{j=0}^{m-n} C_{n+j}^j g^{\otimes j} \circledcirc_j q_{n+j}\Big\rangle \\
&= \sum_{n=0}^{m} \Big\langle \,\cdot\, ^{\otimes n}, \sum_{j=0}^{m-n} C_{n+j}^j g^{\otimes j} \circledcirc_j \Big(\sum_{k=0}^{m-n-j} C_{n+j+k}^k f^{\otimes k} \circledcirc_k \varphi^{\otimes (n+j+k)}\Big)\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{m} \Big\langle \,\cdot\, ^{\otimes n}, \sum_{j=0}^{m-n} \sum_{k=0}^{m-n-j} C^{j}_{n+j} C^{k}_{n+j+k} \langle g, \varphi \rangle^{j} \langle f, \varphi \rangle^{k} \varphi^{\otimes n} \Big\rangle \\
&= \sum_{n=0}^{m} \Big\langle \,\cdot\, ^{\otimes n}, \sum_{j+k=0}^{m-n} \frac{(n+j+k)!}{n!j!k!} (g^{\otimes j} \widehat{\otimes} f^{\otimes k}) \circledcirc_{j+k} \varphi^{\otimes (n+j+k)} \Big\rangle \\
&= \sum_{n=0}^{m} \Big\langle \,\cdot\, ^{\otimes n}, \sum_{s=0}^{m-n} \frac{(n+s)!}{n!s!} \sum_{j+k=s} \frac{s!}{j!k!} (g^{\otimes j} \widehat{\otimes} f^{\otimes k}) \circledcirc_{s} \varphi^{\otimes (n+s)} \Big\rangle \\
&= \sum_{n=0}^{m} \sum_{s=0}^{m-n} \frac{(n+s)!}{n!s!} \sum_{j+k=s} \frac{s!}{j!k!} \big\langle \,\cdot\, ^{\otimes n}, (g^{\otimes j} \widehat{\otimes} f^{\otimes k}) \circledcirc_{s} \varphi^{\otimes (n+s)} \big\rangle \\
&= \sum_{s=0}^{m} \sum_{n=0}^{m-s} \frac{(n+s)!}{n!s!} \sum_{j+k=s} \frac{s!}{j!k!} \big\langle \,\cdot\, ^{\otimes n} \widehat{\otimes} g^{\otimes j} \widehat{\otimes} f^{\otimes k}, \varphi^{\otimes (n+s)} \big\rangle \\
&= \sum_{s=0}^{m} \Big\langle \sum_{j+k=s} \frac{s!}{j!k!} g^{\otimes j} \widehat{\otimes} f^{\otimes k}, \sum_{n=0}^{m-s} C^{s}_{n+s} (\,\cdot\, ^{\otimes n}) \circledcirc_{n} \varphi^{\otimes (n+s)} \Big\rangle.
\end{aligned}
$$

It follows, that the composition $C_V \circ C_U$ is the convolution operator, associated with

$$
V * U = \bigtimes_{n \in \mathbb{Z}_+} \Big\langle \sum_{j+k=n} \frac{n!}{j!k!} g^{\otimes j} \widehat{\otimes} f^{\otimes k}, \,\cdot\, ^{\otimes n} \Big\rangle \in \mathcal{P}'(\mathcal{G}'_\beta). \tag{4}
$$

For any polynomial ultradistribution $U \in \mathcal{P}'(\mathcal{G}'_\beta)$ let us define the formal series

$$
e^{*U} := \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} U^{*n}, \qquad \text{where} \quad U^{*n} := \underbrace{U * \cdots * U}_{n}. \tag{5}
$$

Note, that each partial sum of this series belongs to the space $\mathcal{P}'(\mathcal{G}'_\beta)$.

3 HEAT EQUATION ASSOCIATED WITH THE GROSS LAPLACIAN

Let $\{U_t : t \in J\}$ be a family of elements from the space $\mathcal{P}'(\mathcal{G}'_\beta)$, let J be an arbitrary interval $[0, \alpha]$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$. Let us assume, that the function $t \longmapsto U_t$ is a continuous map from J into $\mathcal{P}'(\mathcal{G}'_\beta)$. Then the function $t \longmapsto \mathcal{F}'^{\otimes}_{\mathcal{P}} U_t$ is continuous map from J into $\mathcal{P}'(E'_\beta)$, where the mapping $\mathcal{F}'^{\otimes}_{\mathcal{P}}$ is defined with formula (2). Therefore, for each $t \in J$ the set $\{\mathcal{F}'^{\otimes}_{\mathcal{P}} U_s : s \in [0, t]\}$ is a compact subset in $\mathcal{P}'(E'_\beta)$. In particular, it is bounded. It follows, that the element

