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PROPERTIES OF POWER SERIES OF ANALYTIC IN A BIDISC FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

We generalized some criteria of boundedness of **L**-index in joint variables for analytic in a bidisc functions, where $\mathbf{L}(z)=(l_1(z_1,z_2),l_2(z_1,z_2))$, $l_j:\mathbb{D}^2\to\mathbb{R}_+$ is a continuous function, $j\in\{1,2\}$, $\mathbb{D}^2$ is a bidisc $\{(z_1,z_2)\in\mathbb{C}^2: |z_1|<1,|z_2|<1\}$. We obtained propositions, which describe a behaviour of power series expansion on a skeleton of a bidisc. The power series expansion is estimated by a dominating homogeneous polynomial with a degree that does not exceed some number, depending only from radii of a bidisc. Replacing universal quantifier by existential quantifier for radii of a bidisc, we also proved sufficient conditions of boundedness of **L**-index in joint variables for analytic functions, which are weaker than necessary conditions.

Key words and phrases: analytic function, bidisc, bounded **L**-index in joint variables, maximum modulus, partial derivative, dominating polynomial, power series.

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1 Introduction

Recently, we introduced a concept of boundedness of **L**-index in joint variables for analytic in a bidisc functions [4]–[6]. There were obtained criteria which describes a local behaviour of partial derivatives, give estimate maximum modulus on a skeleton of bidisc and was proved an analog of Hayman's Theorem.

In a fact, inequality (1) in a definition of function of bounded **L**-index in joint variables (see below) contains coefficients of power series expansion at a point $z=(z_1,z_2)$. M. T. Bordulyak and M. M. Sheremeta [9] considered entire functions and obtained a proposition which describe a behavior of homogeneous polynomials with power series coefficients for functions of bounded **L**-index in joint variables in the case $\mathbf{L}(z)=(l_1(z_1),\ldots,l_n(z_n))$. Recently, we generalized [5] their result for entire functions and $\mathbf{L}(z)=(l_1(z),\ldots,l_n(z))$, where $z\in\mathbb{C}^n$. Replacing universal quantifier by existential quantifier, there was proved also new theorem which provides weaker sufficient conditions of boundedness of **L**-index in joint variables.

This leads to such a natural question: *Is there a counterpart of the mentioned Bordulyak–Sheremeta's criterion for functions that are analytic in an arbitrary polydisc domain?* Our answer to the question is affirmative. In particular, it is proved in Theorems 1 and 2 of this paper for a bidisc.

In this paper, we will prove a necessity of Bordulyak–Sheremeta's criterion for analytic in a bidisc functions and $\mathbf{L}(z)=(l_1(z_1,z_2),l_2(z_1,z_2))$. As sufficiency for analytic in $\mathbb{D}^2$ functions, we will deduce an analog of weaker sufficient conditions of boundedness of **L**-index in joint variables from [5].

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2 MAIN DEFINITIONS AND NOTATIONS

We consider two-dimensional complex space $\mathbb{C}^2$. This helps to distinguish main methods of investigation. We need some standard notations. Denote $\mathbb{R}_+ = [0, +\infty)$, $\mathbf{0} = (0,0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1,1) \in \mathbb{R}^2_+$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $z = (z_1, z_2) \in \mathbb{C}^2$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions

$$AB = (a_1b_1, a_2b_2),\ \ A/B = (a_1/b_1, a_2/b_2),\ b_1 \neq 0,\ b_2 \neq 0,\ \ A^B = a_1^{b_1}a_2^{b_2},\ b \in \mathbb{Z}^2_+,$$

and the notation $A < B$ means that $a_j < b_j$, $j \in \{1,2\}$; the relation $A \leq B$ is defined similarly. For $K = (k_1, k_2) \in \mathbb{Z}^2_+$ denote $\|K\| = k_1 + k_2$, $K! = k_1!k_2!$.

