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## A MULTIDIMENSIONAL GENERALIZATION OF THE RUTISHAUSER QD-ALGORITHM

In this paper the regular multidimensional  $C$ -fraction with independent variables, which is a generalization of regular  $C$ -fraction, is considered. An algorithm of calculation of the coefficients of the regular multidimensional  $C$ -fraction with independent variables correspondence to a given formal multiple power series is constructed. Necessary and sufficient conditions of the existence of this algorithm are established. The above mentioned algorithm is a multidimensional generalization of the Rutishauser  $qd$ -algorithm.

*Key words and phrases:* regular multidimensional  $C$ -fraction with independent variables, correspondence, multiple power series, algorithm.

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### INTRODUCTION

In constructing the branched continued fractions for a given formal multiple power series the concept of correspondence is used. Some general theory of correspondence for functions of one variables is developed in [15, pp. 148–160] (see also [11, pp. 241–274]) and some aspects of it for functions of several variables are considered in [7], [6, pp. 107–109]. As a result, different types of functional fractions are constructed in [1–6, 8–10, 12–14, 16].

In the present paper we construct and investigate an algorithm for the expansion of a given formal multiple power series into a corresponding regular multidimensional  $C$ -fraction with independent variables, which is a generalization of the regular  $C$ -fraction [15, p. 128–129]. It is a further expansion of the results obtained in [2].

### 1 CORRESPONDENCE

Let  $\mathcal{L}$  be set of all formal multiple power series of the form

$$L(\mathbf{z}) = \sum_{|m(N)| \geq 0} c_{m(N)} \mathbf{z}^{m(N)}, \quad (1)$$

where  $m(N) = m_1, m_2, \dots, m_N$  is multiindex,  $m_i \in \mathbb{Z}_+$ ,  $1 \leq i \leq N$ ,  $0(N) = 0, 0, \dots, 0$ ,  $|m(N)| = m_1 + m_2 + \dots + m_N$ ,  $c_{m(N)} \in \mathbb{C}$ ,  $\mathbf{z}^{m(N)} = z_1^{m_1} z_2^{m_2} \dots z_N^{m_N}$ ,  $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^N$ . Obviously, this set forms a ring with unity respect to the operations addition and multiplication of series. We define the mapping  $\lambda : \mathcal{L} \rightarrow \mathbb{N}_0 \cup \{\infty\}$  as follows:  $\lambda(L(\mathbf{z})) = \infty$ , if  $L(\mathbf{z}) \equiv 0$ ;

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$\lambda(L(\mathbf{z})) = n$ , if  $L(\mathbf{z}) \neq 0$ , where  $n$  is the smallest degree of homogeneous polynomial for which  $c_{m(N)} \neq 0$ , that is  $n = |m(N)|$ . We consider the sequence of rational functions

$$f_n(\mathbf{z}) = \frac{P_{m_n}(\mathbf{z})}{Q_{l_n}(\mathbf{z})}, \quad n \geq 1,$$

where  $P_{m_n}(\mathbf{z})$ ,  $Q_{l_n}(\mathbf{z})$  are polynomials of degrees  $m_n$  and  $l_n$  respectively,  $\mathbf{z} \in \mathbb{C}^N$ , moreover,  $Q_{l_n}(0, 0, \dots, 0) \neq 0$ .

The sequence  $\{f_n(\mathbf{z})\}$  corresponds to series (1) at  $\mathbf{z} = (0, 0, \dots, 0)$ , if

$$\lim_{n \rightarrow +\infty} \lambda(L(\mathbf{z}) - L(f_n(\mathbf{z}))) = +\infty,$$

where  $L(f_n(\mathbf{z}))$  is expansion of function  $f_n(\mathbf{z})$  into Taylor series at  $\mathbf{z} = (0, 0, \dots, 0)$ . The order of correspondence of  $f_n(\mathbf{z})$  is defined by the formula  $\nu_n = \lambda(L(\mathbf{z}) - L(f_n(\mathbf{z})))$ . This means that the expansion  $f_n(\mathbf{z})$  into formal multiple power series coincides with  $L(\mathbf{z})$  for all homogeneous polynomials to the degree  $(\nu_n - 1)$  inclusively.

