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APPROXIMATION OF CAPACITIES WITH ADDITIVE MEASURES

For a space of non-additive regular measures on a metric compactum with the Prokhorov-style metric, it is shown that the problem of approximation of arbitrary measure with an additive measure on a fixed finite subspace reduces to linear optimization problem with parameters dependent on the values of the measure on a finite number of sets.

An algorithm for such an approximation, which is more efficient than the straighforward usage of simplex method, is presented.

Key words and phrases: Prokhorov metric, non-additive measure, approximation, compact metric space.

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INTRODUCTION

Capacities were introduced by Choquet [1] and found numerous applications in different branches of mathematics. Spaces of upper semicontinuous capacities on compacta were systematically studied in [5]. In particular, in the latter paper functoriality of the construction of a space of capacities was proved and Prokhorov-style and Kantorovich-Rubinstein-style metrics on the set of capacities on a metric compactum were introduced. Needs of practice require that a capacity can be approximated with capacities of simpler structure or with some convenient properties.

We follow the terminology and notation of [5] and denote by $\exp X$ the set of all non-empty closed subsets of a compactum X. We call a function $c : \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the three following properties hold for all subsets $F, G \underset{\mathrm{cl}}{\subset} X$:

1. $c(\varnothing) = 0$;
2. if $F \subset G$, then $c(F) \leqslant c(G)$ (monotonicity);
3. if $c(F) < a$, then there is an open subset $U \supset F$ such that for all $G \subset U$ the inequality $c(G) < a$ is valid (upper semicontinuity).

If, additionally, $c(X) = 1$ (or $c(X) \leq 1$) holds, then the capacity is called *normalized* (resp. *subnormalized*). We denote by $\overline{M}X$, MX, and $\underline{M}X$ the sets of all capacities on X, of all normalized, and of all subnormalized capacities on X respectively.

It was shown in [5] that MX carries a compact Hausdorff topology with the subbase of all sets of the form

$$O_{-}(F, a) = \{c \in MX \mid c(F) < a\}, \text{where} F \underset{\mathrm{cl}}{\subset} X, a \in I,$$

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and

$$O_+(U,a) = \{c \in MX \mid c(U) > a\}$$
$$= \{c \in MX \mid \text{there is a compactum } F \subset U, c(F) > a\}, \text{where } U \underset{\text{op}}{\subset} X, a \in I.$$

The same formulae determine a subbase of a compact Hausdorff topology on $\underline{M}X$ so that $MX \subset \underline{M}X$ is a subspace.

Previously we have considered the following subclasses of MX:

1) $M_\cap X$ is the set of the so-called $\cap$-*capacities* (or necessity measures) with the property: $c(A \cap B) = \min\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$.

2) $M_\cup X$ is the set of the so-called $\cup$-*capacities* (or possibility measures) with the property: $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$.

3) Class MX_0 of capacities defined *on a closed subspace* $X_0 \subset X$. We regard each capacity c_0 on X_0 as a capacity on X extended with the formula $c(F) = c_0(F \cap X_0)$, $F \underset{\text{cl}}{\subset} X$.

4) Class $M_{Lip}X$ of capacities that are non-expanding w.r.t. the Hausdorff metric on $\exp X$.

Analogous subclasses are defined in $\underline{M}X$ and $\bar{M}X$, with the obvious denotations.

It was proved in [2, 3] that the subsets $M_\cap X$, $M_\cup X$, $M_{Lip}X$, and MX_0 are closed in MX, hence for a compactum X they are compacta as well, similarly for the respective subsets in $\underline{M}X$ and $\bar{M}X$.

We consider the metric on the set $\bar{M}X$ of capacities on a metric compactum (X, d) :

$$\hat{d}(c, c') = \inf\{\varepsilon > 0 \mid c(\bar{O}_\varepsilon(F)) + \varepsilon \geqslant c'(F), c'(\bar{O}_\varepsilon(F)) + \varepsilon \geqslant c(F), \forall F \underset{\text{cl}}{\subset} X\},$$

here $\bar{O}_\varepsilon(F)$ is the closed ε-neighborhood of a subset $F \subset X$. The restrictions of this metric on $\underline{M}X$ and MX are admissible [5].