$$
\int_0^t \mathcal{F}'^{\otimes}_{\mathcal{P}} U_s \, ds,
$$

belongs to the space $\mathcal{P}'(E'_\beta)$ for each $t \in J$. Hence, in the space $\mathcal{P}'(\mathcal{G}'_\beta)$ there exists a unique element, which we denote $\int_0^t U_s \, ds$, such that

$$
\mathcal{F}'^{\otimes}_{\mathcal{P}} \int_0^t U_s \, ds = \int_0^t \mathcal{F}'^{\otimes}_{\mathcal{P}} U_s \, ds.
$$

Moreover, the map $E_t = \int_0^t U_s \, ds$, $t \in J$, is differentiable and satisfies the equality $\frac{\partial}{\partial t} E_t = U_t$.

Let $\{U_t : t \in J\}$ be any described above family of elements from $\mathcal{P}'(\mathcal{G}'_\beta)$. Let us consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} X_t = U_t * X_t, & t \in J, \\ X_0 = P, & P \in \mathcal{P}(\mathcal{G}'_\beta). \end{cases} \tag{6}$$

Theorem 1. *Cauchy problem* (6) *has a unique solution in* $\mathcal{P}(\mathcal{G}'_\beta)$*, which can be presented in the view*

$$X_t = e^{*\int_0^t U_s\,ds} * P, \qquad t \in J, \tag{7}$$

where $e^{*\int_0^t U_s\,ds}$ *is treated in the sense of the formula* (5).

Proof. Using Picard's iteration, the solution X_t of Cauchy problem (6) is written informally in the form (7). Since the polynomial $P \in \mathcal{P}(\mathcal{G}'_\beta)$ has a finite number of addends, a value of $e^{*\int_0^t U_s\,ds} * P$ depends on some partial sum of the series $e^{*\int_0^t U_s\,ds}$. Formula (3) implies that solution (7) belongs to the space $\mathcal{P}(\mathcal{G}'_\beta)$. □

As an application of Theorem 1 we consider the generalized heat equation, associated with the Gross Laplacian.

Let the trace operator τ be defined by

$$\langle \tau, \varphi \widehat{\otimes} \psi \rangle := \int_{\mathbb{R}^d_+} \varphi(t)\psi(t)\,dt, \qquad \varphi, \psi \in \mathcal{G}_\beta.$$

It is clear, that $\tau \in \mathscr{L}(\mathcal{G}_\beta^{\widehat{\otimes}2}, \mathbb{C}) = (\mathcal{G}_\beta^{\widehat{\otimes}2})' \simeq \mathcal{G}_\beta'^{\widehat{\otimes}2}$.

The Gross Laplacian Δ_G by definition (see e.g. [8]) is the following operator

$$\Delta_G : P = \sum_{n=0}^{m} \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle \longmapsto \Delta_G P := \sum_{n=0}^{m-2} (n+2)(n+1) \langle \tau, \varphi^{\otimes 2} \rangle \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{G}_\beta.$$

Theorem 2. *The Gross Laplacian* Δ_G *acts as a convolution operator, i.e.*

$$\frac{1}{2}\Delta_G P = U_\tau * P, \qquad P \in \mathcal{P}(\mathcal{G}'_\beta),$$

where U_τ *is a polynomial ultradistribution from the space* $\mathcal{P}'(\mathcal{G}'_\beta)$*, that corresponds to the element* $(0, 0, \tau, 0, \dots) \in \Gamma(\mathcal{G}'_\beta)$.

Proof. The polynomial ultradistribution U_τ can be written in the form

$$U_\tau = \bigtimes_{n \in \mathbb{Z}_+} \langle u_{\tau,n}, \cdot^{\otimes n} \rangle = \big(0, 0, \langle \tau, \cdot^{\otimes 2} \rangle, 0, \dots\big),$$

where $u_{\tau,n} = \tau$ if $n = 2$ and $u_{\tau,n} = 0$ if $n \neq 2$.

