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POINTS OF NARROWNESS AND UNIFORMLY NARROW OPERATORS

It is known that the sum of every two narrow operators on L_1 is narrow, however the same is false for L_p with $1 < p < \infty$. The present paper continues numerous investigations of the kind. Firstly, we study narrowness of a linear and orthogonally additive operators on Köthe function spaces and Riesz spaces at a fixed point. Theorem 1 asserts that, for every Köthe Banach space E on a finite atomless measure space there exist continuous linear operators $S, T : E \to E$ which are narrow at some fixed point but the sum $S + T$ is not narrow at the same point. Secondly, we introduce and study uniformly narrow pairs of operators $S, T : E \to X$, that is, for every $e \in E$ and every $\varepsilon > 0$ there exists a decomposition $e = e' + e''$ to disjoint elements such that $\|S(e') - S(e'')\| < \varepsilon$ and $\|T(e') - T(e'')\| < \varepsilon$. The standard tool in the literature to prove the narrowness of the sum of two narrow operators $S + T$ is to show that the pair S, T is uniformly narrow. We study the question of whether every pair of narrow operators with narrow sum is uniformly narrow. Having no counterexample, we prove several theorems showing that the answer is affirmative for some partial cases.

Key words and phrases: narrow operator, orthogonally additive operator, Köthe Banach space.

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INTRODUCTION

The class of narrow operators includes some other classes of "small" operators defined on atomless function spaces and Riesz spaces, such as weakly compact, Dunford-Pettis, absolutely summing etc. It was introduced and studied in [11] for function spaces and in [7] for Riesz spaces, however some results on these operators appeared in 80-th years of XXth century. The importance of narrow operators is explained by different geometric implications of their properties, see survey [13] and textbook [14]. Then the notion was naturally generalized to (nonlinear) orthogonally additive operators in [12]. An operator (linear or, more general, orthogonally additive) $T : E \to X$ from an atomless function space or atomless Riesz space E to a topological vector space X is said to be *narrow* if for every $e \in E$ and every neighborhood V of zero in X there exists a decomposition to disjoint summands $e = e' + e''$ such that $T(e') - T(e'') \in V$. Although it would be natural to consider narrowness at a fixed point $e \in E$, no investigation before [12] (2014) took this point into account. However in [12] the authors considered narrowness of an operator T at a fixed point $e \in E$ only for technical reasons to prove the main result.

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One of the most interesting facts concerning narrow operators is that, for some pairs of spaces (E, F) the sum $S + T$ of every two narrow operators $S, T : E \to X$ is narrow, but for other pairs the same is not true. For instance, the sum of every two narrow operators on L_1 is narrow, however every operator on L_p with $1 < p < \infty$ is a sum of two narrow operators. A number of published papers of different authors devoted to the questions of narrowness of a sum of two narrow operators (see, e.g. [2, 7, 8, 11]). A very different situation appears for narrowness at a fixed point. Theorem 1 asserts that for every Köthe Banach space E on a finite atomless measure space there exist continuous linear operators $S, T : E \to E$ which are narrow at some fixed point but the sum $S + T$ is not narrow at the same point.

A very natural proof that the sum $S + T$ of two narrow operators $S, T : E \to X$ is narrow is reduced to the proof that, for every $e \in E$ and every $\varepsilon > 0$ there exists a partition $e = e' \sqcup e''$ (common for both S and T) such that $\|Se' - Se''\| < \varepsilon/2$ and $\|Te' - Te''\| < \varepsilon/2$. This naturally leads us to a new notion of uniformly narrow pair of operators and to the question of whether every pair of narrow operators with narrow sum is uniformly narrow. Having no counterexample, in Section 2 we prove several theorems showing that the answer is affirmative for some partial cases.

Now we give a brief preliminaries on the notions used below. An *F-space* is a complete metric linear space X over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with an invariant metric ρ (i.e., $\rho(x, y) = \rho(x + z, y + z)$ for each $x, y, z \in X$). We set $\|x\| = \rho(x, 0)$, and so, $\rho(x, y) = \|x - y\|$ and call the defined map $\| \cdot \| : X \times X \to [0, +\infty)$ the F-norm of the F-space X. A very important class of F-spaces is the class of Banach spaces. Let (Ω, Σ, μ) be a finite measure space. An F-space E of equivalence classes of measurable functions on Ω is called a *Köthe F-space* if the following conditions hold: (K_i) if $y \in E$ and $|x| \le |y|$ then $x \in E$ and $\|x\| \le \|y\|$; (K_{ii}) $\mathbf{1}_\Omega \in E$. If, moreover, E is a Banach space and (K_{iii}) $E \subseteq L_1(\mu)$ then E is called a *Köthe Banach space*.

By $\mathcal{L}(X, Y)$ we denote the set of all continuous linear operators acting from X to Y.

