ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2016, **8** (2), 211–214
doi:10.15330/cmp.8.2.211-214

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2016, Т.8, №2, С.211–214

VASYLYSHYN T.V.

EXTENSIONS OF MULTILINEAR MAPPINGS TO POWERS OF LINEAR SPACES

We consider the question of the possibility to recover a multilinear mapping from the restriction to the diagonal of its extension to a Cartesian power of a space.

Key words and phrases: multilinear mapping, polarization formula.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: taras.v.vasylyshyn@gmail.com

INTRODUCTION

Let X and Y be linear spaces over the same field $\mathbb{K}$. It is well-known (see e. g. [1, Theorem 1.10]) that every *symmetric* n-linear mapping $A : X^n \to Y$ can be recovered from its restriction to the diagonal $\widehat{A} : X \to Y$, $\widehat{A}(x) = A(x, \ldots, x)$, by means of the so-called Polarization Formula:

$$A(x_1, \ldots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1 \ldots \varepsilon_n \widehat{A}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n).$$

But in general if A is non-symmetric, it cannot be recovered from $\widehat{A}$. For example, if A is alternating, then $\widehat{A}$ is equal to zero. Let us recall that A is called *alternating* if $A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma A(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in X$ and $\sigma \in S_n$, where S_n is the group of all permutations of n elements and $(-1)^\sigma$ is the sign of the permutation σ.

In [1, p. 8] it has been introduced mappings between complex linear spaces, which are linear with respect to some arguments and antilinear with respect to other arguments. If such a mapping is symmetric with respect to "linear" and "antilinear" arguments separately, then it can be recovered from its restriction to the diagonal by means of polarization formulas, proved in [2] and [3]. Note that in this case there are no any requirements of symmetry between "linear" and "antilinear" arguments. In some cases for multilinear mappings there is a similar situation. For example, if $A : X^n \to Y$ is an n-linear mapping, then a mapping $\widetilde{A} : (X^n)^n \to Y$, defined by

$$\widetilde{A}(x_1, \ldots, x_n) = A(x_1^{(1)}, \ldots, x_n^{(n)}),$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(n)}) \in X^n$, $j \in \{1, \ldots, n\}$, is an n-linear mapping too (in general, non-symmetric) and its restriction to the diagonal $\widehat{\widetilde{A}}(x)$ is equal to $A(x^{(1)}, \ldots, x^{(n)})$ for $x = (x^{(1)}, \ldots, x^{(n)}) \in X^n$. Therefore, A and, consequently, $\widetilde{A}$, can be recovered from the restriction of $\widetilde{A}$ to the diagonal.

We consider the question of the possibility of recovering of a multilinear mapping from the restriction to the diagonal of its extension to a power of a space.

УДК 517.98
2010 *Mathematics Subject Classification:* 46J20, 46E15.

1 THE MAIN RESULT

Let $M = (m_{ij})_{i,j=1}^n$ be a matrix of scalars from $\mathbb{K}$. Then for every n-linear mapping $A : X^n \to Y$ a mapping $E_M(A) : (X^n)^n \to Y$, defined by

$$E_M(A)(x_1, \ldots, x_n) = A(m_{11}x_1^{(1)} + \ldots + m_{1n}x_1^{(n)}, \cdots, m_{n1}x_n^{(1)} + \ldots + m_{nn}x_n^{(n)}),$$

where $x_1, \ldots, x_n \in X^n$, is an n-linear mapping. Its restriction to the diagonal is equal to

$$\widehat{E_M(A)}(x) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} m_{1k_1} \ldots m_{nk_n} A(x^{(k_1)}, \ldots, x^{(k_n)}). \tag{1}$$

Note that if $m_{ij} = 1$, $i = 1, \ldots, n$, for the fixed $j \in \{1, \ldots, n\}$, then $E_M(A)$ is an extension of A.

Proposition 1.1. *For every n-linear alternating mapping $A : X^n \to Y$,*

$$\widehat{E_M(A)}(x) = \det(M) A(x^{(1)}, \ldots, x^{(n)}),$$

where $x = (x^{(1)}, \ldots, x^{(n)}) \in X^n$.

Proof. Since A is alternating, $A(x^{(k_1)}, \ldots, x^{(k_n)}) = 0$ if $k_l = k_s$ for some $l \neq s$. Therefore, by (1),

$$\widehat{E_M(A)}(x) = \sum_{\sigma \in S_n} m_{1\sigma(k_1)} \ldots m_{n\sigma(k_n)} A(x^{(\sigma(1))}, \ldots, x^{(\sigma(n))}).$$

Since $A(x^{(\sigma(1))}, \ldots, x^{(\sigma(n))}) = (-1)^\sigma A(x^{(1)}, \ldots, x^{(n)})$, therefore

$$\widehat{E_M(A)}(x) = \sum_{\sigma \in S_n} (-1)^\sigma m_{1\sigma(k_1)} \ldots m_{n\sigma(k_n)} A(x^{(1)}, \ldots, x^{(n)}) = \det(M) A(x^{(1)}, \ldots, x^{(n)}).$$

