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SUPEREXTENSIONS OF THREE-ELEMENT SEMIGROUPS

A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any set $B \supset A$ belongs to $\mathcal{A}$. An upfamily $\mathcal{L}$ of subsets of X is said to be *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. A linked upfamily $\mathcal{M}$ of subsets of X is *maximal linked* if $\mathcal{M}$ coincides with each linked upfamily $\mathcal{L}$ on X that contains $\mathcal{M}$. The *superextension* $\lambda(X)$ consists of all maximal linked upfamilies on X. Any associative binary operation $* : X \times X \to X$ can be extended to an associative binary operation $\circ : \lambda(X) \times \lambda(X) \to \lambda(X)$ by the formula $\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$ for maximal linked upfamilies $\mathcal{L}, \mathcal{M} \in \lambda(X)$. In the paper we describe superextensions of all three-element semigroups up to isomorphism.

Key words and phrases: semigroup, maximal linked upfamily, superextension, projective retraction, commutative.

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INTRODUCTION

In this paper we investigate the algebraic structure of the superextension $\lambda(S)$ of a three-element semigroup S. The thorough study of various extensions of semigroups was started in [11] and continued in [1–7, 12–16]. The largest among these extensions is the semigroup $v(S)$ of all upfamilies on S. A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any subset $B \supset A$ belongs to $\mathcal{A}$. Each family $\mathcal{B}$ of non-empty subsets of X generates the upfamily $\langle B \subset X : B \in \mathcal{B} \rangle = \{A \subset X : \exists B \in \mathcal{B} (B \subset A)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called the Stone-Čech compactification of X, see [17], [20]. An ultrafilter $\{x\}$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle \{x\} \rangle$ generated by the singleton $\{x\}$, and hence we consider $X \subset \beta(X) \subset v(X)$. It was shown in [11] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $\circ : v(S) \times v(S) \to v(S)$ by the formula

$$\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$$

for upfamilies $\mathcal{L}, \mathcal{M} \in v(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $v(S)$.

The semigroup $v(S)$ contains many other important extensions of S. In particular, it contains the semigroup $\lambda(S)$ of maximal linked upfamilies. The space $\lambda(S)$ is well-known in

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General and Categorial Topology as the *superextension* of S, see [19]- [21]. An upfamily $\mathcal{L}$ of subsets of S is *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. The family of all linked upfamilies on S is denoted by $N_2(S)$. It is a subsemigroup of $v(S)$. The superextension $\lambda(S)$ consists of all maximal elements of $N_2(S)$, see [10], [11].

Each map $f : X \to Y$ induces the map

$$\lambda f : \lambda(X) \to \lambda(Y), \quad \lambda f : \mathcal{M} \mapsto \langle f(M) \subset Y : M \in \mathcal{M} \rangle \text{ (see [10]).}$$

A non-empty subset I of a semigroup S is called *an ideal* if $IS \cup SI \subset I$. A semigroup S is called *simple* if S is the unique ideal of S. An element z of a semigroup S is called a *zero* (resp. a *left zero*, a *right zero*) in S if $az = za = z$ (resp. $za = z$, $az = z$) for any $a \in S$. A semigroup S is said to be a *left (right) zeros semigroup* if $ab = a$ ($ab = b$) for any $a, b \in S$. A semigroup S is called a *null semigroup* if there exists an element $c \in S$ such that $xy = c$ for any $x, y \in S$. By O_n, LO_n and RO_n we denote a null semigroup, a left zero semigroups and a right zero semigroup of order n respectively. Following the algebraic tradition, we denote by C_n the cyclic group of order n.

Let S be a semigroup and $e \notin S$. The binary operation defined on S can be extended to $S \cup \{e\}$ putting $es = se = s$ for all $s \in S \cup \{e\}$. The notation S^{+1} denotes a monoid $S \cup \{e\}$ obtained from S by adjoining an extra identity e (regardless of whether S is or is not a monoid). Analogous to the above construction, for every semigroup S one can define S^{+0}, a semigroup with attached an extra zero to S.