For an arbitrary capacity c on a metric compactum X, explicit constructions for the closest to c point in the four above subclasses were presented in [3, 4].

Now we consider probably the most important class of *additive* regular measures.

Our goal is to approximate a capacity c on a metric compactum X with an additive measure on a *finite subspace* of X. Such measures are dense in the space $\overline{P}X$ of all finite additive regular measures and have nice representation as linear combinations of Dirac measures.

1 ALGORITHM FOR APPROXIMATION OF A CAPACITY WITH AN ADDITIVE MEASURE ON A FINITE SUBSPACE

Consider a capacity c on a metric compactum (X, d) and a finite subspace $X_0 = \{x_1, x_2, \ldots, x_n\} \subset X$. We are going to find the distance between $c \in \bar{M}X$ and the subspace $\overline{P}X_0 \subset \bar{M}X$, in particular to find an additive measure m on X_0 that is (almost) the closest to c with respect to the distance $\hat{d}$.

The inequality $\hat{d}(c, m) \leqslant \varepsilon$ means that there is $0 \leqslant z \leqslant \varepsilon$ satisfying

$$\begin{cases} m(A) \leqslant c(\bar{O}_\varepsilon A) + z, \\ c(A) \leqslant m(\bar{O}_\varepsilon A) + z \end{cases}$$

for all $A \underset{\text{cl}}{\subset} X$. Obviously it is sufficient to verify the first inequality $m(A) \leqslant c^+_\varepsilon(A) + z$, where we denote $c^+_\varepsilon = c(\bar{O}_\varepsilon(A))$, only for all $A \subset X_0$. Similarly, for the second condition we verify

$c(B) \leqslant m(A) + z$ for all $B \subset X$ and $A \subset X_0$ such that $(\bar{O}_\varepsilon B) \cap X_0 \subset A$. This is equivalent to $m(A) \geqslant c_\varepsilon^-(A) - z$ for all $A \subset X_0$, where

$$c_\varepsilon^-(A) = c(X \setminus \bar{O}_\varepsilon(X_0 \setminus A)) = \sup\{c(B) \mid B \underset{\mathrm{cl}}{\subset} X, B \cap \bar{O}_\varepsilon(X_0 \setminus A) = \varnothing\}.$$

Obviously $c_\varepsilon^-(A) \leqslant c_\varepsilon^+(A)$ for all $A \subset X_0$.

All additive measures on X_0 are of the form $m = y_1\delta_{x_1} + y_2\delta_{x_2} + \cdots + y_n\delta_{x_n}$. Thus, to find the least z that satisfies the above conditions for some m, we have to solve the linear programming problem w.r.t. the variables $y_1, y_2, \ldots, y_n, z \geqslant 0$:

$$\begin{cases} y_1, y_2, \ldots, y_n, z \geqslant 0, \\ \sum_{x_i \in A} y_i \leqslant c_\varepsilon^+(A) + z & \text{for all } A \subset X_0, \\ \sum_{x_i \in A} y_i \geqslant c_\varepsilon^-(A) - z & \text{for all } A \subset X_0, \\ z \to \min, \end{cases}$$

which we rewrite as follows:

$$\begin{cases} y_1, y_2, \ldots, y_n, z \geqslant 0, \\ -\sum_{x_i \in A} y_i + z \geqslant -c_\varepsilon^+(A) & \text{for all } A \subset X_0, \\ \quad \sum_{x_i \in A} y_i + z \geqslant \quad c_\varepsilon^-(A) & \text{for all } A \subset X_0, \\ z \to \min. \end{cases}$$