Let E be a Riesz space (in particular, a Köthe F-space) and X a vector space. A map $T : E \to X$ is called an *orthogonally additive operator* if $T(x + y) = T(x) + T(y)$ for all $x, y \in E$ with $x \perp y$ (for Köthe F-space it means that x and y have disjoint supports). If, moreover, X is a Riesz space then an order bounded orthogonally additive operator $T : E \to X$ is called an *abstract Uryson operator*. We refer the reader to [4, 5, 6, 10] and the bibliography therein for examples and some usual facts on orthogonally additive operators. An element y of a Riesz space E is called a *fragment* (in another terminology, a *component*) of an element $x \in E$, provided $y \perp (x - y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x. A net $(x_\alpha)_{\alpha \in \Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\mathrm{o}} x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \le u_\beta$ for all $\beta \in \Lambda$. The equality $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ if $i \ne j$. Note that in this case one has that $x_i \sqsubseteq x$ for all i. If E is a Riesz space and $e \in E^+$ then by $\mathfrak{F}_e$ we denote the set of all fragments of e. We say that a net $(x_\alpha)_{\alpha \in \Lambda}$ in E *up-laterally converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\ell\uparrow} x$) if $x_\alpha \xrightarrow{\mathrm{o}} x$ and $x_\alpha \sqsubseteq x_\beta$ as $\alpha < \beta$. A function $f : E \to F$ between Riesz spaces is said to be up-laterally continuous if for every net $(x_\alpha)_{\alpha \in \Lambda}$ in E and every $x \in E$ the condition $x_\alpha \xrightarrow{\ell\uparrow} x$ implies $f(x_\alpha) \xrightarrow{\ell\uparrow} f(x)$ in F.

An element e of a Riesz space E is called a *projection element* if the band B_e generated by e is a projection band. A Riesz space E is said to have the *principal projection property* if every element of E is a projection element. For instance, every Dedekind σ-complete Riesz space has the principal projection property. An element $u \ne 0$ of a Riesz space E is called an *atom* whenever $0 \le x \le |u|$, $0 \le y \le |u|$ and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$. Evidently, if $u \in E$ is an atom then $\mathfrak{F}_u = \{0, u\}$. A Riesz space without a nonzero atom is said to be *atomless*.

1 POINTS OF NARROWNESS

Below we give main definitions of narrow operators adapted to the idea to consider narrowness at a fixed point.

Definition 1.1 (of a narrow map). *Let E be a Riesz space and X be a topological vector space. A function $f : E \to X$ is said to be:*

- ***narrow at a point*** *$e \in E$ if for every neighborhood of zero U in X there exists a decomposition $e = e_1 \sqcup e_2$ such that $f(e_1) - f(e_2) \in U$. The set of all points of E at which f is narrow is denoted by $\mathcal{N}(f)$;*
- ***narrow*** *if $\mathcal{N}(f) = E$.*

Observe that, for linear maps the definition is equivalent to the following one. A linear operator $T : E \to X$ is said to be *narrow at a point* $e \in E$ if for every neighborhood of zero U in X there exists $f \in E$ such that $|f| = |e|$ and $Tf \in U$.

Definition 1.2 (of a strictly narrow map). *Let E be a Riesz space and X be a set. A function $f : E \to X$ is said to be*

- ***strictly narrow at a point*** *$e \in E$ if there exists a decomposition $e = e_1 \sqcup e_2$ such that $f(e_1) = f(e_2)$. The set of all points of E at which f is strictly narrow is denoted by $\mathcal{N}^s(f)$;*
- ***strictly narrow*** *if $\mathcal{N}^s(f) = E$.*

Likewise, if X is a linear space, a linear operator $T : E \to X$ is strictly narrow at a point $e \in E$ if and only if there exists $f \in E$ such that $|f| = |e|$ and $Tf = 0$.

Definition 1.3 (of an order narrow map). *Let E, X be Riesz spaces. A function $f : E \to X$ is said to be:*

- ***order narrow at a point*** *$e \in E$ if there is a net of decompositions $e = e'_\lambda \sqcup e''_\lambda$, $\lambda \in \Lambda$ such that $(f(e'_\lambda) - f(e''_\lambda)) \xrightarrow{\text{o}} 0$ in X. The set of all points of E at which f is order narrow is denoted by $\mathcal{N}^o(f)$;*
- order narrow *if $\mathcal{N}^o(f) = E$.*

Similarly, a linear operator $T : E \to X$ is order narrow at a point $e \in E$ if and only if there exists a net $f_\alpha \in E$ with $|f_\alpha| = |e|$ for all indices α such that $Tf_\alpha \xrightarrow{\text{o}} 0$.

Observe that a narrow (in any sense) function sends any atom to zero. So, to avoid triviality one may consider atomless Köthe F-spaces and atomless Riesz spaces to be the domain spaces of narrow maps. Another simple observation is that 0 is a point of narrowness of any map in any sense of narrowness.

Obviously, if X is a topological vector space then every strictly narrow (at a point, on a set) function is narrow. So, $\mathcal{N}^s(f) \subseteq \mathcal{N}(f)$ for any map $f : E \to X$. Similarly, if X is a Riesz space then every strictly narrow (at a point, on a set) function is order narrow. So, $\mathcal{N}^s(f) \subseteq \mathcal{N}^o(f)$

for any map $f : E \to X$. If one considers a compact linear operator T with zero kernel acting from a Köthe F-space E to an F-space X then $\mathcal{N}^s(T) = \{0\}$, however $\mathcal{N}(T) = E$, because every compact operator is narrow [14, Proposition 2.1]. If, moreover, X is an order continuous Banach lattice then $\mathcal{N}^o(T) = E$ as well, because in this case every narrow operator is order narrow [14, Proposition 10.9].