The bidisc $\{z \in \mathbb{C}^2 :\ |z_j - z_j^0| < r_j,\ j = 1,2\}$ is denoted by $\mathbb{D}^2(z^0, R)$, its skeleton $\{z \in \mathbb{C}^2 :\ |z_j - z_j^0| = r_j,\ j = 1,2\}$ is denoted by $\mathbb{T}^2(z^0, R)$, and the closed bidisc $\{z \in \mathbb{C}^2 :\ |z_j - z_j^0| \leq r_j,\ j = 1,2\}$ is denoted by $\mathbb{D}^2[z^0, R]$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} :\ |z| < 1\}$. For $p, q \in \mathbb{Z}_+$ and partial derivative of analytic in $\mathbb{D}^2$ function $F(z)$ we will use the notation

$$F^{(p,q)}(z) = F^{(p,q)}(z_1, z_2) := \frac{\partial^{p+q}F(z_1, z_2)}{\partial z_1^p \partial z_2^q}.$$

Let $\mathbf{L}(z) = (l_1(z), l_2(z))$, where $l_j(z) : \mathbb{D}^2 \to \mathbb{R}_+$ is a continuous function such that for all $z \in \mathbb{D}^2$: $l_j(z) > \beta/(1 - |z_j|)$, $j \in \{1,2\}$, where $\beta > 1$ is a some constant, $\boldsymbol{\beta} := (\beta, \beta)$. S.N. Strochyk, M.M. Sheremeta, V.O. Kushnir [14], [20] imposed a similar condition for a function $l : \mathbb{D} \to \mathbb{R}_+$ and $l : G \to \mathbb{R}_+$, where G is arbitrary domain in $\mathbb{C}$.

An analytic function $F\colon \mathbb{D}^2 \to \mathbb{C}$ is called a function of *bounded* $\mathbf{L}$*-index (in joint variables)*, if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z = (z_1, z_2) \in \mathbb{D}^2$ and for all $(p_1, p_2) \in \mathbb{Z}^2_+$

$$\frac{1}{p_1!p_2!}\frac{|F^{(p_1,p_2)}(z)|}{l_1^{p_1}(z)l_2^{p_2}(z)} \leq \max\left\{\frac{1}{k_1!k_2!}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} : 0 \leq k_1 + k_2 \leq n_0\right\}. \tag{1}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* $F(z)$ and is denoted by $N(F, \mathbf{L}, \mathbb{D}^2) = n_0$. This is an analog of definition of entire function of bounded $\mathbf{L}$-index or bounded index ($\mathbf{L} \equiv 1$) in joint variables in $\mathbb{C}^2$ (see [3], [9, 10], [16, 17, 18]) and a definition of analytic in a domain function of bounded index [12]. Note that a primary definition of entire in $\mathbb{C}$ function of bounded index was supposed by B. Lepson [15]. Other approach (so-called L-index in a direction) is considered in [7, 8].

By $Q(\mathbb{D}^2)$ we denote the class of functions $\mathbf{L}$, which satisfy the condition for all $r_j \in [0, \beta]$, $j \in \{1,2\}$

$$0 < \lambda_{1,j}(R) \leq \lambda_{2,j}(R) < \infty,$$

where

$$\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{D}^2} \inf\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$$

$$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{D}^2} \sup\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2\left[z^0, R/\mathbf{L}(z^0)\right]\right\}.$$

It is easy to prove that the function $L_1(z_1, z_2) = (\beta'/(1 - |z_1|), \beta'/(1 - |z_2|))$ belongs to $Q(\mathbb{D}^2)$, where $\beta' > \beta$. Other possible methods to construct these functions are considered in [1].