We embed the set $\operatorname{Exp} X_0$ into $\mathbb{R}^n$ by identifying each subset $A \subset X_0$ with the vector containing 1 at all i-th positions such that $x_i \in A$ and 0 at all other positions. E.g., $\varnothing$ is represented by $(0, \ldots, 0)$, and X_0 by $(1, \ldots, 1)$. By $-\operatorname{Exp} X_0$ we denote the set of the opposites to elements of $\operatorname{Exp} X_0 \subset \mathbb{R}^n$. Define a function $c_\varepsilon : \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0) \to \mathbb{R}$ by the formula

$$c_\varepsilon(A) = \begin{cases} c_\varepsilon^-(A), & A \in \operatorname{Exp} X_0, \\ -c_\varepsilon^+(-A), & A \in (-\operatorname{Exp} X_0). \end{cases}$$

The common element $\varnothing = (0, \ldots, 0) \in \operatorname{Exp} X_0 \cap (-\operatorname{Exp} X_0)$ leads to no contradiction because $c_\varepsilon^-(\varnothing) = c_\varepsilon^+(\varnothing) = 0$.

We also denote by $(A|1)$ the vector obtained by appending a trailing 1 to the sequence $A = (a_1, a_2, \ldots, a_n) \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$. Then the linear optimization problem can we written as

$$\begin{cases} y_1, y_2, \ldots, y_n, z \geqslant 0, \\ (A|1) \cdot (y_1, y_2, \ldots, y_n, z) \geqslant c_\varepsilon(A) \text{ for all } A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0), \\ z \to \min. \end{cases}$$

It has a straightforward geometric interpretation: of all functionals of the form

$$\gamma(t_1, t_2, \ldots, t_n) = y_1 t_1 + y_2 t_2 + \cdots + y_n t_n + z$$

such that $\gamma(A) \geqslant c_\varepsilon(A)$ for all $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$, choose one with the minimal z, i.e., with the least value $\gamma(\vec{0})$. Now it is clear that, due to monotonicity of the function c_ε, the restrictions $y_1, y_2, \ldots, y_n \geqslant 0$ can be dropped. Observe also that the restriction $z \geqslant 0$ is equivalent to

$$(\varnothing|1) \cdot (y_1, y_2, \ldots, y_n, z) \geqslant c_\varepsilon(\varnothing),$$

hence can be dropped as well.

Geometric arguments also show that the problem is solved if affinely independent

$$A_1, A_2, \dots, A_{n+1} \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$$

are found such that $\vec{0}$ is in their convex hull (in the sequel we call such $A_1, A_2, \dots, A_{n+1}$ *basic subsets*), and the solutions $y_1, y_2, \dots, y_n, z$ of the system

$$\begin{cases} (A_1|1) \cdot (y_1, y_2, \dots, y_n, z) &= c_\varepsilon(A_1), \\ (A_2|1) \cdot (y_1, y_2, \dots, y_n, z) &= c_\varepsilon(A_2), \\ \dots \\ (A_{n+1}|1) \cdot (y_1, y_2, \dots, y_n, z) &= c_\varepsilon(A_{n+1}) \end{cases}$$

satisfy

$$(A|1) \cdot (y_1, y_2, \dots, y_n, z) \geqslant c_\varepsilon(A)$$

for all $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$.

Therefore we propose the following algorithm, which essentially is equivalent to the simplex algorithm, but is better suited for our needs. Choose initial basic subsets, e.g., $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, …, $A_n = \{x_n\}$, $A_{n+1} = -\{x_n\}$, then calculate $y_1, y_2, \dots, y_n, z$ as

$$(y_1, y_2, \dots, y_n, z)^T = \big(M(A_1, A_2, \dots, A_n)\big)^{-1} (c(A_1), c(A_2), \dots, c(A_{n+1}))^T,$$

where $(-)^T$ means transposition, and

$$M(A_1, A_2, \dots, A_n) = \begin{bmatrix} A_1 & | & 1 \\ A_2 & | & 1 \\ \dots & & \dots \\ A_{n+1} & | & 1 \end{bmatrix},$$

i.e., it is the matrix with the rows $(A_1|1)$, $(A_2|1)$, …, $(A_{n+1}|1)$.