Let us recall that a *semilattice* is a commutative idempotent semigroup. Idempotent semigroups are called *bands*. So, in a band each element x is an *idempotent*, which means that $xx = x$. By L_n we denote the linear semilattice $\{0, 1, \ldots, n\}$ of order n, endowed with the operation of minimum. A semigroup S is called *Clifford* if it is a union of groups.

A semigroup $\langle a \rangle = \{a^n\}_{n \in \mathbb{N}}$ generated by a single element a is called *monogenic* or *cyclic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$. A finite monogenic semigroup $S = \langle a \rangle$ also has very simple structure (see [8], [18]). There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \ldots, a^{m+r-1}\}$ and $m + r - 1 = |S|$;
- for any $i, j \in \omega$ the equality $a^{r+i} = a^{r+j}$ holds if and only if $i \equiv j \mod m$;
- $C_m = \{a^r, a^{r+1}, \ldots, a^{m+r-1}\}$ is a cyclic and maximal subgroup of S with the neutral element $e = a^n \in C_m$, where m divides n.

We denote by $C_{r,m}$ a finite monogenic semigroup of index r and period m.

An *isomorphism* between S and S' is one-to-one function $\varphi : S \to S'$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. If there exist an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted $S \cong S'$. An *antiisomorphism* between S and S' is one-to-one function $\varphi : S \to S'$ such that $\varphi(xy) = \varphi(y)\varphi(x)$ for all $x, y \in S$. If there exist an antiisomorphism between S and S', then S and S' are said to be *antiisomorphic*, denoted $S \cong_a S'$. If $(S, *)$ is a semigroup, then $(S, \circ)$, where $x \circ y = y * x$, is a semigroup as well. The semigroups $(S, *)$ and $(S, \circ)$ are called *dual*. It is easy to see that dual semigroups are antiisomorphic.

There are exactly five pairwise non-isomorphic semigroups having two elements: C_2, L_2, O_2, LO_2, RO_2. The superextension $\lambda(S)$ of two-element semigroups S consists of two principal ultrafilters and therefore $\lambda(S) \cong S$.

In this paper we concentrate on describing the structure of the superextensions $\lambda(S)$ of three-element semigroups S. Among 19683 different operations on a three-element set $S = \{a, b, c\}$ there are exactly 113 operations which are associative, see [9]. In other words, there exist exactly 113 three-element semigroups, and many of these are isomorphic so that there are essentially only 24 pairwise non-isomorphic semigroups of order 3.

1 Projective retractions and superextensions

In this section we will apply some properties of proretract semigroups to study the structure of the superextensions of semigroups.

A subset R of a set X is called a *retract* if there exists a *retraction* of X onto R, that is a map of X onto R which leaves each element of R fixed. A retraction $r : S \to T$ of a semigroup S onto a subsemigroup T of S is called a *projective retraction* if $xy = r(x)r(y)$ for any $x, y \in S$. A semigroup S is said to be a *proretract-semigroup* provided that there exists a projective retraction $r : S \to T$ of S onto some proper subsemigroup T of S. In this case T will be called a *projective retract* of S under a projective retraction r, and S will be called a *proretract extension* of T under a projective retraction r. If $r : S \to T$ is a projective retraction of a semigroup S onto a subsemigroup T of S, then r is a homomorphism and T is an ideal of S.

If a semigroup S is simple, then it is not a proretract-semigroup. In particular, groups, left zero and right zero semigroups are not proretract-semigroups.

Proposition 1. *A finite monogenic semigroup $C_{r,m}$ of index r and period m is a proretract-semigroup if and only if $r = 2$.*

Proof. Let $C_{r,m} = \{a, a^2, \dots, a^r, \dots, a^{r+m-1} \mid a^{r+m} = a^m\}$. If $r = 1$, then $C_{r,m}$ is simple and thus it is not a proretract-semigroup.

Let $r = 2$. Consider the map $\varphi : C_{2,m} \to C_m = \{a^2, \dots, a^{m+1}\}$, $\varphi(s) = es$, where e is the identity of the maximal subgroup C_m of $C_{2,m}$. Then $st \in C_m$ and $st = eset = \varphi(s)\varphi(t)$ for any $s, t \in C_{2,m}$. Consequently, φ is a projective retraction.