Let $z^0 \in \mathbb{D}^2$. We develop an analytic in $\mathbb{D}^2$ function $F(z)$ in the power series written in a diagonal form

$$F(z) = \sum_{k_1+k_2=0}^{\infty} p_{k_1+k_2}((z_1-z_1^0),(z_2-z_2^0)) = \sum_{k=0}^{\infty}\sum_{j_1+j_2=k} b_{j_1,j_2}(z_1-z_1^0)^{j_1}(z_2-z_2^0)^{j_2}, \qquad (2)$$

where p_k are homogeneous polynomials of degree k. The polynomial $p_{k_0}, k_0 \in \mathbb{Z}_+$, is called a dominating polynomial in the power series expansion (2) on $\mathbb{T}^2(z^0,R)$ if for every $z \in \mathbb{T}^2(z^0,R)$ the next inequality holds:

$$\left|\sum_{k_1+k_2\neq k^0} p_{k_1+k_2}((z_1-z_1^0),(z_2-z_2^0))\right| \le \frac{1}{2}\max\{|b_{j_1,j_2}|r_1^{j_1}r_2^{j_2}\colon j_1+j_2=k^0\},$$

where $b_{j_1,j_2} = \frac{F^{(j_1,j_2)}(z^0)}{j_1!j_2!}$.

3 Some property of power expansion of analytic in a bidisc function of bounded L-index in joint variables

Theorem 1. *Let $\beta > 1$, $\mathbf{L} \in Q(\mathbb{D}^2)$. If an analytic function F in $\mathbb{D}^2$ has bounded $\mathbf{L}$-index in joint variables then there exists $p \in \mathbb{Z}_+$ that for all $d \in (0;\beta]$ there exists $\eta(d) \in (0;d)$ such that for each $z^0 \in \mathbb{D}^2$ and some $r = r(d,z^0) \in (\eta(d),d)$, $k^0 = k^0(d,z^0) \le p$ the polynomial p_{k^0} is the dominating polynomial in the series* (2) *on $\mathbb{T}^2(z^0, \frac{R}{\mathbf{L}(z^0)})$ with $R=(r,r)$.*

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N = N(F,\mathbf{L},\mathbb{D}^2) < +\infty$ and n_0 be $\mathbf{L}$-index in joint variables at a point $z^0 \in \mathbb{D}^2$. Then for each $z^0 \in \mathbb{D}^2$ $n_0 \le N$. We put

$$a^*_{j_1,j_2} = \frac{|b_{j_1,j_2}|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} = \frac{|F^{(j_1,j_2)}(z^0)|}{j_1!j_2!l_1^{j_1}(z^0)l_2^{j_2}(z^0)},\ a_k = \max\{a^*_{j_1,j_2}\colon j_1+j_2=k\},$$
$$c = 2((N+1)^3 + 6(N+3)!).$$

Let $d \in (0;\beta]$ be an arbitrary number. We put $r_m = \frac{d}{(d+1)c^m}, m \in \mathbb{Z}_+$ and denote

$$\mu_m = \max\{a_k r_m^k\colon k \in \mathbb{Z}_+\},\ s_m = \min\{k\colon a_k r_m^k = \mu_m\}.$$

Since z^0 is a fixed point the inequality $a^*_{k_1,k_2} \le \max\{a^*_{j_1,j_2}\colon j_1+j_2 \le n_0\}$ is valid for all $(k_2,k_2) \in \mathbb{Z}_+^2$. Then $a_k \le a_{n_0}$ for all $k \in \mathbb{Z}_+$. Hence, for all $k > n_0$ in view of $r_0 < 1$ we have $a_k r_0^k < a_{n_0} r_0^{n_0}$. This implies $s_0 \le n_0$. Since $cr_m = r_{m-1}$, we obtain that for each $k > s_{m-1}$

$$a_{s_{m-1}} r_m^{s_{m-1}} = a_{s_{m-1}} r_{m-1}^{s_{m-1}} c^{-s_{m-1}} \ge a_k r_{m-1}^k c^{-s_{m-1}} = a_k r_m^k c^{k-s_{m-1}} \ge c a_k r_m^k. \qquad (3)$$