The connections between narrowness and order narrowness of a map is not so obvious, however it can be easily deduced from the arguments of [7]. Recall that a Banach lattice E is said to be *order continuous* if for each net (x_α) in E the condition $x_\alpha \downarrow 0$ implies that $\|x_\alpha\| \to 0$. Note that in this case the weaker condition $x_\alpha \xrightarrow{\text{o}} 0$ also implies that $\|x_\alpha\| \to 0$.

Proposition 1.1. *Let E be a Riesz space and X a Banach lattice. Then*

(1) *every narrow at a point $e \in E$ map $f : E \to X$ is order narrow at e;*

(2) *if, moreover, X is order continuous then every order narrow at a point $e \in E$ map $f : E \to X$ is narrow at e;*

(3) *there exists an order narrow positive operator $T \in \mathcal{L}(L_\infty)$ that is not narrow.*

Proof. (1) For each $n \in \mathbb{N}$ we choose a decomposition $e = e_n' \sqcup e_n''$ with $\|f(e_n') - f(x_n'')\| < 2^{-n}$ and set $u_n = \sum_{k \geq n} |f(e_k') - f(x_k'')|$ (the series obviously satisfies Cauchy's condition and hence converges). To show that $(f(e_n') - f(e_n'')) \xrightarrow{\text{o}} 0$ is a standard technical exercise.

(2) Let f be order narrow at e. We choose a net of decompositions $e = e_\lambda' \sqcup e_\lambda''$, $\lambda \in \Lambda$ such that $(f(e_\lambda') - f(e_\lambda'')) \xrightarrow{\text{o}} 0$. By the definition of an order continuous Banach lattice, $\|f(e_\lambda') - f(e_\lambda'')\| \to 0$, and thus, f is narrow at e.

(3) See Example 3.3 of [7]. □

The following two propositions are simple exercises.

Proposition 1.2. *Let E be a Riesz space and X a topological vector space.*

1. *For a linear operator $T : E \to X$ the following assertions are equivalent:*

   (i) *T is narrow;*

   (ii) $E^+ \subseteq \mathcal{N}(T)$.

2. *For an orthogonally additive operator $T : E \to X$ the following are equivalent:*

   (i) *T is narrow;*

   (ii) $E^+ \cup E^- \subseteq \mathcal{N}(T)$.

Similar statements are true for strictly narrow and order narrow operators.

Remark that the condition $E^+ \subseteq \mathcal{N}(T)$ for an orthogonally additive operator T does not imply that T is narrow, as the following simple example shows: $Tx = x^-$ for all $x \in E$.

Proposition 1.3. *Let E be a Riesz space and X a topological vector space.*

1. *Assume $T : E \to X$ is a linear operator.*

   (a) *If $e, f \in E$, $e \in \mathcal{N}(T)$ and $|f| = |e|$ then $f \in \mathcal{N}(T)$.*

(b) If $e_1, e_2 \in \mathcal{N}(T)$, $e_1 \perp e_2$ and $a, b \in \mathbb{R}$ then $ae_1 + be_2 \in \mathcal{N}(T)$.

2. *Assume $T : E \to X$ is an orthogonally additive operator. If $e_1, e_2 \in \mathcal{N}(T)$ and $e_1 \perp e_2$ then $e_1 + e_2 \in \mathcal{N}(T)$.*

Similar statements are true for strictly narrow and order narrow operators.

Proposition 1.4. *Let E be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ), X a topological vector space, $T : E \to X$ a uniformly continuous orthogonally additive operator. Then the set of narrowness $\mathcal{N}(T)$ is closed in E.*

Proof. Let e belong to the F-norm closure of $\mathcal{N}(T)$. We show that T is narrow at e. Let V be any neighborhood of zero in X. Choose a neighborhood of zero V_1 in X so that $V_1 + V_1 + V_1 \subseteq V$ and $\delta > 0$ so that if $x, y \in E$ with $\|x - y\| < \delta$ then $T(x) - T(y) \in V_1$. Now choose $e_1 \in \mathcal{N}(T)$ so that $\|e_1 - e\| < \delta$ and choose a decomposition $e_1 = e_1' \sqcup e_1''$ so that $T(e_1') - T(e_1'') \in V_1$. Set $\Omega' = \operatorname{supp} e_1'$, $\Omega'' = \Omega \setminus \Omega'$, $e' = e \cdot \mathbf{1}_{\Omega'}$ and $e'' = e \cdot \mathbf{1}_{\Omega''}$. Then $e = e' \sqcup e''$. We show that $Te' - Te'' \in V$. Indeed, observe that

$$\|e' - e_1'\| = \big\|e \cdot \mathbf{1}_{\Omega'} - e_1 \cdot \mathbf{1}_{\Omega'}\big\| \le \|e - e_1\| < \delta$$

and analogously $\|e'' - e_1''\| < \delta$. Then $Te' - Te_1' \in V_1$ and $Te'' - Te_1'' \in V_1$. Hence,

$$Te' - Te'' = (Te' - Te_1') + (Te_1' - Te_1'') + (Te_1'' - Te'') \in V_1 + V_1 + V_1 \subseteq V. \qquad \square$$

Next we provide an example of a linear operator the set of narrowness of which coincides with the set of all functions with constant modulus.