Let $r > 2$. Suppose that $\varphi : C_{r,m} \to I$ is a projective retraction onto some proper ideal I of S. Then $aa = \varphi(a)\varphi(a)$. In monogenic semigroups of index $r > 2$ the equality $a^2 = \varphi(a)^2$ is possible only in the case $\varphi(a) = a$. Since φ is a homomorphism, then φ leaves each element of $C_{r,m}$ fixed. Therefore, $I = C_{r,m}$, a contradiction. □

Let us note that for a subsemigroup T of a semigroup S the homomorphism $i : \lambda(T) \to \lambda(S)$, $i : \mathcal{A} \to \langle \mathcal{A} \rangle_S$ is injective, and thus we can identify the semigroup $\lambda(T)$ with the subsemigroup $i(\lambda(T)) \subset \lambda(S)$. Therefore, for each family $\mathcal{B}$ of non-empty subsets of T we identify the upfamilies

$$\langle \mathcal{B} \rangle_T = \{A \in T \mid \exists B \in \mathcal{B} (B \subset A)\} \in \lambda(T) \quad \text{and} \quad \langle \mathcal{B} \rangle_S = \{A \in S \mid \exists B \in \mathcal{B} (B \subset A)\} \in \lambda(S).$$

In the following proposition we show that proretract-semigroup property is preserved by superextensions.

Proposition 2. *If $r : S \to T$ is a projective retraction of a semigroup S onto a subsemigroup T of S, then $\lambda r : \lambda(S) \to \lambda(T)$ is a projective retraction of the superextension $\lambda(S)$ onto $\lambda(T)$.*

Proof. Let $\mathcal{L}, \mathcal{M} \in \lambda(S)$. Then

$$\begin{aligned}\lambda r(\mathcal{L}) \circ \lambda r(\mathcal{M}) &= \Big\langle \bigcup_{a \in r(L)} a * r(M)_a : r(L) \in \lambda r(\mathcal{L}),\ \ \{r(M)_a\}_{a \in r(L)} \subset \lambda r(\mathcal{M}) \Big\rangle \\ &= \Big\langle \bigcup_{a \in L} r(a) * r(M)_a : L \in \mathcal{L},\ \ \{r(M)_a\}_{a \in L} \subset \lambda r(\mathcal{M}) \Big\rangle \\ &= \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle = \mathcal{L} \circ \mathcal{M}.\end{aligned}$$

□

Corollary 1. *If S is a proretract-semigroup, then $\lambda(S)$ is a proretract-semigroup as well.*

In the next section we show that there exists a semigroup S that is not a proretract-semigroup, but the superextension $\lambda(S)$ is a proretract-semigroup.

Theorem 1. *If S is a null semigroup, then $\lambda(S)$ is a null semigroup as well.*

Proof. Let S be a null semigroup. So there exists $c \in S$ such that $xy = c$ for all $x, y \in S$. Then the map $r : S \to \{c\}$, $r(s) = c$ for any $s \in S$, is a projective retraction. According to Proposition 2 the map $\lambda r : \lambda(S) \to \lambda\{c\} = \{\langle\{c\}\rangle\}$ is a projective retraction as well. Therefore,

$$\mathcal{L} \circ \mathcal{M} = \lambda r(\mathcal{L}) \circ \lambda r(\mathcal{M}) = \langle\{c\}\rangle \circ \langle\{c\}\rangle = \langle\{c\}\rangle$$

for any $\mathcal{L}, \mathcal{M} \in \lambda(S)$. Consequently $\lambda(S)$ is a null semigroup. □

A semigroup S is said to be an *almost null semigroup* if there exist the distinct elements $a, c \in S$ such that $aa = a$ and $xy = c$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$.

Theorem 2. *If S is an almost null semigroup, then $\lambda(S)$ is an almost null semigroup as well.*

Proof. Let S be an almost null semigroup, so there exist the elements $a, c \in S$, $c \neq a$, such that $aa = a$ and $xy = c$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$. Then the map $r : S \to \{a, c\}$, $r(a) = a$ and $r(s) = c$ for any $s \neq a$, is a projective retraction. According to Proposition 2 the map $\lambda r : \lambda(S) \to \lambda\{a, c\}$ is a projective retraction as well. It is easy to see that the semigroup $\lambda\{a, c\} = \{\langle\{a\}\rangle, \langle\{c\}\rangle\} \cong \{a, c\}$ is isomorphic to the semilattice $L_2 = \{0, 1\}$ with operation of minimum.