From (3) it follows that $s_m \le s_{m-1}$ for all $m \in \mathbb{N}$. Thus, we can rewrite

$$\mu_0 = \max\{a_k r_0^k\colon k \le n_0\},\ \mu_m = \max\{a_k r_m^k\colon k \le s_{m-1}\}.$$

We denote

$$\mu_0^* = \max\{a_k r_0^k\colon s_0 \ne k \le n_0\}, \quad \mu_m^* = \max\{a_k r_m^k\colon s_m \ne k \le s_{m-1}\},$$
$$s_0^* = \min\{k\colon k \ne s_0, a_k r_0^k = \mu_0^*\}, \quad s_m^* = \min\{k\colon k \ne s_m, a_k r_m^k = \mu_m^*\}, m \in \mathbb{N}$$

and we will show that there exists $m_0 \in \mathbb{Z}_+$ such, that

$$\frac{\mu^*_{m_0}}{\mu_{m_0}} \leq \frac{1}{c}. \tag{4}$$

Suppose that for all $m \in \mathbb{Z}_+$ the next inequality holds

$$\frac{\mu^*_m}{\mu_m} > \frac{1}{c}. \tag{5}$$

If $s^*_m < s_m$ ($s^*_m \neq s_m$ in view of definition) then we have

$$a_{s^*_m} r^{s^*_m}_{m+1} = \frac{a_{s^*_m} r^{s^*_m}_m}{c^{s^*_m}} = \frac{\mu^*_m}{c^{s^*_m}} > \frac{\mu_m}{c^{s^*_m+1}} = \frac{a_{s_m} r^{s_m}_m}{c^{s^*_m+1}} = \frac{a_{s_m} r^{s_m}_{m+1}}{c^{s^*_m+1-s_m}} \geq a_{s_m} r^{s_m}_{m+1},$$

and for all $k > s^*_m, k \neq s_m$, similarly,

$$a_{s^*_m} r^{s^*_m}_{m+1} = \frac{a_{s^*_m} r^{s^*_m}_m}{c^{s^*_m}} \geq \frac{a_k r^k_m}{c^{s^*_m}} \geq \frac{a_k r^k_m}{c^{k-1}} = \frac{c a_k r^k_m}{c^k} = c a_k r^k_{m+1},$$

i.e. $a_{s^*_m} r^{s^*_m}_{m+1} > a_k r^k_{m+1}$ for all $k > s^*_m$. Hence,

$$s_{m+1} \leq s^*_m \leq s_m - 1. \tag{6}$$

On the contrary, if $s_m < s^*_m \leq s_{m-1}$ then the equality $s_{m+1} = s_m$ may hold. But in this case the inequalities $s^*_{m+1} \leq s_m$ and $s^*_m \neq s_{m+1}$ imply that $s^*_{m+1} < s_{m+1}$, $s^*_{m+1} \neq s_{m+1}$. Instead of (6) we have the inequality $s_{m+2} \leq s^*_{m+1} \leq s_{m+1} - 1 = s_m - 1$. Hence, if for all $m \in \mathbb{Z}_+$ estimate (5) is true then for all $m \in \mathbb{Z}_+$ either inequality $s_{m+1} \leq s_m - 1$ or $s_{m+2} \leq s_m - 1$ holds, i.e. $s_{m+2} \leq s_m - 1$, because $s_{m+2} \leq s_{m+1}$. It implies that

$$s_m \leq s_{m-2} - 1 \leq \ldots \leq s_{m-2\lfloor \frac{m}{2} \rfloor} - \left\lfloor \frac{m}{2} \right\rfloor \leq s_0 - \left\lfloor \frac{m}{2} \right\rfloor \leq n_0 - \left\lfloor \frac{m}{2} \right\rfloor \leq N - \left\lfloor \frac{m}{2} \right\rfloor,$$