Let us introduce the following set of multiindices

$$\mathcal{J} = \{m(N) : m(N) = m_1, m_2, \dots, m_N, m_p \in \mathbb{Z}_+, 1 \leq p \leq N\}.$$

And now, let us define arithmetical operations on the set  $\mathcal{J}$  componentwise. If

$$r(N) = r_1, r_2, \dots, r_N \in \mathcal{J}, \quad s(N) = s_1, s_2, \dots, s_N \in \mathcal{J}, \quad k \in \mathbb{Z}_+,$$

then

$$r(N) + s(N) = r_1 + s_1, r_2 + s_2, \dots, r_N + s_N, \quad kr(N) = kr_1, kr_2, \dots, kr_N.$$

We consider the regular multidimensional C-fraction with independent variables

$$\frac{a_0}{1 + \prod_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)} z_{i_k}}{1}} = \frac{a_0}{1 + \sum_{i_1=1}^N \frac{a_{i(1)} z_{i_1}}{1 + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1 + \dots}}}, \quad (2)$$

where  $i(k) = i_1, i_2, \dots, i_k$  is multiindex,  $a_0 \neq 0$ ,  $a_{i(k)} \neq 0$ ,  $k \geq 1$ ,  $1 \leq i_n \leq i_{n-1}$ ,  $1 \leq n \leq k$ ,  $i_0 = N$ ,  $\mathbf{z} \in \mathbb{C}^N$ .

Let  $e_0 = 0, 0, \dots, 0$ ,  $e_r = \delta_{r,1}, \delta_{r,2}, \dots, \delta_{r,N}$  be a multiindex,  $\delta_{r,s}$  be a Kronecker symbol,  $1 \leq r, s \leq N$ . Let us introduce the following sets of multiindices

$$\begin{aligned} \mathcal{I} &= \{i(k) : i(k) = i_1, i_2, \dots, i_k, 1 \leq i_p \leq i_{p-1}, 1 \leq p \leq k, k \geq 1, i_0 = N\}, \\ \mathcal{I}^* &= \{\mathbf{i}_{i(k)}^N : \mathbf{i}_{i(k)}^N = e_{i_1} + e_{i_2} + \dots + e_{i_k}, i(k) \in \mathcal{I}\} \end{aligned}$$

and the mapping  $\varphi : \mathcal{I} \rightarrow \mathcal{I}^*$ , such that  $\varphi(i(k)) = \mathbf{i}_{i(k)}^N$  for all  $i(k) \in \mathcal{I}$  (we can show that the mapping  $\varphi$  is bijective).

Let  $a_0 = b_0$ ,  $a_{i(k)} = b_{\mathbf{i}_{i(k)}^N}$ ,  $i(k) \in \mathcal{I}$ ,  $\mathbf{i}_{i(k)}^N \in \mathcal{I}^*$ . Then we write fraction (2) in the form

$$b_0 \left( 1 + \prod_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{b_{\mathbf{i}_{i(k)}^N} z_{i_k}}{1} \right)^{-1}, \quad (3)$$

where  $b_0 \neq 0$ ,  $b_{i_{i(k)}^N} \neq 0$ ,  $\mathbf{i}_{i(k)}^N \in \mathcal{I}^*$ ,  $\mathbf{z} \in \mathbb{C}^N$ .

Let

$$g_n(\mathbf{z}) = b_0 \left( 1 + \prod_{k=1}^{n-1} \sum_{i_k=1}^{i_{k-1}} \frac{b_{i_{i(k)}^N} z_{i_k}}{1} \right)^{-1}$$

be the  $n$ th approximant of regular multidimensional C-fraction with independent variables (3),  $n \geq 1$ .