It is obvious that $\langle\{a\}\rangle \circ \langle\{a\}\rangle = \langle\{a\}\rangle$. If $\mathcal{A} \neq \langle\{a\}\rangle$, then there exists $A \in \mathcal{A}$ such that $a \notin A$ and therefore $r(A) = c$. This implies that $\lambda r(\mathcal{A}) = \{\langle\{c\}\rangle\}$. If $(\mathcal{L}, \mathcal{M}) \in \lambda(S) \times \lambda(S) \setminus \{(\langle\{a\}\rangle, \langle\{a\}\rangle)\}$, then $\lambda r(\mathcal{L}) = \langle\{c\}\rangle$ or $\lambda r(\mathcal{M}) = \langle\{c\}\rangle$. Therefore, $\mathcal{L} \circ \mathcal{M} = \lambda r(\mathcal{L}) \circ \lambda r(\mathcal{M}) = \langle\{c\}\rangle$. Consequently, $\lambda(S)$ is an almost null semigroup. □

Theorem 3. *If S is a left (right) zero semigroup, then $\lambda(S)$ is a left (right) zero semigroup as well.*

Proof. Let S be a left zero semigroup. Then

$$\mathcal{L} \circ \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle = \Big\langle \bigcup_{a \in L} \{a\} : L \in \mathcal{L} \Big\rangle = \mathcal{L}$$

for any $\mathcal{L}, \mathcal{M} \in \lambda(S)$. Thus $\lambda(S)$ is a left zero semigroup as well.

For a right zero semigroup the proof is similar. □

2 SUPEREXTENSIONS OF COMMUTATIVE SEMIGROUPS OF ORDER 3

In this section we describe the structure of superextensions of commutative three-element semigroups. Among 24 pairwise non-isomorphic semigroups of order 3 there are 12 commutative semigroups.

For a semigroup $S = \{a, b, c\}$ the semigroup $\lambda(S)$ contains the three principal ultrafilters $\langle\{a\}\rangle, \langle\{b\}\rangle, \langle\{c\}\rangle$ and the maximal linked upfamily $\triangle = \langle\{a, b\}, \{a, c\}, \{b, c\}\rangle$. Since semigroups S and $\{\langle\{a\}\rangle, \langle\{b\}\rangle, \langle\{c\}\rangle\}$ are isomorphic, then we can assume that $\lambda(S) = S \cup \{\triangle\}$.

In the sequel we will describe the structure of superextensions of three-element semigroups $S = \{a, b, c\}$ defined by Cayley tables using the formula

$$\mathcal{L} \circ \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle$$

of product of maximal linked upfamilies $\mathcal{L}, \mathcal{M} \in \lambda(S)$.

The superextension $\lambda(C_3)$ (described by the following Cayley table) of the cyclic group C_3 is isomorphic to $(C_3)^{+0}$ and therefore $\lambda(C_3)$ is a commutative Clifford semigroup. The thorough study of superextensions of groups was started in [7] and continued in [1–3].

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	b	c	a	$\triangle$
c	c	a	b	$\triangle$
$\triangle$	$\triangle$	$\triangle$	$\triangle$	$\triangle$

The superextensions of monogenic semigroups were studied in [13]. The cyclic semigroup $C_{2,2}$ is a proretract extension of cyclic subgroup $\{b, c\} \cong C_2$ under retraction $\varphi : \{a, b, c\} \to \{b, c\}$ with $\varphi(a) = c$. The superextension $\lambda(C_{2,2})$ is also a proretract extension of $\lambda\{b, c\} \cong \{b, c\}$ according to Proposition 2. The monogenic semigroup $C_{3,1}$ is not a proretract-semigroup by Proposition 1, but its superextension $\lambda(C_{3,1})$ is a proretract extension of $C_{3,1}$ under retraction $r : \lambda(C_{3,1}) \to C_{3,1}$ with $r(\triangle) = c$, and, therefore, $\lambda(C_{3,1})$ is a proretract-semigroup. Here are the Cayley tables of $\lambda(C_{2,2})$ and $\lambda(C_{3,1})$ respectively:

$\cdot$	a	b	c	$\triangle$
a	b	c	b	b
b	c	b	c	c
c	b	c	b	b
$\triangle$	b	c	b	b

$\cdot$	a	b	c	$\triangle$
a	b	c	c	c
b	c	c	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

The following Cayley tables for the semigroups $\lambda((C_2)^{+0})$ and $\lambda((C_2)^{+1})$, where $C_2 \cong \{a, b\}$, imply that

$$\lambda((C_2)^{+0}) \cong \{a, b, \triangle\}^{+0} \cong ((C_2)^{+0})^{+0}$$

and

$$\lambda((C_2)^{+1}) \cong \{a, b, \triangle\}^{+1} \cong ((C_2)^{+1})^{+1} :$$

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	b	a	c	$\triangle$
c	c	c	c	c
$\triangle$	$\triangle$	$\triangle$	c	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	a	a
b	b	a	b	b
c	a	b	c	$\triangle$
$\triangle$	a	b	$\triangle$	$\triangle$

The superextensions of a null semigroup and an almost null semigroup are a null semigroup and an almost null semigroup as well according to Theorems 1 and 2:

$\cdot$	a	b	c	$\triangle$
a	c	c	c	c
b	c	c	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

$\cdot$	a	b	c	$\triangle$
a	a	c	c	c
b	c	c	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

The following Cayley tables for the semigroups $\lambda((O_2)^{+0})$ and $\lambda((O_2)^{+1})$ imply that

$$\lambda((O_2)^{+0}) \cong \{a,b,\triangle\}^{+0} \cong (O_3)^{+0} \qquad \text{and} \qquad \lambda((O_2)^{+1}) \cong \{a,b,\triangle\}^{+1} \cong (O_3)^{+1}.$$

The semigroups $(O_2)^{+0}$ and $\lambda((O_2)^{+0})$ are proretract extensions of the subsemigroup $\{b,c\} \cong L_2$.

$\cdot$	a	b	c	$\triangle$
a	b	b	c	b
b	b	b	c	b
c	c	c	c	c
$\triangle$	b	b	c	b

$\cdot$	a	b	c	$\triangle$
a	b	b	a	b
b	b	b	b	b
c	a	b	c	$\triangle$
$\triangle$	b	b	$\triangle$	b

The superextensions of semilattices were studied in [4]. The following Cayley tables imply that $\lambda(L_3) \cong L_4$ is a linear semilattice, but the superextension of the non-linear semilattice is its proretract extension and it is not even a Clifford semigroup:

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	b	b	c	b
c	c	c	c	c
$\triangle$	$\triangle$	b	c	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	c	c	c
b	c	b	c	c
c	c	c	c	c
$\triangle$	c	c	c	c

The structure of the superextension of the last commutative semigroup is shown in the following table. This semigroup and its superextension are proretract extensions of the subgroup $\{a,c\} \cong C_2$.

$\cdot$	a	b	c	$\triangle$
a	c	a	a	a
b	a	c	c	c
c	a	c	c	c
$\triangle$	a	c	c	c

3 SUPEREXTENSIONS OF NON-COMMUTATIVE SEMIGROUPS OF ORDER 3

There are 12 pairwise non-isomorphic non-commutative three-element semigroups. Non-commutative semigroups are divided into the pairs of dual semigroups that are antiisomorphic.

The superextension of a left (right) zero semigroup is a left (right) zero semigroup as well according to Theorem 3. Therefore $\lambda(LO_3) \cong LO_4$ and $\lambda(RO_3) \cong RO_4$.

