

ANSARI A.H.<sup>1</sup>, BINBASIOGLU D.<sup>2</sup>, TURKOGLU D.<sup>3</sup>

## COUPLED COINCIDENCE POINT RESULTS FOR CONTRACTION OF C-CLASS MAPPINGS IN ORDERED UNIFORM SPACES

In the literature there is a lot of works related to fixed point theory. The theory has many applications and some authors are interested in these applications in various spaces. In 2009, Altun I. and Imdad M. defined the order relation on uniform spaces and the concept of compatibility of mappings. Later Ansari A.H. defined the C-class function concept. In this paper, we take some ultra altering distance and C-class functions, then we prove some coupled coincidence point theorems for a mapping providing mixed  $g$ -monotonicity property in ordered uniform spaces. We also give the appropriate examples.

*Key words and phrases:* coupled coincidence point, C-class mapping, ordered uniform space.

<sup>1</sup> Department of Mathematics, Islamic Azad University, 31485-313, Karaj, Iran

<sup>2</sup> Department of Mathematics, Gaziosmanpasa University, 60250, Tokat, Turkey

<sup>3</sup> Department of Mathematics, Gazi University, 06500, Ankara, Turkey

E-mail: mathanalsisamir4@gmail.com (Ansari A.H.), demetbinbasi@hotmail.com (Binbasioglu D.), dturkoglu@gazi.edu.tr (Turkoglu D.)

### INTRODUCTION AND PRELIMINARIES

In the literature there is a lot of works related to fixed point theory. Some of them are fixed or common fixed point results in uniform space (e.g. [1–3, 12]). Lately, Aamri M. and El Moutawakil D. [1] have introduced the concept of  $E$ -distance function on uniform spaces and utilize it to improve some well known results of the existing literature involving both  $E$ -contractive or  $E$ -expansive mappings. Later, Altun I. and Imdad M. [3] have introduced a partial ordering on uniform spaces utilizing  $E$ -distance function and have used the same to prove a fixed point theorem for single-valued non-decreasing mappings on ordered uniform spaces.

In this paper, we use the C-class function defined by Ansari A.H. [4], the order relation on uniform spaces defined by Altun I. and Imdad M. [3] and the concept of compatibility of mappings, then we prove coupled coincidence point theorems in ordered uniform spaces. We also discuss an example.

Now, we mention some relevant definitions and properties from the foundation of uniform spaces. We call a pair  $(X, \vartheta)$  to be a uniform space which consists of a non-empty set  $X$  together with a uniformity  $\vartheta$ , wherein the latter begins with a special kind of filter on  $X \times X$ , whose all elements contain the diagonal  $\Delta = \{(x, x) : x \in X\}$ . If  $V \in \vartheta$  and  $(x, y) \in V$ ,  $(y, x) \in V$ , then  $x$  and  $y$  are said to be  $V$ -close. Also a sequence  $\{x_n\}$  in  $X$  is said to be a Cauchy sequence with regard to uniformity  $\vartheta$  if for any  $V \in \vartheta$ , there exists  $N \geq 1$  such that  $x_n$  and  $x_m$  are  $V$ -close for

$m, n \geq N$ . A uniformity  $\vartheta$  defines a unique topology  $\tau(\vartheta)$  on  $X$  for which the neighborhoods of  $x \in X$  are the sets  $V(x) = \{y \in X : (x, y) \in V\}$  when  $V$  runs over  $\vartheta$ .

A uniform space  $(X, \vartheta)$  is said to be Hausdorff if and only if the intersection of all the  $V \in \vartheta$  reduces to diagonal  $\Delta$  of  $X$ , i.e.  $(x, y) \in V$  for  $V \in \vartheta$  implies  $x = y$ . Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity  $\vartheta$  is said to be symmetrical if  $V = V^{-1} = \{(y, x) : (x, y) \in V\}$ . Since each  $V \in \vartheta$  contains a symmetrical  $W \in \vartheta$  and if  $(x, y) \in W$  then  $x$  and  $y$  are both  $W$  and  $V$ -close and then one may assume that each  $V \in \vartheta$  is symmetrical. When topological concepts are mentioned in the context of a uniform space  $(X, \vartheta)$ , they are naturally interpreted with respect to the topological space  $(X, \tau(\vartheta))$ .

In the sequel we shall require the following definitions and lemmas.

**Definition 1** ([1]). Let  $(X, \vartheta)$  be a uniform space. A function  $p : X \times X \rightarrow \mathbb{R}^+$  is said to be an  $E$ -distance if

(p1) for any  $V \in \vartheta$  there exists  $\delta > 0$  such that  $p(z, x) \leq \delta$  and  $p(z, y) \leq \delta$  for some  $z \in X$ , imply  $(x, y) \in V$ ,

(p2)  $p(x, y) \leq p(x, z) + p(z, y)$  for any  $x, y, z \in X$ .

The following lemma embodies some useful properties of  $E$ -distance.