i.e. $s_m < 0$ if only $m > 2N+1$, which is impossible. Therefore, there exists $m_0 \leq 2N+1$ such that (4) holds. We put $r = r_{m_0}$, $\eta(d) = \frac{d}{(d+1)c^{2(N+1)}}$, $p = N$ and $k_0 = s_{m_0}$. Then for all $j_1 + j_2 \neq k_0 = s_{m_0}$ on $\mathbb{T}^2(z^0, \frac{r}{\mathbf{L}(z^0)})$ in view (4) we have

$$|b_{j_1,j_2}||z_1 - z^0_1|^{j_1}|z_2 - z^0_2|^{j_2} = a^*_{j_1,j_2} r^{j_1+j_2} \leq a_{j_1+j_2} r^{j_1+j_2} \leq \mu^*_{m_0} \leq \frac{1}{c}\mu_{m_0} \leq \frac{1}{c} a_{s_{m_0}} r^{s_{m_0}}_{m_0} = \frac{1}{c} a_{k_0} r^{k_0}.$$

Thus, on $\mathbb{T}^2(z^0, \frac{r}{\mathbf{L}(z^0)})$ we obtain

$$\begin{aligned} \left| \sum_{j_1+j_2 \neq k_0} b_{j_1,j_2}(z_1 - z^0_1)^{j_1}(z_2 - z^0_2)^{j_2} \right| &\leq \sum_{j_1+j_2 \neq k_0} a^*_{j_1,j_2} r^{j_1+j_2} \leq \sum_{k=0, k\neq k_0}^{\infty} a_k (k+1)^2 r^k \\ &= \sum_{k=0,\ k\neq s_{m_0}}^{s_{m_0-1}} a_k (k+1)^2 r^k + \sum_{k=s_{m_0-1}+1}^{\infty} a_k (k+1)^2 r^k. \end{aligned} \tag{7}$$

We will estimate two sums in (7). From (4) it follows that $\mu^*_{m_0} \leq \frac{1}{c}\mu_{m_0}$ or

$$\max\{a_k r^k_{m_0} : k \neq s_{m_0},\ k \leq s_{m_0-1}\} \leq \frac{1}{c} \max\{a_k r^k_{m_0} : k \neq s_{m_0}, k \leq s_{m_0-1}\},$$

i. e. $a_k r^k \le \frac{1}{c} a_{k_0} r^{k_0}$. Then

$$\sum_{k=0,\ k\ne s_{m_0}}^{s_{m_0-1}} a_k (k+1)^2 r^k \le \frac{a_{k_0} r^{k_0}}{c} \sum_{k=0}^{N} (k+1)^2 \le \frac{a_{k_0} r^{k_0}}{c} (N+1)^3. \tag{8}$$

For each k the inequality $a_k r^k_{m_0-1} \le \mu_{m_0-1}$ holds and, hence,

$$a_k r^k_{m_0} = \frac{a_k r^k_{m_0-1}}{c^k} \le \frac{\mu_{m_0-1}}{c^k}. \tag{9}$$