The correspondence of fraction (3) to series (1) means that the sequence of approximants  $\{g_n(\mathbf{z})\}$  corresponds to  $L(\mathbf{z})$ .

## 2 ALGORITHM

We shall construct and investigate the algorithm for the expansion of the formal multiple power series (1) into the corresponding regular multidimensional C-fraction with independent variables (3).

Let  $c_{0(N)} \neq 0$  and

$$R_{e_0}(\mathbf{z}) = \sum_{|m(N)| \geq 0} \frac{c_{m(N)}}{c_{0(N)}} \mathbf{z}^{m(N)}.$$

Next, let

$$R'_{e_0}(\mathbf{z}) = \sum_{|m(N)| \geq 0} c_{m(N)}^{(e_0)} \mathbf{z}^{m(N)} \quad (4)$$

be reciprocal to series  $R_{e_0}(\mathbf{z})$ . The coefficient of FMPS (4) are uniquely determined by recurrent formulas

$$c_{m(N)}^{(e_0)} = - \sum_{|r(N)|=1}^{|m(N)|} c_{m(N)-r(N)}^{(e_0)} \frac{c_{r(N)}}{c_{0(N)}}, \quad m_j \geq 0, \quad 1 \leq j \leq N, \quad |m(N)| \geq 1, \quad (5)$$

where  $c_{0(N)}^{(e_0)} = 1$ , moreover,  $c_{m(N)}^{(e_0)} = 0$ , if here exist an index  $j$ ,  $1 \leq j \leq N$ , such that  $n_j < 0$ .

By condition  $c_{e_j}^{(e_0)} \neq 0$ ,  $2 \leq j \leq N$ , we write the series (4) in the form

$$R'_{e_0}(\mathbf{z}) = P_{e_1}(z_1) + \sum_{j=2}^N c_{e_j}^{(e_0)} z_j R_{e_j}(\mathbf{z}),$$

where

$$P_{e_1}(z_1) = \sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} c_{m(N)}^{(e_0)} z_1^{m_1}, \quad R_{e_j}(\mathbf{z}) = \sum_{\substack{|r(N)| \geq 0 \\ r_i=0, j+1 \leq i \leq N}} \frac{c_{e_j+r(N)}^{(e_0)}}{c_{e_j}^{(e_0)}} \mathbf{z}^{r(N)}.$$

Then  $L(\mathbf{z})$  can be written

$$L(\mathbf{z}) = \frac{c_{0(N)}}{P_{e_1}(z_1) + \sum_{j=2}^N c_{e_j}^{(e_0)} z_j R_{e_j}(\mathbf{z})}.$$

Let

$$\sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} c_{m(N)} z_1^{m_1}$$

be a normal series (for the notion of normality of formal power series, see [15, pp. 185-190]). Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers  $q_{i(k)}^{(n)}, e_{i(k)}^{(n)}, i_p = 1, 1 \leq p \leq k, k \geq 1, n \geq 0$ , of  $qd$ -table for  $h = 0$  :

$$\begin{array}{ccccccc}
 & & q_{i(1)+j(h)}^{(0)} & & & & \\
 e_{i(0)+j(h)}^{(1)} & & & e_{i(2)+j(h)}^{(0)} & & & \\
 & q_{i(1)+j(h)}^{(1)} & & q_{i(2)+j(h)}^{(0)} & & & \\
 e_{i(0)+j(h)}^{(2)} & & e_{i(1)+j(h)}^{(1)} & & e_{i(2)+j(h)}^{(0)} & & \\
 & q_{i(1)+j(h)}^{(2)} & & q_{i(2)+j(h)}^{(1)} & & \vdots & \ddots \\
 e_{i(0)+j(h)}^{(3)} & & e_{i(1)+j(h)}^{(2)} & & \vdots & & \\
 \vdots & q_{i(1)+j(h)}^{(3)} & \vdots & & & & \\
 & \vdots & & & & & 
 \end{array} \tag{6}$$