Let the polynomial $P \in \mathcal{P}(\mathcal{G}'_\beta)$ be of the form $P = \sum_{n=0}^{m} \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle$, $\varphi \in \mathcal{G}_\beta$. Using equalities (3), we obtain the required result

$$\begin{aligned} U_\tau * P = \sum_{n=0}^{m} \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{m-n} C_{n+k}^{k} u_{\tau,k} \circledcirc_k \varphi^{\otimes(n+k)} \Big\rangle &= \sum_{n=0}^{m-2} \big\langle \cdot^{\otimes n}, C_{n+2}^{2} \tau \circledcirc_2 \varphi^{\otimes(n+2)} \big\rangle \\ &= \sum_{n=0}^{m-2} C_{n+2}^{2} \langle \tau, \varphi^{\otimes 2} \rangle \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle = \frac{1}{2}\Delta_G P. \end{aligned}$$

□

Theorem 3. *Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta_G X_t, & t \in J, \\ X_0 = P, & P \in \mathcal{P}(\mathcal{G}'_\beta), \end{cases} \tag{8}$$

for heat equation, associated with the Gross Laplacian, has a unique solution in $\mathcal{P}(\mathcal{G}'_\beta)$ *given by*

$$X_t = e^{*tU_\tau} * P, \qquad t \in J.$$

Proof. Theorem 2 allows us to rewrite the heat equation in the view $\frac{\partial}{\partial t}X_t = U_\tau * X_t$. It follows from Theorem 1 that the Cauchy problem has a unique solution given by

$$X_t = e^{*\int_0^t U_\tau\, ds} * P = e^{*tU_\tau} * P.$$

We can rewrite it in explicit form. Using formula (4), let us find $(tU_\tau)^{*n}$. For $n = 2$ we obtain

$$(tU_\tau) * (tU_\tau) = \bigtimes_{n\in\mathbb{Z}_+} \Big\langle t^2 \sum_{j+k=n} \frac{n!}{j!k!} u_{\tau,j}\widehat{\otimes} u_{\tau,k}, \cdot^{\otimes n} \Big\rangle = \big(0,0,0,0,\frac{4!}{2!2!}t^2\langle \tau^{\otimes 2}, \cdot^{\otimes 4}\rangle, 0, \dots\big),$$

since $u_{\tau,n}$ does not vanish only for $n = 2$. Using the mathematical induction, it is easy to prove that

$$(tU_\tau)^{*n} = \big(\underbrace{0,\dots,0}_{2n}, \frac{(2n)!}{2^n} t^n \langle \tau^{\otimes n}, \cdot^{\otimes 2n}\rangle, 0, \dots\big).$$

It follows

$$\begin{aligned} e^{*tU_\tau} &= \sum_{n\in\mathbb{Z}_+} \frac{1}{n!}(tU_\tau)^{*n} = \sum_{n\in\mathbb{Z}_+} \frac{1}{n!}\big(\underbrace{0,\dots,0}_{2n}, \frac{(2n)!}{2^n} t^n \langle \tau^{\otimes n}, \cdot^{\otimes 2n}\rangle, 0, \dots\big) \\ &= \big(1, 0, t\langle \tau, \cdot^{\otimes 2}\rangle, 0, 3t^2\langle \tau^{\otimes 2}, \cdot^{\otimes 4}\rangle, 0, \dots, 0, \underbrace{\frac{(2n)!}{n!}\frac{t^n}{2^n}\langle \tau^{\otimes n}, \cdot^{\otimes 2n}\rangle}_{2n\text{-th place}}, 0, \dots\big). \end{aligned} \tag{9}$$

It only remains to find the convolution $e^{*tU_\tau} * P$. Let the polynomial $P \in \mathcal{P}(\mathcal{G}'_\beta)$ be written in the form $P = \sum_{n=0}^m \langle \cdot^{\otimes n}, \varphi^{\otimes n}\rangle$, $\varphi \in \mathcal{G}_\beta$. For any $n \in \mathbb{Z}_+$ let us denote $e_{2n} := \frac{(2n)!}{n!}\frac{t^n}{2^n}\tau^{\otimes n}$ and $e_{2n+1} := 0$. Then e^{*tU_τ} can be rewritten as $e^{*tU_\tau} = \bigtimes_{n\in\mathbb{Z}_+}\langle e_n, \cdot^{\otimes n}\rangle$. Therefore, we obtain