Using (9) and (4) we deduce

$$\begin{aligned}
\sum_{k=s_{m_0-1}+1}^{\infty} a_k (k+1)^2 r^k &\le \mu_{m_0-1} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)^2 \frac{1}{c^k} = a_{s_{m_0-1}} r^{s_{m_0-1}}_{m_0-1} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)^2 \frac{1}{c^k} \\
&= a_{s_{m_0-1}} \frac{r^{s_{m_0-1}}_{m_0-1}}{c^{s_{m_0-1}}} c^{s_{m_0-1}} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)^2 \frac{1}{c^k} \le a_{s_{m_0-1}} r^{s_{m_0-1}}_{m_0} c^{s_{m_0-1}} \sum_{k=s_{m_0-1}+1}^{\infty} (k+1)(k+2) \frac{1}{c^k} \\
&\le \frac{a_{s_{m_0}} r^{s_{m_0}}}{c} c^{s_{m_0-1}} \left(\sum_{k=s_{m_0-1}+1}^{\infty} x^{k+2} \right)^{(2)} \Bigg|_{x=\frac{1}{c}} = \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \left(\frac{x^{s_{m_0-1}+3}}{1-x} \right)^{(2)} \Bigg|_{x=\frac{1}{c}} \\
&= \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} \Bigg(\frac{(s_{m_0-1}+3)(s_{m_0-1}+2) x^{s_{m_0-1}+1}}{1-x} + \frac{2(s_{m_0-1}+3) x^{s_{m_0-1}+2}}{(1-x)^2} \\
&\quad + \frac{2x^{s_{m_0-1}+3}}{(1-x)^3} \Bigg) \Bigg|_{x=\frac{1}{c}} \le \frac{a_{k_0} r^{k_0}}{c} c^{s_{m_0-1}} 2(s_{m_0-1}+3)(s_{m_0-1}+2) \sum_{j=0}^{2} \frac{x^{s_{m_0-1}+1+j}}{(1-x)^{1+j}} \Bigg|_{x=\frac{1}{c}} \\
&\le \frac{a_{k_0} r^{k_0}}{c} 2(N+3)! \sum_{j=0}^{2} \frac{1}{(c-1)^{1+j}} \le \frac{a_{k_0} r^{k_0}}{c} 6(N+3)!,
\end{aligned} \tag{10}$$

because $c \ge 2$. Hence, from (8) and (10) we obtain

$$\begin{aligned}
\left| \sum_{j_1+j_2\ne k_0} b_{j_1,j_2} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} \right| &\le \frac{a_{k_0} r^{k_0}}{c} (N+1)^3 + 6 \frac{a_{k_0} r^{k_0}}{c} (N+3)! \\
&= \frac{a_{k_0} r^{k_0}}{c} ((N+1)^3 + 6(N+3)!) = \frac{1}{2} a_{k_0} r^{k_0}.
\end{aligned}$$

Therefore, the polynomial p_{k^0} is the dominating polynomial in the series (2) on the skeleton $\mathbb{T}^2\big(z^0, \frac{R}{\mathbf{L}(z^0)}\big)$. $\square$

Theorem 2. *Let $\beta > 1$, $\mathbf{L} \in Q(\mathbb{D}^2)$. If there exist $p \in \mathbb{Z}_+$, $d \in (0;1]$, $\eta \in (0;d)$ such that for each $z^0 \in \mathbb{D}^2$ and some $R = (r_1, r_2)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1,2\}$, and certain $k^0 = k^0(d, z^0) \le p$ the polynomial p_{k^0} is the dominating polynomial in the series* (2) *on $\mathbb{T}^2(z^0, R/\mathbf{L}(z^0))$ then the analytic in $\mathbb{D}^2$ function F has bounded $\mathbf{L}$-index in joint variables.*

Proof. Suppose that there exist $p \in \mathbb{Z}_+$, $d \le 1$ and $\eta \in (0,d)$ such that for each $z^0 \in \mathbb{D}^2$ and some $R = (r_1, r_2)$ with $r_j = r_j(d, z^0) \in (\eta, d)$, $j \in \{1,2\}$, and $k_0 = k_0(d, z^0) \le p$ the polynomial p_{k^0} is the dominating polynomial in the series (2) on $\mathbb{T}^2(z^0, R/\mathbf{L}(z^0))$. Let us to denote $r_0 = max\{r_1, r_2\}$. Then

$$\left| \sum_{j_1+j_2\ne k_0} b_{j_1,j_2} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} \right| = \left| F(z) - \sum_{j_1+j_2=k_0} b_{j_1,j_2} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} \right| \le \frac{a_{k_0} r_0^{k_0}}{2}. \tag{11}$$

Using (11) and Cauchy's inequality we have:

$$|b_{j_1,j_2}(z_1-z_1^0)^{j_1}(z_2-z_2^0)^{j_2}| = a^*_{j_1,j_2} r_1^{j_1} r_2^{j_2} \le \frac{a_{k_0} r_0^{k_0}}{2}$$

for all $j_1, j_2 \in \mathbb{Z}_+$, i.e. for all $k_1+k_2=k\neq k_0$

$$a_k r_1^{k_1} r_2^{k_2} \le \frac{a_{k_0} r_0^{k_0}}{2}. \tag{12}$$

Suppose that F is not a function of bounded $\mathbf{L}$-index in joint variables.