the entries of which are defined by the initial conditions

$$e_{i(0)+j(h)}^{(n)} = 0, \quad q_{i(1)+j(h)}^{(n)} = \frac{c_{m(N)+e_{i_1}+e_{j_h}}^{(j(h)-e_{j_h})}}{c_{m(N)+e_{j_h}}^{(j(h)-e_{j_h})}}, \quad |m(N)| = m_{i_1} = n, \quad n \geq 0, \tag{7}$$

moreover,

$$q_{i(1)}^{(n)} = \frac{c_{m(N)+e_{i_1}}}{c_{m(N)}}, \quad |m(N)| = m_{i_1} = n, \quad n \geq 0,$$

and the rhombus rule

$$\begin{aligned}
 e_{i(r)+j(h)}^{(n)} + q_{i(r)+j(h)}^{(n)} &= q_{i(r)+j(h)}^{(n+1)} + e_{i(r-1)+j(h)}^{(n+1)}, \quad r \geq 1, n \geq 0, \\
 e_{i(r)+j(h)}^{(n)} q_{i(r+1)+j(h)}^{(n)} &= q_{i(r)+j(h)}^{(n+1)} e_{i(r)+j(h)}^{(n+1)}, \quad r \geq 1, n \geq 0,
 \end{aligned} \tag{8}$$

The procedure of calculation of the elements of table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) is called the Rutishauser  $qd$ -algorithm [15, p. 227].

We put  $b_{i(2k-1)}^{(0)} = -q_{i(2k-1)}^{(0)}, b_{i(2k)}^{(0)} = -e_{i(2k)}^{(0)}, i_p = 1, 1 \leq p \leq k, k \geq 1$ . According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{m_1=0}^{\infty} \frac{c_{m(N)} z_1^{m_1}}{c_{0(N)}} \sim \left( 1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i(k)} z_1}{1} \right)^{-1}.$$

Here the symbol " $\sim$ " means the correspondence between the series and the fraction. Moreover, according to Lemma 3 [4] we have

$$P_{e_1}(z_1) \sim 1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i(k)} z_1}{1},$$

since the series  $P_{e_1}(z_1)$  is reciprocal to series

$$\sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} \frac{c_{m(N)}}{c_{0(N)}} z_1^{m_1}.$$

Thus we can write

$$L(\mathbf{z}) \sim \frac{c_{0(N)}}{1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i(k)} z_1}{1} + \sum_{j=2}^N c_{e_j^{(e_0)}} z_j R_{e_j}(\mathbf{z})}.$$

Let  $l$  be an arbitrary natural number, moreover,  $2 \leq l \leq N$ . Next, let

$$\sum_{\substack{m_l=0 \\ m_j=0, j \neq l, 1 \leq j \leq N}}^{\infty} c_{m(N)} z_l^{m_l}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers  $q_{i(k)}^{(n)}, e_{i(k)}^{(n)}, i_p = l, 1 \leq p \leq k, k \geq 1, n \geq 0$ , of  $qd$ -table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for  $h = 0$ .

We put  $b'_{i(2k-1)} = -q_{i(2k-1)}^{(0)}, b'_{i(2k)} = -e_{i(2k)}^{(0)}, i_p = l, 1 \leq p \leq k, k \geq 1$ . According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_l=0 \\ m_j=0, j \neq l, 1 \leq j \leq N}}^{\infty} \frac{c_{m(N)} z_l^{m_l}}{c_{0(N)}} \sim \left( 1 + \prod_{\substack{k=1 \\ i_p=l, 1 \leq p \leq k}}^{\infty} \frac{b'_{i(k)} z_l}{1} \right)^{-1}.$$

Since

$$\frac{c_{m(N)}^{(e_0)}}{c_{0(N)}} = -\frac{c_{m(N)}}{c_{0(N)}} = b'_{i(1)}, \quad m_l = 1, m_j = 0, j \neq l, 1 \leq j \leq N, i_1 = l,$$

then we put  $b_{i(1)} = b'_{i(1)}, i_1 = l$ .