$\cdot$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
$\triangle$	$\triangle$	$\triangle$	$\triangle$	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	a	b	c	$\triangle$
c	a	b	c	$\triangle$
$\triangle$	a	b	c	$\triangle$

The following Cayley tables for the semigroups $\lambda((LO_2)^{+0})$ and $\lambda((RO_2)^{+0})$ imply that

$$\lambda((LO_2)^{+0}) \cong \{a, b, \triangle\}^{+0} \cong (LO_3)^{+0}$$

and

$$\lambda((RO_2)^{+0}) \cong \{a, b, \triangle\}^{+0} \cong (RO_3)^{+0} :$$

$\cdot$	a	b	c	$\triangle$
a	a	a	c	a
b	b	b	c	b
c	c	c	c	c
$\triangle$	$\triangle$	$\triangle$	c	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	c	$\triangle$
b	a	b	c	$\triangle$
c	c	c	c	c
$\triangle$	a	b	c	$\triangle$

The following Cayley tables for the semigroups $\lambda((LO_2)^{+1})$ and $\lambda((RO_2)^{+1})$ imply that

$$\lambda((LO_2)^{+1}) \cong \{a, b, \triangle\}^{+1} \cong (\{a, b\}^{+1})^{+1} \cong ((LO_2)^{+1})^{+1}$$

and

$$\lambda((RO_2)^{+1}) \cong \{a, b, \triangle\}^{+1} \cong (\{a, b\}^{+1})^{+1} \cong ((RO_2)^{+1})^{+1} :$$

$\cdot$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	a	b	c	$\triangle$
$\triangle$	a	b	$\triangle$	$\triangle$

$\cdot$	a	b	c	$\triangle$
a	a	b	a	a
b	a	b	b	b
c	a	b	c	$\triangle$
$\triangle$	a	b	$\triangle$	$\triangle$

The following three-element semigroups and its superextensions are proretract extensions of its subsemigroups, which are isomorphic to LO_2 and RO_2 respectively:

$\cdot$	a	b	c	$\triangle$
a	c	c	c	c
b	b	b	b	b
c	c	c	c	c
$\triangle$	c	c	c	c

$\cdot$	a	b	c	$\triangle$
a	c	b	c	c
b	c	b	c	c
c	c	b	c	c
$\triangle$	c	b	c	c

Other two pairs of non-Clifford non-commutative dual superextensions of three-element semigroups are given by the following Cayley tables:

$\cdot$	a	b	c	$\triangle$
a	c	c	c	c
b	a	b	c	$\triangle$
c	c	c	c	c
$\triangle$	c	c	c	c

$\cdot$	a	b	c	$\triangle$
a	c	a	c	c
b	c	b	c	c
c	c	c	c	c
$\triangle$	c	$\triangle$	c	c

$\cdot$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	a	a	c	a
$\triangle$	a	a	$\triangle$	a

$\cdot$	a	b	c	$\triangle$
a	a	b	a	a
b	a	b	a	a
c	a	b	c	$\triangle$
$\triangle$	a	b	a	a

The last two three-element semigroups are the examples of non-commutative bands whose superextensions are not Clifford semigroups.

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Гаврилків В.М. *Суперрозширення трьохелементних напівгруп* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 28–36.

Сім'я $\mathcal{A}$ непорожніх підмножин множини X називається *монотонною*, якщо для кожної множини $A \in \mathcal{A}$ довільна множина $B \supset A$ належить $\mathcal{A}$. Монотонна сім'я $\mathcal{L}$ підмножин множини X називається *зчепленою*, якщо $A \cap B \neq \varnothing$ для всіх $A, B \in \mathcal{L}$. Зчеплена монотонна сім'я $\mathcal{M}$ підмножин множини X є *максимальною зчепленою*, якщо $\mathcal{M}$ збігається з кожною зчепленою монотонною сім'єю $\mathcal{L}$ на X, яка містить $\mathcal{M}$. *Суперрозширення* $\lambda(X)$ складається з усіх максимальних зчеплених монотонних сімей на X. Кожна асоціативна бінарна операція $* : X \times X \to X$ продовжується до асоціативної бінарної операції $\circ : \lambda(X) \times \lambda(X) \to \lambda(X)$ за формулою $\mathcal{L} \circ \mathcal{M} = \left\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \{M_a\}_{a \in L} \subset \mathcal{M} \right\rangle$ для максимальних зчеплених монотонних сімей $\mathcal{L}, \mathcal{M} \in \lambda(X)$. У цій статті описуються суперрозширення всіх трьохелементних напівгруп з точністю до ізоморфізму.

Ключові слова і фрази: напівгрупа, максимальна зчеплена система, суперрозширення, проективна ретракція, комутативність.