Let $\mathbf{L}\in Q(\mathbb{D}^2)$. It is known [6] that an analytic function F in $\mathbb{D}^2$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p\in\mathbb{Z}_+$ and $c\in\mathbb{R}_+$ such that for each $z=(z_1,z_2)\in\mathbb{D}^2$ the next inequality holds

$$\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)} : j_1+j_2=p+1\right\} \le c\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} : k_1+k_2\le p\right\}.$$

This statement and its generalizations [19, 13, 9, 2, 6] are analogs of known Hayman's Theorem [11] in theory of functions of bounded index. Then by the Hayman Theorem for all $p_1\in\mathbb{Z}_+$ and $c\ge 1$ there exists $z^0\in\mathbb{D}^2$ such that the next inequality holds:

$$\max\left\{\frac{|F^{(j_1,j_2)}(z^0)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} : j_1+j_2=p_1+1\right\} > c\max\left\{\frac{|F^{(k_1,k_2)}(z^0)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)} : k_1+k_2\le p_1\right\}.$$

We put $p_1=p$ and $c=\left(\frac{(p+1)!}{\eta^{p+1}}\right)^2$. Then for this $z^0(p_1,c)$ we obtain:

$$\max\left\{\frac{|F^{(j_1,j_2)}(z^0)|}{j_1!j_2!l_1^{j_1}(z^0)l_2^{j_2}(z^0)} : j_1+j_2=p+1\right\} > \frac{1}{\eta^{p+1}}\max\left\{\frac{|F^{(k_1,k_2)}(z^0)|}{k_1!k_2!l_1^{k_1}(z^0)l_2^{k_2}(z^0)} : k_1+k_2\le p\right\},$$

i.e. $a_{p+1} > \frac{a_{k_0}}{\eta^{p+1}}$ and, hence, $a_{p+1}r_0^{p+1} > \frac{a_{k_0}r_0^{p+1}}{\eta^{p+1}} \ge a_{k_0}r_0^{k_0}$. This is a contradiction with (12). Therefore, F is of bounded $\mathbf{L}$-index in joint variables. □

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Бандура А. І, Петречко Н. В. *Властивості степеневих рядів аналітичних у бікрузі функцій обмеженого* **L**-*індексу за сукупністю змінних* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 6–12.

Нами узагальнено деякі критерії обмеженості **L**-індексу за сукупністю змінних для аналітичних у бікрузі функцій, де $\mathbf{L}(z) = (l_1(z_1, z_2), l_2(z_1, z_2))$, $l_j : \mathbb{D}^2 \to \mathbb{R}_+$ — неперервна функція, $j \in \{1, 2\}$, $\mathbb{D}^2$ — бікруг $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. Отримані твердження описують поводження розвинення у степеневий ряд на кістяку бікруга. При цьому сума відповідного степеневого ряду оцінена через домінувальний однорідний многочлен, степінь якого не перевищує деякого числа, залежного тільки від радіусів бікруга. Замінюючи квантор загальності на квантор існування для значень радіусів бікруга, ми також доводимо достатні умови обмеженості **L**-індексу за сукупністю змінних для аналичних функцій, які слабші за необхідні умови.

Ключові слова і фрази: аналітична функція, бікруг, обмежений **L**-індекс за сукупністю змінних, максимум модуля, частинна похідна, головний многочлен, степеневий ряд.