□

Let us consider recovering of multilinear mappings, which in general are neither symmetric nor alternating. It can be easily seen that if M is a diagonal matrix, then

$$\widehat{E_M(A)}(x) = m_{11} \ldots m_{nn} A(x^{(1)}, \ldots, x^{(n)})$$

for every n-linear mapping A. Let us construct a non-diagonal matrix M' such that *every* n-linear mapping A can be recovered from $\widehat{E_{M'}(A)}$. Let

$$M' = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & -1 & 1 & \ldots & 1 \\ 1 & 1 & -1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & -1 \end{pmatrix}.$$

For $k \in \{1, \ldots, n\}$ let $i_k : X \to X^n$, $i_k(x) = (\underbrace{0, \ldots, 0}_{k-1}, x, 0, \ldots, 0)$.

Theorem 1. *The n-linear mapping A can be recovered from $\widehat{E_{M'}(A)}$ by means of the formula:*

$$A(x_1,\ldots,x_n) = \frac{1}{2^{2n-1}} \sum_{j_2,\ldots,j_n=0}^{1} (-1)^{j_2+\ldots+j_n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \widehat{E_{M'}(A)} \times \left(\varepsilon_1 i_1(x_1) + \varepsilon_2 p^{(2)}_{j_2}(x_2) + \ldots + \varepsilon_n p^{(n)}_{j_n}(x_n)\right), \tag{2}$$

where

$$p^{(k)}_{j_k}(x) = \begin{cases} i_1(x), & \text{if } j_k = 0, \\ i_k(x), & \text{if } j_k = 1 \end{cases}$$

for $k \in \{2,\ldots,n\}$.

Proof. Let $y_1 = i_1(x_1), y_2 = p^{(2)}_{j_2}(x_2), \ldots, y_n = p^{(n)}_{j_n}(x_n)$. Notice that

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \widehat{E_{M'}(A)}(\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n) = \sum_{k_1,\ldots,k_n=1}^{n} E_{M'}(A)(y_{k_1},\ldots,y_{k_n}) \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \varepsilon_{k_1}\ldots\varepsilon_{k_n}$$

and

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \varepsilon_{k_1}\ldots\varepsilon_{k_n} = \begin{cases} 2^n, & \text{if } k_1 \neq \ldots \neq k_n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{1}{2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \widehat{E_{M'}(A)}(\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n) = \sum_{\sigma\in S_n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n)}).$$

For $\sigma \in S_n$ such that $\sigma(n) = n$ we have

$$\sum_{j_n=0}^{1} (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},y_{\sigma(n)}) = \sum_{j_n=0}^{1} (-1)^{j_n} E_{M'}(A)\left(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},p^{(n)}_{j_n}(x_n)\right)$$
$$= E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_1(x_n)) - E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_n(x_n))$$
$$= 2E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_1(x_n)).$$

For $\sigma \in S_n$ such that $\sigma(n) \neq n$ we have

$$\sum_{j_n=0}^{1} (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n)}) = \sum_{j_n=0}^{1} (-1)^{j_n} \left(y_{\sigma(1)},\ldots,p^{(n)}_{j_n}(x_n),\ldots,y_{\sigma(n)}\right)$$
$$= E_{M'}(A)(y_{\sigma(1)},\ldots,i_1(x_n),\ldots,y_{\sigma(n)}) - E_{M'}(A)(y_{\sigma(1)},\ldots,i_n(x_n),\ldots,y_{\sigma(n)}) = 0.$$

Therefore, the right-hand side of (2) is equal to

$$\frac{1}{2^{n-2}} \sum_{j_2,\ldots,j_{n-1}=0}^{1} (-1)^{j_2+\ldots+j_{n-1}} \sum_{\sigma\in S_n,\, \sigma(n)=n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_1(x_n)).$$

After applying this method $n-1$ times we obtain that the right-hand side of (2) is equal to $E_{M'}(A)(i_1(x_1),i_1(x_2),\ldots,i_1(x_n))$, which is equal to $A(x_1,x_2,\ldots,x_n)$. □

REFERENCES

[1] Mujica J. Complex Analysis in Banach Spaces. North Holland, 1986.

[2] Vasylyshyn T.V., Zagorodnyuk A.V. *Polarization formula for* (p, q)*-polynomials on a complex normed space*. Methods Funct. Anal. Topology 2011, **17** (1), 75–83.

[3] Vasylyshyn T.V., Zagorodnyuk A.V. *Polarization formula and polarization inequality for* (p, q)*-linear mappings*. Carpathian Math. Publ. 2009, **1** (2), 128–144. (in Ukrainian)

Received 30.10.2016

Василишин Т.В. *Продовження мультилінійних відображень на степені лінійних просторів* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 211–214.

Розглянуто питання можливості відновлення мультилінійного відображення зі звуження на діагональ продовження цього відображення на деякий декартів степінь простору.

Ключові слова і фрази: мультилінійне відображення, поляризаційна формула.