Thus we can write

$$L(\mathbf{z}) \sim \frac{c_{0(N)}}{1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i(k)} z_1}{1} + \sum_{j_1=2}^N b_{j_1} z_{j_1} R_{j_1}(\mathbf{z})}.$$

Again, let  $l$  be an arbitrary natural number, moreover,  $2 \leq l \leq N$ . Next, let

$$R'_{e_l}(\mathbf{z}) = \sum_{\substack{|m(N)| \geq 0 \\ m_i=0, l+1 \leq i \leq N}} c_{m(N)}^{(e_l)} \mathbf{z}^{m(N)} \tag{9}$$

be reciprocal to series  $R_{e_l}(\mathbf{z})$ . The coefficients of series (9) are uniquely determined by recurrent formulas for  $m_i = 0$ ,  $j_h + 1 \leq i \leq N$ ,  $|m(N)| \geq 1$ , and  $\mathbf{j}_{j(h)}^N = e_l$

$$c_{m(N)}^{(\mathbf{j}_{j(h)}^N)} = - \sum_{|r(N)|=1}^{|m(N)|} c_{m(N)-r(N)}^{(\mathbf{j}_{j(h)}^N)} \frac{c_{r(N)+e_{j_h}}^{(\mathbf{j}_{j(h)}^N - e_{j_h})}}{c_{e_{j_h}}^{(\mathbf{j}_{j(h)}^N - e_{j_h})}}, \tag{10}$$

where  $c_{0(N)}^{(\mathbf{j}_{j(h)}^N)} = 1$ , moreover,  $c_{n(N)}^{(\mathbf{j}_{j(h)}^N)} = 0$ , if here exist an index  $p$ ,  $1 \leq p \leq N$ , such that  $n_p < 0$ .

By condition  $c_{e_j}^{(e_l)} \neq 0$ ,  $2 \leq j \leq l$ , we write the series (9) in the form

$$R'_{e_l}(\mathbf{z}) = P_{e_l+e_1}(z_1) + \sum_{j=2}^l c_{e_j}^{(e_l)} z_j R_{e_l+e_j}(\mathbf{z}),$$

where

$$P_{e_l+e_1}(z_1) = \sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} c_{m(N)}^{(e_l)} z_1^{m_1}, \quad R_{e_l+e_j}(\mathbf{z}) = \sum_{\substack{|r(N)| \geq 0 \\ r_i=0, j+1 \leq i \leq N}} \frac{c_{e_j+r(N)}^{(e_l)}}{c_{e_j}^{(e_l)}} \mathbf{z}^{r(N)}.$$

Then  $R_{e_l}(\mathbf{z})$  can be written as follows

$$R_{e_l}(\mathbf{z}) = \left( P_{e_l+e_1}(z_1) + \sum_{j=2}^l c_{e_j}^{(e_l)} z_j R_{e_l+e_j}(\mathbf{z}) \right)^{-1}.$$

Let

$$\sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} c_{m(N)+e_l}^{(e_0)} z_1^{m_1}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers  $q_{i_{i(k)}^N+e_l}^{(n)}$ ,  $e_{i_{i(k)}^N+e_l}^{(n)}$ ,  $i_p = 1$ ,  $1 \leq p \leq k$ ,  $k \geq 1$ ,  $n \geq 0$ , of  $qd$ -table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for  $\mathbf{j}_{j(h)}^N = e_l$ .