$$\begin{aligned} e^{*tU_\tau} * P &= \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{m-n} C_{n+k}^k e_k \circledcirc_k \varphi^{\otimes(n+k)} \Big\rangle = \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} C_{n+2k}^{2k} e_{2k} \circledcirc_{2k} \varphi^{\otimes(n+2k)} \Big\rangle \\ &= \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{(2k)!n!}\frac{(2k)!}{k!}\frac{t^k}{2^k}\langle \tau^{\otimes k}, \varphi^{\otimes 2k}\rangle \varphi^{\otimes n} \Big\rangle \\ &= \sum_{n=0}^m \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!}\frac{t^k}{2^k}\langle \tau^{\otimes k}, \varphi^{\otimes 2k}\rangle \langle \cdot^{\otimes n}, \varphi^{\otimes n}\rangle, \end{aligned}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the floor function.

Hence, if the polynomial P from (8) has the form $P = \sum_{n=0}^m \langle \cdot^{\otimes n}, p_n\rangle$, $p_n \in \mathcal{G}_\beta^{\widehat{\otimes} n}$, then the solution of Cauchy problem for heat equation associated with the Gross Laplacian can be expressed as

$$X_t = \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!}\frac{t^k}{2^k}\tau^{\otimes k} \circledcirc_{2k} p_{n+2k} \Big\rangle.$$

□

4 SEMIGROUP GENERATED BY THE GROSS LAPLACIAN

Our next goal is to construct an one-parameter semigroup $\{\mathsf{G}_t : t \geq 0\}$ with the infinitesimal generator $\frac{1}{2}\Delta_G$. This semigroup can be formally expressed as $\mathsf{G}_t = e^{t\frac{1}{2}\Delta_G}$.

Since $\frac{1}{2}\Delta_G P = U_\tau * P$, results of previous section imply

$$\mathsf{G}_t P := \sum_{n=0}^{m} \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle \,\cdot\,^{\otimes n}, \varphi^{\otimes n} \rangle, \tag{10}$$

where $P = \sum_{n=0}^{m} \langle \,\cdot\,^{\otimes n}, \varphi^{\otimes n} \rangle$, $\varphi \in \mathcal{G}_\beta$.

Proposition 4.1. *The mapping* $\mathbb{R}_+ \ni t \longmapsto \mathsf{G}_t \in \mathscr{L}(\mathcal{P}(\mathcal{G}'_\beta))$*, where* G_t *is defined by formula* (10)*, is a strongly continuous one-parameter semigroup of continuous linear operators from* $\mathcal{P}(\mathcal{G}'_\beta)$ *into itself with infinitesimal generator* $\frac{1}{2}\Delta_G$*.*

Proof. Formula (10) can be rewritten as

$$\mathsf{G}_t P = P + \sum_{n=0}^{m-2} \sum_{k=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle \,\cdot\,^{\otimes n}, \varphi^{\otimes n} \rangle, \tag{11}$$

therefore the equality $\mathsf{G}_0 = I_{\mathcal{P}(\mathcal{G}'_\beta)}$ is clear.

Formulas (4), (9) and the following equalities

$$\mathsf{G}_t \mathsf{G}_s = e^{t\frac{1}{2}\Delta_G} e^{s\frac{1}{2}\Delta_G} = e^{*tU_\tau} * e^{*sU_\tau} = e^{*(t+s)U_\tau} = e^{(t+s)\frac{1}{2}\Delta_G} = \mathsf{G}_{t+s}$$

imply the semigroup property $\mathsf{G}_t \mathsf{G}_s = \mathsf{G}_{t+s}$.

To prove the strong continuity of the semigroup, we need to show that for any $P \in \mathcal{P}(\mathcal{G}'_\beta)$ the function $t \longmapsto \mathsf{G}_t P$ is continuous. Using representation (11), we obtain

$$\begin{aligned}
\lim_{t\to 0} \sup_f |\mathsf{G}_t P - P| &= \lim_{t\to 0} \sup_f \Big| \sum_{n=0}^{m} \sum_{k=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle f^{\otimes n}, \varphi^{\otimes n} \rangle \Big| \\
&\leq \lim_{t\to 0} \sup_f \sum_{n=0}^{m} \sum_{k=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{|t|^k}{2^k} \big|\langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle\big| \big|\langle f^{\otimes n}, \varphi^{\otimes n} \rangle\big| \\
&= \sum_{n=0}^{m} \sup_f \big|\langle f^{\otimes n}, \varphi^{\otimes n} \rangle\big| \lim_{t\to 0} \sum_{k=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{|t|^k}{2^k} \big|\langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle\big| = 0.
\end{aligned}$$