Example 1. *Let (Ω, Σ, μ) be an atomless probability space (that is, a measure space with $\mu(\Omega) = 1$), $1 \le p < \infty$. Let $\Omega = A \sqcup B$ be any partition to measurable sets A, B. Then for the operator $T \in \mathcal{L}\big(L_p(\mu)\big)$ given by*

$$Tx = x - \Big(\int_\Omega rx\, d\mu\Big) r, \quad \text{where } r = \mathbf{1}_A - \mathbf{1}_B, \ \ x \in L_p(\mu)$$

one has $\mathcal{N}^s(T) = \mathcal{N}(T) = \{e \in E : |e(\omega)| = \lambda \text{ a.e. on } \Omega, \ \lambda \in \mathbb{R}\}$.

Proof. The inclusion $\{e \in E : |e(\omega)| = \lambda \text{ a.e. on } \Omega, \ \lambda \in \mathbb{R}\} \subseteq \mathcal{N}^s(T)$ follows from the observation that $T(\lambda r) = 0$ and $|\lambda r| = |e|$ for any element $e \in E$ with $|e(\omega)| = \lambda$ a.e. on Ω. To show that T is not narrow at each point $e \in E$ with $|e| \ne \lambda r$, $\lambda \in \mathbb{R}$, consider any element of the form $x = e \cdot \mathbf{1}_C - e \cdot \mathbf{1}_D$, where $\Omega = C \sqcup D$ (i.e., an arbitrary element $x \in E$ with $|x| = |e|$). Set $F_1 = A \cap C$, $F_2 = A \cap D$, $F_3 = B \cap C$ and $F_4 = B \cap D$. Then

$$\alpha \stackrel{\text{def}}{=} \int_\Omega rx\, d\mu = \int_{F_1} e\, d\mu - \int_{F_2} e\, d\mu - \int_{F_3} e\, d\mu + \int_{F_4} e\, d\mu,$$

which implies $|\alpha| \le \int_\Omega |e|\, d\mu = \|e\|_{L_1(\mu)}$.

Hence,

$$\|Tx\| = \|x - \alpha r\| \ge \|x\| - |\alpha|\|r\| = \|e\| - |\alpha| \ge \|e\|_{L_p(\mu)} - \|e\|_{L_1(\mu)}. \tag{1}$$

If we assume that T is narrow at e then by (1), $\|e\|_{L_p(\mu)} - \|e\|_{L_1(\mu)} = 0$ which yields that $|e|$ is a constant. $\square$

The following theorem provides an example of narrow at a fixed point operators on an arbitrary Köthe Banach space with nonnarrow sum at the same point.

Theorem 1. *Let E be a Köthe Banach space on a finite atomless measure space (Ω, Σ, μ). Then there are continuous linear operators $T_1, T_2 \in \mathcal{L}(E)$ each of which is strictly narrow at the point $\mathbf{1} = \mathbf{1}_\Omega$, however the sum $T_1 + T_2$ is not narrow at $\mathbf{1}$.*

Proof. Assume for simplicity of the notation that $\mu(\Omega) = 1$ and $\|\mathbf{1}\| = 1$. Decompose $\Omega = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$ with measure $\mu(A_i) = 1/4$ each. Set $r_1 = \mathbf{1}_{A_1} + \mathbf{1}_{A_2} - \mathbf{1}_{A_3} - \mathbf{1}_{A_4}$ and $r_2 = \mathbf{1}_{A_1} - \mathbf{1}_{A_2} + \mathbf{1}_{A_3} - \mathbf{1}_{A_4}$. Define operators $T_1, T_2 \in \mathcal{L}(E)$ by setting

$$T_i x = x - \Big(\int_\Omega r_i x \, d\mu \Big) \, r_i, \quad x \in E, \quad i = 1, 2.$$

It is immediately that T_i are strictly narrow at $\mathbf{1}$, because $T_i r_i = 0$, $i = 1, 2$. We show that $T_1 + T_2$ is not narrow at $\mathbf{1}$. Let $r \in E$ be any element of the form $r = \mathbf{1}_A - \mathbf{1}_B$, where $A, B \in \Sigma$ with $\Omega = A \sqcup B$. We set $D_k = A \cap A_k$ and $F_k = B \cap A_k$ for $k = 1, 2, 3, 4$. Then set

$$\lambda_i = \int_\Omega r r_i \, d\mu, \quad i = 1, 2.$$

Taking into account that $\mu(D_k) + \mu(F_k) = 1/4$ for all k, we obtain

$$\begin{aligned} \lambda_1 &= \mu(D_1) + \mu(D_2) - \mu(D_3) - \mu(D_4) - \mu(F_1) - \mu(F_2) + \mu(F_3) + \mu(F_4) \\ &= 2\mu(D_1) + 2\mu(D_2) - 2\mu(D_3) - 2\mu(D_4) \end{aligned} \tag{2}$$