We put  $b_{i_{i(2k-1)}^N+e_l} = -q_{i_{i(2k-1)}^N+e_l}^{(0)}$ ,  $b_{i_{i(2k)}^N+e_l} = -e_{i_{i(2k)}^N+e_l}^{(0)}$ ,  $i_p = l$ ,  $1 \leq p \leq k$ ,  $k \geq 1$ . According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} \frac{c_{m(N)+e_l}^{(e_0)}}{c_{e_l}^{(e_0)}} z_1^{m_1} \sim \left( 1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i_{i(k)}^N+e_l} z_1}{1} \right)^{-1}.$$

Since the series  $P_{e_l+e_1}(z_1)$  is reciprocal to series

$$\sum_{\substack{m_1=0 \\ m_j=0, 2 \leq j \leq N}}^{\infty} \frac{c_{m(N)+e_l}^{(e_0)}}{c_{e_l}^{(e_0)}} z_1^{m_1},$$

then according to Lemma 3 [4] we obtain

$$P_{e_l+e_1}(z_1) \sim 1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i_{i(k)}+e_l} z_1}{1}.$$

Let  $t$  be an arbitrary natural number, moreover,  $2 \leq t \leq l - 1$ . Next, let

$$\sum_{\substack{m_t=0 \\ m_j=0, j \neq t, 1 \leq j \leq N}}^{\infty} c_{m(N)+e_l}^{(e_0)} z_t^{m_t}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers  $q_{i_{i(k)}+e_l}^{(n)}, e_{i_{i(k)}+e_l}^{(n)}, i_p = t, 1 \leq p \leq k, k \geq 1, n \geq 0$ , of  $qd$ -table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for  $\mathbf{j}_{j(h)}^N = e_l$ .

We put  $b'_{i_{i(2k-1)}+e_l} = -q_{i_{i(2k-1)}+e_l}^{(0)}, b'_{i_{i(2k)}+e_l} = -e_{i_{i(2k)}+e_l}^{(0)}, i_p = t, 1 \leq p \leq k, k \geq 1$ . According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_t=0 \\ m_j=0, j \neq t, 1 \leq j \leq N}}^{\infty} \frac{c_{m(N)+e_l}^{(e_0)}}{c_{e_l}^{(e_0)}} z_t^{m_t} \sim \left( 1 + \prod_{\substack{k=1 \\ i_p=t, 1 \leq p \leq k}}^{\infty} \frac{b'_{i_{i(k)}+e_l} z_t}{1} \right)^{-1}.$$

Since the series  $P_{e_l+e_r}(z_t)$  is reciprocal to series

$$\sum_{\substack{m_t=0 \\ m_j=0, j \neq t, 1 \leq j \leq N}}^{\infty} \frac{c_{m(N)+e_l}^{(e_0)}}{c_{e_l}^{(e_0)}} z_t^{m_t},$$

then according to Lemma 3 [4] we obtain

$$P_{e_l+e_r}(z_t) \sim 1 + \prod_{\substack{k=1 \\ i_p=t, 1 \leq p \leq k}}^{\infty} \frac{b'_{i_{i(k)}+e_l} z_t}{1}.$$

Since

$$c_{e_t}^{(e_l)} = -\frac{c_{e_l+e_t}^{(e_0)}}{c_{e_l}^{(e_0)}} = -\frac{c_{0(N)} c_{e_l+e_t} - c_{e_l}^2}{c_{e_l} c_{0(N)}} = b'_{e_l+e_t}, \quad c_{e_l}^{(e_l)} = -\frac{c_{2e_l}^{(e_0)}}{c_{e_l}^{(e_0)}} = -\frac{c_{0(N)} c_{2e_l} - c_{e_l}^2}{c_{e_l} c_{0(N)}} = b'_{2e_l},$$

than we put  $b_{e_l+e_t} = b'_{e_l+e_t}, b_{2e_l} = b'_{2e_l}$ .

Thus we can write

$$L(\mathbf{z}) \sim \frac{c_{0(N)}}{Q_{\mathbf{j}_{j(0)}^N}(z_1) + \sum_{j_1=2}^N \frac{b_{\mathbf{j}_{j(1)}^N} z_{j_1}}{Q_{\mathbf{j}_{j(1)}^N}(z_1) + \sum_{j_2=2}^{j_1} b_{\mathbf{j}_{j(2)}^N} z_{j_2} R_{\mathbf{j}_{j(2)}^N}(\mathbf{z})}},$$

where

$$Q_{j(h)}^N(z_1) = 1 + \prod_{\substack{k=1 \\ i_p=1, 1 \leq p \leq k}}^{\infty} \frac{b_{i(k)+j(h)}^N z_1}{1}, \quad h \geq 0,$$

moreover,  $j_r \neq 1, 1 \leq r \leq h, \mathbf{j}_{j(h)}^N \in \mathcal{I}^*$ , if  $h \geq 1$ .