We will permanently need the inverse matrix

$$\big(M(A_1, A_2, \dots, A_n)\big)^{-1} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1,n+1} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2,n+1} \\ \dots & \dots & \ddots & \dots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{n,n+1} \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \end{bmatrix}.$$

For any $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$ the column $\big(M(A_1, A_2, \dots, A_n)\big)^{-1} (A|1)^T$ consists of the coefficients $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_{n+1} = 1$ and $\alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_{n+1} A_{n+1} = A$ (in the above sense). In particular, $\mu_1 A_1 + \mu_2 A_2 + \dots + \mu_{n+1} A_{n+1} = \varnothing$, and $\lambda_{i1} A_1 + \lambda_{i2} A_2 + \dots + \lambda_{i,n+1} A_{n+1} = \{x_i\}$ for all $1 \leqslant i \leqslant n$.

Now, having $y_1, y_2, \dots, y_n, z$ calculated, compare the differences

$$c_\varepsilon(A) - (A|1)(y_1, y_2, \dots, y_n, z)$$

for all $A \in \operatorname{Exp} X_0 \cup (-\operatorname{Exp} X_0)$. If the basic subsets $A_1, A_2, \dots, A_{n+1}$ provide a solution, then all the differences are not greater than 0. Otherwise find the greatest difference $\Delta =$

$c_\varepsilon(A') - (A'|1)(y_1, y_2, \ldots, y_n, z)$, which is positive, and replace with A' a subset A_i such that $\vec{0}$ is in the convex hull of $A_1, A_2, \ldots, A_{i-1}, A', A_{i+1}, \ldots, A_{n+1}$.

Let $(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})^T = \big(M(A_1, A_2, \ldots, A_n)\big)^{-1}(A'|1)^T$, hence $A' = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_{n+1} A_{n+1}$, then

$$A_i = \frac{1}{\alpha_i}A' - \frac{\alpha_1}{\alpha_i}A_1 - \cdots - \frac{\alpha_{i-1}}{\alpha_i}A_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}A_{i+1} - \frac{\alpha_{n+1}}{\alpha_i}A_{n+1}.$$

Therefore

$$\begin{aligned}\varnothing = \big(\mu_1 - \mu_i\frac{\alpha_1}{\alpha_i}\big)A_1 + \cdots + \big(\mu_{i-1} - \mu_i\frac{\alpha_{i-1}}{\alpha_i}\big)A_{i-1} + \big(\mu_{i+1} - \mu_i\frac{\alpha_{i+1}}{\alpha_i}\big)A_{i+1} \\ + \cdots + \big(\mu_{n+1} - \mu_i\frac{\alpha_{n+1}}{\alpha_i}\big)A_{n+1} + \frac{\mu_i}{\alpha_i}A'.\end{aligned}$$

The coefficients in the new decomposition of $\varnothing$ should be nonnegative, hence $\alpha_i > 0$ is required, as well as either $\alpha_j \leqslant 0$ or $\mu_j - \mu_i\dfrac{\alpha_j}{\alpha_i} \geqslant 0$ for all $j \neq i$. If $\alpha_j > 0$, then the latter inequality is equivalent to $\dfrac{\mu_j}{\alpha_j} \geqslant \dfrac{\mu_i}{\alpha_i}$. Hence $\dfrac{\mu_i}{\alpha_i}$ should be the least of $\dfrac{\mu_j}{\alpha_j}$ for $1 \leqslant j \leqslant n+1$ such that $\alpha_j > 0$.

Now we replace A_i with $A'_i = A'$, and the inverse matrix

$$\big(M(A_1, A_2, \ldots, A_{i-1}, A'_i, A_{i+1}, \ldots, A_n)\big)^{-1} = \begin{bmatrix} \lambda'_{11} & \lambda'_{12} & \cdots & \lambda'_{1,n+1} \\ \lambda'_{21} & \lambda'_{22} & \cdots & \lambda'_{2,n+1} \\ \cdots & \cdots & \ddots & \cdots \\ \lambda'_{n1} & \lambda'_{n2} & \cdots & \lambda'_{n,n+1} \\ \mu'_1 & \mu'_2 & \cdots & \mu'_{n+1} \end{bmatrix}$$

is adjusted accordingly:

$$\begin{aligned} \mu'_i &= \frac{\mu_i}{\alpha_i}, \qquad \mu'_j = \mu_j - \alpha_j\frac{\mu_i}{\alpha_i}, && 1 \leqslant j \leqslant n+1,\ j \neq i, \\ \lambda'_{ki} &= \frac{\lambda_{ki}}{\alpha_i}, \qquad \lambda'_{kj} = \lambda_{kj} - \alpha_j\frac{\lambda_{ki}}{\alpha_i}, && 1 \leqslant k, j \leqslant n+1,\ j \neq i. \end{aligned}$$