and analogously

$$\lambda_2 = 2\mu(D_1) - 2\mu(D_2) + 2\mu(D_3) - 2\mu(D_4). \tag{3}$$

Since $|\lambda_i| \le 1$ for $i = 1, 2$ and E is a Köthe Banach space,

$$\begin{aligned} \|(T_1 + T_2)\, r\| = \|2r - \lambda_1 r_1 - \lambda_2 r_2\| = \big\| &(2 - \lambda_1 - \lambda_2)\mathbf{1}_{D_1} + (2 - \lambda_1 + \lambda_2)\mathbf{1}_{D_2} \\ &+ (2 + \lambda_1 - \lambda_2)\mathbf{1}_{D_3} + (2 + \lambda_1 + \lambda_2)\mathbf{1}_{D_4} + (-2 - \lambda_1 - \lambda_2)\mathbf{1}_{F_1} \\ &+ (-2 - \lambda_1 + \lambda_2)\mathbf{1}_{F_2} + (-2 + \lambda_1 - \lambda_2)\mathbf{1}_{F_3} + (-2 + \lambda_1 + \lambda_2)\mathbf{1}_{F_4} \big\| \\ \ge \max\Big\{ &(2 - \lambda_1 - \lambda_2)\|\mathbf{1}_{D_1}\|, (2 - \lambda_1 + \lambda_2)\|\mathbf{1}_{D_2}\|, (2 + \lambda_1 - \lambda_2)\|\mathbf{1}_{D_3}\|, \\ &(2 + \lambda_1 + \lambda_2)\|\mathbf{1}_{D_4}\|, (2 + \lambda_1 + \lambda_2)\|\mathbf{1}_{F_1}\|, (2 + \lambda_1 - \lambda_2)\|\mathbf{1}_{F_2}\|, \\ &(2 - \lambda_1 + \lambda_2)\|\mathbf{1}_{F_3}\|, (2 - \lambda_1 - \lambda_2)\|\mathbf{1}_{F_4}\| \Big\}. \end{aligned}$$

Since $\mathbf{1} = \mathbf{1}_{D_1} + \mathbf{1}_{D_2} + \mathbf{1}_{D_3} + \mathbf{1}_{D_4} + \mathbf{1}_{F_1} + \mathbf{1}_{F_2} + \mathbf{1}_{F_3} + \mathbf{1}_{F_4}$, one of the summands has norm at least $1/8$. Of course, it is a matter of similar cases, which one. Say, $\|\mathbf{1}_{D_1}\| \ge 1/8$. Then

$$\|(T_1 + T_2)\, r\| \ge (2 - \lambda_1 - \lambda_2)\|\mathbf{1}_{D_1}\| \ge (2 - \lambda_1 - \lambda_2)/8.$$

Fix any $\varepsilon > 0$ and assume that r is chosen so that $\|(T_1 + T_2)\, r\| < \varepsilon$. Then by the above,

$$2 - \lambda_1 - \lambda_2 < 8\varepsilon. \tag{4}$$

We claim that $\lambda_i > 1 - 8\varepsilon$ for $i = 1, 2$. Indeed, if $\lambda_1 \le 1 - 8\varepsilon$ then $2 - \lambda_1 - \lambda_2 \ge 1 - \lambda_1 \ge 8\varepsilon$, which contradicts (4). Analogously, $\lambda_2 > 1 - 8\varepsilon$. Then by (2),

$$\mu(D_1) + \mu(D_2) - \mu(D_3) - \mu(D_4) = \frac{\lambda_1}{2} \ge \frac{1}{2} - 4\varepsilon \tag{5}$$

and by (3),

$$\mu(D_1) - \mu(D_2) + \mu(D_3) - \mu(D_4) = \frac{\lambda_2}{2} \geq \frac{1}{2} - 4\varepsilon. \tag{6}$$

Averaging (5) and (6), one gets $\frac{1}{4} \geq \mu(D_1) \geq \mu(D_1) - \mu(D_4) \geq \frac{1}{2} - 4\varepsilon$, which implies $\varepsilon \geq 1/16$. Thus, $T_1 + T_2$ is not narrow at $\mathbf{1}$. □

The following statement characterizes the set of strict narrowness of linear maps.

Proposition 1.5. *Let E be a Riesz space, X a linear space and $T : E \to X$ a linear operator. Then*

$$\mathcal{N}^s(T) = \Big\{ x \in E : \ (\exists e \in \ker T)\, |x| = |e| \Big\}.$$

Proof. Let $x \in \mathcal{N}^s(T)$. Choose a decomposition $x = x' \sqcup x''$ so that $T(x') = T(x'')$. Then for $e = x' - x''$ one has that $|e| = |x|$ and $e \in \ker T$.