Next, computing the coefficients

$$c_{m(N)}^{(\mathbf{j}_{j(h)}^N)}, \quad m_i = 0, j_h + 1 \leq i \leq N, |m(N)| \geq 1, j_r \neq 1, 1 \leq r \leq h, \mathbf{j}_{j(h)}^N \in \mathcal{I}^*,$$

by recurrent formulas (10) and continuing process of iteration under the conditions that the series

$$\sum_{\substack{m_l=0 \\ m_i=0, i \neq l, 1 \leq i \leq N}}^{\infty} c_{m(N)} z_l^{m_l}, \quad \sum_{\substack{m_t=0 \\ m_i=0, i \neq t, 1 \leq i \leq N}}^{\infty} c_{m(N)+e_p}^{(e_0)} z_t^{m_t}, \quad \sum_{\substack{m_r=0 \\ m_i=0, i \neq r, 1 \leq i \leq N}}^{\infty} c_{m(N)+e_{j_h}}^{(\mathbf{j}_{j(h)}^N)} z_r^{m_r}, \quad (11)$$

where  $1 \leq l \leq N, 1 \leq t \leq p-1, 2 \leq p \leq N, 1 \leq r \leq j_h-1, j_r \neq 1, 1 \leq r \leq h, \mathbf{j}_{j(h)}^N \in \mathcal{I}^*$ , are normal, for series (1) we obtain fraction (3), where  $c_0 = c_{0(N)}, b_{i(2k-1)+j(h)}^N = -q_{i(k)+j(h)}^{(0)}$ ,  $b_{i(2k)+j(h)}^N = -e_{i(k)+j(h)}^{(0)}, i_p = n, 1 \leq p \leq k, 1 \leq n \leq j_h-1, k \geq 1, j_r \neq 1, 1 \leq r \leq h, \mathbf{j}_{j(h)}^N \in \mathcal{I}^*$  (the numbers  $q_{i(k)+j(h)}^{(0)}, e_{i(k)+j(h)}^{(0)}, i_p = n, 1 \leq p \leq k, 1 \leq n \leq j_h-1, k \geq 1, j_r \neq 1, 1 \leq r \leq h, \mathbf{j}_{j(h)}^N \in \mathcal{I}^*$ , are the diagonal elements of the  $qd$ -table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8)).

Thus, if the coefficients of the formal multiple power series (1) are given, then the recurrent algorithm of calculation of the coefficients of the regular multidimensional C-fraction with independent variables (3) is constructed. This algorithm is a multidimensional generalization of Rutishauser  $qd$ -algorithm [15, p. 227]. The correspondence of fraction (3) to series (1) can be proved by a scheme proposed in [5].

Hence, the following theorem holds:

**Theorem.** *The regular multidimensional C-fraction with independent variables (3) corresponds to the given formal multiple power series (1) if and only if the formal power series (11) are normal.*

We remark that some examples of functions of two variables represented by regular two-dimensional C-fractions with independent variables are given in [2].