Now look how $y_1, y_2, \ldots, y_n, z$ have changed. Taking into account

$$\begin{aligned} z &= \mu_1 c_\varepsilon(A_1) + \cdots + \mu_{i-1}c_\varepsilon(A_{i-1}) + \mu_i c_\varepsilon(A_i) \\ &\quad + \mu_{i+1}c_\varepsilon(A_{i+1}) + \cdots + \mu_{n+1}A_{n+1}, \\ z' &= \big(\mu_1 - \alpha_1\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_1) + \cdots + \big(\mu_{i-1} - \alpha_{i-1}\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_{i-1}) + \frac{\mu_i}{\alpha_i}c_\varepsilon(A'_i) \\ &\quad + \big(\mu_{i+1} - \alpha_{i+1}\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_{i+1}) + \cdots + \big(\mu_{n+1} - \alpha_{n+1}\frac{\mu_i}{\alpha_i}\big)c_\varepsilon(A_{n+1}), \end{aligned}$$

obtain

$$z' - z = \frac{\mu_i}{\alpha_i}\big(c_\varepsilon(A'_i) - (\alpha_1 c_\varepsilon(A_1) + \cdots + \alpha_{n+1}c_\varepsilon(A_{n+1}))\big) = \frac{\mu_i}{\alpha_i}\cdot\Delta.$$

Similarly

$$y'_k - y_k = \frac{\lambda_{ki}}{\alpha_i}\big(c_\varepsilon(A'_i) - (\alpha_1 c_\varepsilon(A_1) + \cdots + \alpha_{n+1}c_\varepsilon(A_{n+1}))\big) = \frac{\lambda_{ki}}{\alpha_i}\cdot\Delta.$$

This simplifies calculation of z' and all y'_k. We iterate the above step until $\Delta = 0$. The final value of z, which we denote $z(\varepsilon)$, is the least z such that

$$\begin{cases} m(A) \leqslant c(\bar{O}_\varepsilon A) + z, \\ c(A) \leqslant m(\bar{O}_\varepsilon A) + z \end{cases}$$

for some $m \in \overline{P}X_0$ and all $A \underset{\mathrm{cl}}{\subset} X$.

Observe that $z(\varepsilon)$ is non-increasing with respect to ε, hence the distance between c and $\overline{P}X_0$ is the least ε such that $z(\varepsilon) \leqslant \varepsilon$. This distance is not greater than $z(0)$, therefore it is easy to bisect the segment $[0, z(0)]$ to find the distance and an approximating additive measure with arbitrary precision.

2 Concluding remarks

The proposed algorithm was implemented as a C program and tested on data sets with cardinality of X_0 up to 10.

However, each iteration of the presented algorithm requires previously calculated values of a capacity for all $2^{\text{cardinality of the space}}$ subsets, which is not appropriate even for $\geqslant 40$ points. Hence, to handle subspaces of greater cardinality, we need to cut memory and time requirements using the metric structure and the only reliable property of a capacity, i.e., its monotonicity. This requires deeper investigation combining both topological properties of non-additive measures, e.g., their dimensional characteristics, and computational aspects.

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Для простору неадитивних регулярних мір на метричному компакті з відстанню в стилі Прохорова показано, що задача наближення довільної міри адитивною мірою на фіксованому скінченному підпросторі зводиться до задачі лінійної оптимізації з параметрами, залежними від значень вихідної міри на скінченному числі множин.

Запропоновано алгоритм такого наближення, ефективніший порівняно з прямолінійним застосуванням симплекс-методу.

Ключові слова і фрази: метрика Прохорова, неадитивна міра, апроксимація, компактний метричний простір.