Assume $e \in \ker T$ $x \in E$ and $|x| = |e|$. Then

$$e = (x^+ \wedge e^+) \sqcup (x^- \wedge e^+) \sqcup \big(-(x^+ \wedge e^-)\big) \sqcup \big(-(x^- \wedge e^-)\big) \tag{7}$$

and

$$x = (x^+ \wedge e^+) \sqcup \big(-(x^- \wedge e^+)\big) \sqcup (x^+ \wedge e^-) \sqcup \big(-(x^- \wedge e^-)\big). \tag{8}$$

Then setting $x' = (x^+ \wedge e^+) - (x^- \wedge e^-)$ and $x'' = -(x^- \wedge e^+) + (x^+ \wedge e^-)$, we obtain $x = x' \sqcup x''$ and by (7) and (8),

$$0 = Te = T(x^+ \wedge e^+) + T(x^- \wedge e^+) - T(x^+ \wedge e^-) - T(x^- \wedge e^-) = Tx' - Tx''.$$ □

In particular, $\mathcal{N}^s(T)$ need not be a linear subspace of E. For instance, if $\ker T$ is the set of all constant functions then $\mathcal{N}^s(T)$ equals the set of all functions with constant modulus.

Remark that Proposition 1.5 is not longer true for orthogonally additive operators due to the obvious example $Tx = x^-$ for which $\mathcal{N}^s(T) = E^+$. To provide more examples for orthogonally additive operators we recall some necessary information from [9]. Given any two elements x, y of a Riesz space E, by xy we denote the greatest lower bound of the two-element set $\{x, y\}$ in E with respect to the lateral order $u \sqsubseteq v$ on E, if it exists. If E is a Riesz space of functions then

$$xy(t) = \begin{cases} x(t), & \text{if } x(t) = y(t); \\ 0, & \text{if } x(t) \neq y(t). \end{cases}$$

A Riesz space is said to have the intersection property if every two-point subset $\{x, y\}$ of E has the lateral infimum xy. In particular, the principal projection property implies the intersection property [9].

Example 2. *Let E be a Riesz space with the intersection property and $e \in E$. Then the function $T : E \to E$ given by $Tx = ex$ is an orthogonally additive operator with $\mathcal{N}^s(T) = \{0\} \cup (E \setminus \mathfrak{F}_e)$.*

Example 3. *Let E be a Riesz space with the intersection property and $e \in E$. Then the function $T : E \to E$ given by $Tx = x - ex$ is an orthogonally additive operator with $\mathcal{N}^s(T) = \mathfrak{F}_e$.*

The following example [7, Example 4.2] shows that, a continuous linear functional on an atomless Banach lattice may have the only zero point of narrowness.

Example 4. *There is a continuous linear functional $f \in L_\infty^*$ for which $\mathcal{N}(f) = \mathcal{N}^o(f) = \{0\}$.*

Proof. Denote by $\mathcal{B}$ the Boolean algebra of Borel subsets of $[0,1]$ equals up to measure null sets. Let $\mathcal{U}$ be any ultrafilter on $\mathcal{B}$. Then the linear functional $f : E \to \mathbb{R}$ defined by

$$f(x) = \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A x \, d\mu$$

is obviously bounded. However it is not narrow in any sense at every nonzero point. Indeed, for each $x \in L_\infty \setminus \{0\}$ of the form $x = \mathbf{1}_A - \mathbf{1}_B$ where $[0,1] = A \sqcup B$ one has $f(x) = \pm 1$ depending on whether $A \in \mathcal{U}$ or $B \in \mathcal{U}$. □

2 UNIFORMLY NARROW PAIRS OF OPERATORS

Below we define a uniformly narrow pair of operators; even though one can consider an arbitrary uniformly narrow set of operators.

Definition 2.1. *Let E be a Riesz space and X be an F-space. We say that an orthogonally additive operators $S, T : E \to X$ are* ***uniformly narrow*** *if for every $e \in E$ and every $\varepsilon > 0$ there exists a partition $e = e' \sqcup e''$ such that $\|Se' - Se''\| < \varepsilon$ and $\|Te' - Te''\| < \varepsilon$.*

As was noted in the introduction, a simple argument shows that, if orthogonally additive operators $S, T : E \to X$ are uniformly narrow then the sum $S + T$ is narrow. The following question naturally arises.

Problem 1. *Let E be a Riesz space and X be an F-space. Are the following assertions equivalent for every pair of narrow linear (orthogonally additive operators) $S, T : E \to X$?*

(i) *$S + T$ is narrow;*

(ii) *S, T are uniformly narrow.*

Although we do not know any example of spaces with negative answer to Problem 1, we present below an affirmative solution for some partial cases. We refer the reader to [1] for further standard terminology concerning operators on Riesz spaces.

We say that a Banach space X has the *contains its square* if there are a subspace Y of X and a decomposition $Y = X_1 \oplus X_2$ onto subspaces X_1, X_2 isomorphic to X.