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### REFERENCES

[1] Dmytryshyn R.I. *Associated branched continued fractions with two independent variables*. Ukrainian Math. J. 2015, **66** (9), 1312–1323. doi:10.1007/s11253-015-1011-6 (translation of Ukrain. Mat. Zh. 2014, **66** (9), 1175–1184. (in Ukrainian))



- [2] Dmytryshyn R.I. *Two-dimensional generalization of the Rutishauser  $qd$ -algorithm*. J. Math. Sci. 2015, **208** (3), 301–309. doi:10.1007/s10958-015-2447-9 (translation of Mat. Met. Fiz.-Mekh. Polya 2013, **56** (4), 33–39. (in Ukrainian))
- [3] Dmytryshyn R.I. *Regular two-dimensional  $C$ -fraction with independent variables for double power series*. Bukovinskij Mat. Zh. 2013, **1** (1-2), 55–57. (in Ukrainian)
- [4] Dmytryshyn R.I. *The two-dimensional  $g$ -fraction with independent variables for double power series*. J. Approx. Theory. 2012, **164** (12), 1520–1539. doi:10.1016/j.jat.2012.09.002
- [5] Dmytryshyn R.I. *The multidimensional generalization of Bauer's  $g$ -algorithm*. Carpathian Math. Publ. 2012, **4** (2), 247–260. (in Ukrainian)
- [6] Kuchminska Ch.Yo. *Two-Dimensional Continued Fractions*. Pidstryhach Institute of Applied Problems of Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv, 2010. (in Ukrainian)
- [7] Hoyenko N.P. *Correspondence principle and convergence of sequences of analytic functions of several variables*. Math. Bull. Shevchenko Sci. Soc. 2007, **4**, 42–48. (in Ukrainian)
- [8] Dmytryshyn R.I. *The multidimensional generalization of  $g$ -fractions and their application*. J. Comput. Appl. Math. 2004, **164-165**, 265–284. doi:10.1016/S0377-0427(03)00642-3
- [9] Baran O.E., Bodnar D.I. *The expansion of multiple power series into multidimensional  $C$ -fraction with independent variables*. Volynskij Mat. Visn. 1999, **6**, 15–20. (in Ukrainian)
- [10] Bodnar D.I. *Multidimensional  $C$ -fractions*. J. Math. Sci. 1998, **90** (5), 2352–2359. doi:10.1007/BF02433965 (translation of Mat. Met. Fiz.-Mekh. Polya 1996, **39** (3), 39–66. (in Ukrainian))
- [11] Lorentzen L., Waadeland H. *Continued Fractions with Applications*. In: Studies in Computational Mathematics, 3. North-Holland, Amsterdam, London, New-York, Tokyo, 1992.
- [12] Bodnar D.I. *Corresponding branched continued fractions with linear partial numerators for double power series*. Ukrain. Mat. Zh. 1991, **43** (4), 474–482. (in Russian)
- [13] Cuyt A., Verdonk B. *A review of branched continued fraction theory for the construction of multivariate rational approximations*. Appl. Numer. Math. 1988, **4**, 263–271. doi: 10.1016/0168-9274(83)90006-5
- [14] Siemaszko W. *Branched continued fractions for double power series*. J. Comput. Appl. Math. 1980, **6** (2), 121–125. doi:10.1016/0771-050X(80)90005-4
- [15] Jones W.B., Thron W.J. *Continued Fractions: Analytic Theory and Applications*. In: Encyclopedia of Mathematics and its Applications, 11. Addison-Wesley, London, Amsterdam, Don Mills, Ontario, Sydney, Tokyo, 1980.
- [16] Murphy J.F., O'Donohoe M.R. *A two-variable generalisation of the Stieltjes-type continued fractions*. J. Comput. Appl. Math. 1978, **4** (3), 181–190. doi:10.1016/0771-050X(78)90002-5

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Розглядається регулярний багатовимірний  $C$ -дріб з нерівнозначними змінними, який є узагальненням регулярного  $C$ -дріб. Побудовано алгоритм обчислення коефіцієнтів багатовимірної  $C$ -дріб з нерівнозначними змінними, відповідного заданому формальному кратному степеневому ряду, який є узагальненням  $qd$ -алгоритму Рутисхаузера. Встановлено необхідні та достатні умови існування такого алгоритму.

*Ключові слова і фрази:* регулярний багатовимірний  $C$ -дріб з нерівнозначними змінними, відповідність, кратний степеневий ряд, алгоритм.