It remains to show that the Gross Laplacian is the generator of the semigroup G_t. Using representation (11), we can write

$$\begin{aligned}
\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = &\sum_{n=0}^{m} \sum_{k=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^{k-1}}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle \,\cdot\,^{\otimes n}, \varphi^{\otimes n} \rangle \\
&- \sum_{n=0}^{m-2} \frac{(n+2)(n+1)}{2} \langle \tau, \varphi^{\otimes 2} \rangle \langle \,\cdot\,^{\otimes n}, \varphi^{\otimes n} \rangle.
\end{aligned}$$

Note, that $\lfloor\frac{m-n}{2}\rfloor = 0$ for $n = m-1$ and for $n = m$. So, we can rewrite the above equality

$$\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m-2}\Bigg(\sum_{k=1}^{\lfloor\frac{m-n}{2}\rfloor}\frac{(n+2k)!}{k!n!}\frac{t^{k-1}}{2^k}\langle\tau^{\otimes k},\varphi^{\otimes 2k}\rangle - \frac{(n+2)(n+1)}{2}\langle\tau,\varphi^{\otimes 2}\rangle\Bigg)\langle\,\cdot\,^{\otimes n},\varphi^{\otimes n}\rangle.$$

It is clear that $\frac{(n+2k)!}{k!n!}\frac{t^{k-1}}{2^k}\langle\tau^{\otimes k},\varphi^{\otimes 2k}\rangle = \frac{(n+2)(n+1)}{2}\langle\tau,\varphi^{\otimes 2}\rangle$ with $k = 1$, therefore

$$\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m-2}\Bigg(\sum_{k=2}^{\lfloor\frac{m-n}{2}\rfloor}\frac{(n+2k)!}{k!n!}\frac{t^{k-1}}{2^k}\langle\tau^{\otimes k},\varphi^{\otimes 2k}\rangle\Bigg)\langle\,\cdot\,^{\otimes n},\varphi^{\otimes n}\rangle.$$

Note, that $\lfloor\frac{m-n}{2}\rfloor = 1$ for $n = m-2$ and for $n = m-3$. So, we obtain

$$\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m-4}\Bigg(\sum_{k=2}^{\lfloor\frac{m-n}{2}\rfloor}\frac{(n+2k)!}{k!n!}\frac{t^{k-1}}{2^k}\langle\tau^{\otimes k},\varphi^{\otimes 2k}\rangle\Bigg)\langle\,\cdot\,^{\otimes n},\varphi^{\otimes n}\rangle.$$

From the above formula we can derive the required result

$$\lim_{t\to 0}\sup_f\Big|\frac{\mathsf{G}_t P(f) - P(f)}{t} - \frac{1}{2}\Delta_G P(f)\Big| \le \sum_{n=0}^{m-4}\sup_f\big|\langle f^{\otimes n},\varphi^{\otimes n}\rangle\big|\lim_{t\to 0}\sum_{k=2}^{\lfloor\frac{m-n}{2}\rfloor}\frac{(n+2k)!}{k!n!}\frac{|t|^{k-1}}{2^k}|\langle\tau^{\otimes k},\varphi^{\otimes 2k}\rangle| = 0.$$

□

Corollary 4.1. *Cauchy problem*

$$\begin{cases}\frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta_G X_t, & t\in J,\\ X_0 = P, & P\in\mathcal{P}(\mathcal{G}'_\beta),\end{cases}$$

for heat equation associated with the Gross Laplacian has a unique solution in $\mathcal{P}(\mathcal{G}'_\beta)$ *given by*

$$X_t = \mathsf{G}_t P, \qquad t\in J.$$

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Шарин С.В. *Застосування функціонального числення до розв'язання задачі Коші для нескінченновимірного рівняння теплопровідності* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 313–322.

У цій роботі ми вивчаємо нескінченновимірне рівняння теплопровідності, породжене лапласіаном Ґросса. Використовуючи метод функціонального числення, ми отримуємо розв'язок відповідної задачі Коші у просторі поліноміальних ультрадиференційовних функцій. Також розглянуто напівгруповий підхід розв'язання такої задачі.

Ключові слова і фрази: нескінченновимірне рівняння теплопровідності, лапласіан Ґросса, простір поліноміальних ультрадиференційовних функцій, простір поліноміальних ультрарозподілів.