Theorem 2. *Let E be a Riesz space and X be a Banach space containing its square. Let the sum of every two narrow linear bounded operators from E to X is narrow. Then every pair $S, T : E \to X$ of narrow linear bounded operators is uniformly narrow.*

Proof. Let Y be a subspace of X, $Y = X_1 \oplus X_2$ with subspaces X_1, X_2 isomorphic to X. Let $\tau_i : X \to X_i$ be isomorphisms, $i = 1, 2$. Let $S, T : E \to X$ be narrow linear operators. Then the linear operators $S', T' : E \to Y \subseteq X$ defined by setting $S' = \tau_1 \circ S$ and $T' = \tau_2 \circ T$ are narrow as compositions of a narrow operator from the right by a bounded operator from the left. By the assumption, the operator $A = S' + T'$ is narrow. Denote by P the projection of Y onto X_1 parallel to X_2 and by Q the projection of Y onto X_2 parallel to X_1. Observe that $P \circ A = S'$ and $Q \circ A = T'$. Given any $e \in E^+$ and $\varepsilon > 0$, we choose a decomposition $e = e' \sqcup e''$ such that

$$\|Ae' - Ae''\| < \frac{\varepsilon}{\|\tau^{-1}\| \max\{\|P\|, \|Q\|\}}.$$

Then

$$\|Se' - Se''\| \le \|\tau^{-1}\| \|\tau(Se' - Se'')\| = \|\tau^{-1}\| \|S'e' - S'e''\|$$
$$= \|\tau^{-1}\| \|P(Ae' - Ae'')\| \le \|\tau^{-1}\| \|P\| \|Ae' - Ae''\| < \varepsilon.$$

Analogously, $\|Te' - Te''\| < \varepsilon$. ☐

For example, the assumptions of Theorem 2 are valid for $E = F = L_1$ (see [2] or [14, Theorem 7.46] for the fact that a sum of every two narrow operators on L_1 is narrow).

We say that a Banach lattice X *regularly contains its square* if there are a subspace Y of X and a decomposition $Y = X_1 \oplus X_2$ onto subspaces X_1, X_2 isomorphic to X by means of regular isomorphisms $\tau_i : X \to X_i$, $i = 1, 2$.

Theorem 3. *Let E be a Riesz space and X be a Banach lattice regularly containing its square. Let the sum of every two narrow regular linear operators from E to X is narrow. Then every pair $S, T : E \to X$ of narrow regular linear operators is uniformly narrow.*

Proof. Let Y be a subspace of X, $Y = X_1 \oplus X_2$ with subspaces X_1, X_2 isomorphic to X by means of regular isomorphisms $\tau_i : X \to X_i$, $i = 1, 2$. Let $S, T : E \to X$ be narrow regular linear operators. Then the linear operators $S', T' : E \to Y \subseteq X$ defined by setting $S' = \tau_1 \circ S$ and $T' = \tau_2 \circ T$ are narrow regular as compositions of a narrow regular operator from the right by a bounded regular operator from the left. By the assumption, the operator $A = S' + T'$ is narrow. Starting from this point, the proof is the same as that of Theorem 2. ☐

Corollary 2.1. *Let E, F be order continuous Banach lattices with E atomless and F regularly containing its square. Then every pair of narrow regular operator $S, T : E \to F$ is uniformly narrow.*

Proof. Accordingly to Theorem 11.8 of [7] (see also [14, Theorem 10.41]), the set of all narrow regular linear operators is a band in the Riesz space of all regular linear operators from E to F. In particular, the sum of every two narrow regular linear operators from E to X is narrow. By Theorem 3, every pair of narrow regular operator $S, T : E \to F$ is uniformly narrow. ☐

Now we pass to orthogonally additive operators. Let E and F be Riesz spaces. An orthogonally additive operator $T : E \to F$ is called:

- *positive* provided $Tx \ge 0$ holds in F for all $x \in E$;
- *order bounded* it T maps order bounded sets in E to order bounded sets in F.

Observe that if $T : E \to F$ is a positive orthogonally additive operator and $x \in E$ is such that $T(x) \neq 0$ then $T(-x) \neq -T(x)$ (otherwise both $T(x) \ge 0$ and $T(-x) \ge 0$ would imply $T(x) = 0$). Thus, this positivity turns out to be more restrictive than the usual one for linear operators because the only linear operator which is positive in the above sense is zero.

A positive orthogonally additive operator need not be order bounded. Indeed, every function $T : \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ is an orthogonally additive operator, and obviously, not each of them is order bounded.

Banach lattices E and F are said to be *Riesz isomorphic* if there exists a *Riesz isomorphism* $\tau : E \to F$, that is, an isomorphism between Banach spaces such that both τ and τ^{-1} are order preserving operators.

We say that a Banach lattice X *contains its Riesz square* if there are a subspace Y of X and a decomposition $Y = X_1 \oplus X_2$ onto subspaces X_1, X_2 Riesz isomorphic to X and, moreover, the corresponding projections of Y onto X_i parallel to X_{3-i} are order continuous. For example, the Banach lattice $L_p[0,1]$ with $1 \le p \le \infty$ obviously contains its Riesz square.

Theorem 4. *Let E be an atomless Riesz space and F be an order continuous Banach lattice containing its Riesz square. Let the sum of every two narrow up-laterally continuous abstract Uryson operators from E to X is narrow. Then every pair $S, T : E \to X$ of narrow up-laterally continuous abstract Uryson operators is uniformly narrow.*

Proof. By [12, Lemma 2.7], under the assumptions on E and F, an abstract Uryson operator $B : E \to F$ is narrow if and only if B is order narrow. Let Y be a subspace of X, $Y = X_1 \oplus X_2$ and $\tau_i : X \to X_i$ be Riesz isomorphisms, $i = 1, 2$. Let $S, T : E \to X$ be narrow up-laterally continuous abstract Uryson operators. Then the maps $S', T' : E \to Y \subseteq X$ defined by setting $S' = \tau_1 \circ S$ and $T' = \tau_2 \circ T$ are narrow up-laterally continuous abstract Uryson operators as compositions of such an operator from the right by a bounded regular operator from the left. By the theorem assumptions, the operator $A = S' + T'$ is narrow and so, is order narrow. Denote by P the projection of Y onto X_1 parallel to X_2 and by Q the projection of Y onto X_2 parallel to X_1. Observe that $P \circ A = S'$ and $Q \circ A = T'$. Given any $e \in E^+$ and $\varepsilon > 0$, we choose a net of decompositions $e = e'_\alpha \sqcup e''_\alpha$ with $(Ae'_\alpha - Ae''_\alpha) \xrightarrow{\text{o}} 0$. Since the operators τ^{-1} and P are order continuous,

$$Se'_\alpha - Se''_\alpha = \tau^{-1}(S'e'_\alpha - S'e''_\alpha) = \tau^{-1}P(Ae'_\alpha - Ae''_\alpha) \xrightarrow{\text{o}} 0.$$

By the order continuity of F, $\|Se'_\alpha - Se''_\alpha\| \to 0$. Analogously, $\|Te'_\alpha - Te''_\alpha\| \to 0$. We choose α so that $\|Se'_\alpha - Se''_\alpha\| < \varepsilon$ and $\|Te'_\alpha - Te''_\alpha\| < \varepsilon$. □

As a consequence of [12, Theorem 8.2], we obtain the following assertion.

Corollary 2.2. *Let E be an atomless Riesz space with the principal projection property and F be an order continuous Banach lattice containing its Riesz square. Then every pair $S, T : E \to X$ of narrow up-laterally continuous abstract Uryson operators is uniformly narrow.*

Proof. By [12, Lemma 2.7], under the assumptions on E and F, an abstract Uryson operator $B : E \to F$ is narrow if and only if B is order narrow. So, by [12, Theorem 8.2], the sum of every two narrow up-laterally continuous abstract Uryson operators from E to X is narrow. Then apply Theorem 4. □

Recall that an operator $T \in \mathcal{L}(E, X)$ from a Köthe Banach space E on a finite atomless measure space (Ω, Σ, μ) to a Banach space X is called *hereditarily narrow* if for every $A \in \Sigma$, $\mu(A) > 0$ and every atomless sub-σ-algebra $\mathcal{F}$ of $\Sigma(A)$ the restriction of T to $E(\mathcal{F})$ is narrow (here $\Sigma(A) = \{B \in \Sigma : B \subseteq A\}$ and $E(\mathcal{F}) = \{x \in E(A) : x \text{ is } \mathcal{F} - \text{measurable}\}$). We refer the reader to [14, Section 11.1] for more information on hereditarily narrow operators.

Proposition 2.1. *Let E be a Köthe Banach space on $[0,1]$ with an absolutely continuous norm and X be a Banach space. If $S \in \mathcal{L}(E, X)$ is a hereditarily narrow operator and $T \in \mathcal{L}(E, X)$ is a narrow operator then the pair S, T is uniformly narrow.*

The proof of Proposition 2.1 just repeats the proof of [14, Proposition 11.2] (see also [3]).

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Відомо, що сума довільних двох вузьких операторів на L_1 є вузькою, проте для просторів L_p з $1 < p < \infty$ аналогічне твердження хибне. Дана стаття продовжує численні дослідження на цю тему. По-перше, ми вивчаємо вузькість лінійних та ортогонально адитивних операторів на функціональних просторах Кете і векторних ґратках у фіксованій точці. Теорема 1 стверджує, що для кожного банахового простору Кете на просторі зі скінченною безатомною мірою існують лінійні неперервні оператори $S, T : E \to E$, які є вузькими у деякій фіксованій точці, проте сума $S + T$ не є вузькою у цій же самій точці. По-друге, ми уводимо і досліджуємо одностайно вузькі пари операторів $S, T : E \to X$, тобто, для кожного $e \in E$ та кожного $\varepsilon > 0$ існує розклад $e = e' + e''$ на диз'юнктні елементи такий, що $\|S(e') - S(e'')\| < \varepsilon$ та $\|T(e') - T(e'')\| < \varepsilon$. Стандартний метод в літературі доведення вузькості суми двох вузьких операторів $S + T$ полягає в тому, щоби показати, що пара S, T є одностайно вузькою. Ми вивчаємо питання, чи кожна пара вузьких операторів з вузькою сумою є одностайно вузькою. Не маючи жодного контрприкладу, ми доводимо кілька теорем, які надають позитивну відповідь для деяких часткових випадків.

Ключові слова і фрази: вузький оператор, ортогонально адитивний оператор, банахів простір Кете.