**Lemma 1** ([1, 2]). Let  $(X, \vartheta)$  be a Hausdorff uniform space and  $p$  be an  $E$ -distance on  $X$ . Let  $\{x_n\}$  and  $\{y_n\}$  be arbitrary sequences in  $X$  and  $\{\alpha_n\}, \{\beta_n\}$  be sequences in  $\mathbb{R}^+$  converging to 0. Then, for  $x, y, z \in X$ , the following holds.

(a) If  $p(x_n, y) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for all  $n \in \mathbb{N}$ , then  $y = z$ . In particular, if  $p(x, y) = 0$  and  $p(x, z) = 0$ , then  $y = z$ .

(b) If  $p(x_n, y_n) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for all  $n \in \mathbb{N}$ , then  $\{y_n\}$  converges to  $z$ .

(c) If  $p(x_n, x_m) \leq \alpha_n$  for all  $m > n$ , then  $\{x_n\}$  is a  $p$ -Cauchy sequence in  $(X, \vartheta)$ .

Let  $(X, \vartheta)$  be a uniform space equipped with  $E$ -distance  $p$ . A sequence in  $X$  is  $p$ -Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

**Definition 2** ([1, 2]). Let  $(X, \vartheta)$  be a uniform space and  $p$  be an  $E$ -distance on  $X$ . Then

(i)  $X$  said to be  $S$ -complete if for every  $p$ -Cauchy sequence  $\{x_n\}$  there exists  $x \in X$  with  $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ ,

(ii)  $X$  is said to be  $p$ -Cauchy complete if for every  $p$ -Cauchy sequence  $\{x_n\}$  there exists  $x \in X$  with  $\lim_{n \rightarrow \infty} x_n = x$  with respect to  $\tau(\vartheta)$ ,

(iii)  $f : X \rightarrow X$  is  $p$ -continuous if  $\lim_{n \rightarrow \infty} p(x_n, x) = 0$  implies  $\lim_{n \rightarrow \infty} p(fx_n, fx) = 0$ ,

(iv)  $f : X \rightarrow X$  is  $\tau(\vartheta)$ -continuous if  $\lim_{n \rightarrow \infty} x_n = x$  with respect to  $\tau(\vartheta)$  implies  $\lim_{n \rightarrow \infty} fx_n = fx$  with respect to  $\tau(\vartheta)$ .

**Remark 1** ([1]). Let  $(X, \vartheta)$  be a Hausdorff uniform space and let  $\{x_n\}$  be a  $p$ -Cauchy sequence. Suppose that  $X$  is  $S$ -complete, then there exists  $x \in X$  such that  $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ . Then Lemma 1 (b) gives that  $\lim_{n \rightarrow \infty} x_n = x$  with respect to the topology  $\tau(\vartheta)$  which shows that  $S$ -completeness implies  $p$ -Cauchy completeness.

**Lemma 2** ([3]). Let  $(X, \vartheta)$  be a Hausdorff uniform space,  $p$  be  $E$ -distance on  $X$  and  $\varphi : X \rightarrow \mathbb{R}$ . Define the relation " $\preceq$ " on  $X$  as follows;

$$x \preceq y \Leftrightarrow x = y \text{ or } p(x, y) \leq \varphi(x) - \varphi(y).$$

Then " $\preceq$ " is a (partial) order on  $X$  induced by  $\varphi$ .

**Definition 3** ([6]). We call an element  $(x, y) \in X \times X$  a coupled fixed point of the mapping  $T : X \times X \rightarrow X$  if  $T(x, y) = x$ ,  $T(y, x) = y$ .

**Definition 4** ([11]). An element  $(x, y) \in X \times X$  is called a coupled coincidence point of a mapping  $T : X \times X \rightarrow X$  and  $g : X \rightarrow X$  if  $T(x, y) = g(x)$ ,  $T(y, x) = g(y)$ .

**Definition 5** ([11]). Let  $X$  be a non-empty set and  $T : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . We say  $T$  and  $g$  are commutative if  $g(T(x, y)) = T(g(x), g(y))$  for any  $x, y \in X$ .

**Definition 6** ([7]). Let  $(X, \vartheta)$  be a Hausdorff uniform space,  $p$  be  $E$ -distance on  $X$ . The mappings  $T$  and  $g$ , where  $T : X \times X \rightarrow X$  and  $g : X \rightarrow X$ , are said to be compatible if

$$\lim_{n \rightarrow \infty} p(g(T(x_n, y_n)), T(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n \rightarrow \infty} p(g(T(y_n, x_n)), T(g(y_n), g(x_n))) = 0$$

whenever  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $X$ , such that  $\lim_{n \rightarrow \infty} T(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x$  and  $\lim_{n \rightarrow \infty} T(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ , for any  $x, y \in X$  are satisfied.

In 2014, the concept of  $C$ -class functions (see Definition 7) was introduced by A.H. Ansari in [4] that is pivotal result in fixed point theory. Also see [5, 8, 9].

**Definition 7.** A mapping  $f : [0, \infty)^2 \rightarrow \mathbb{R}$  is called  $C$ -class function if it is continuous and satisfies following axioms:

- (1)  $f(s, t) \leq s$ ;
- (2)  $f(s, t) = s$  implies that either  $s = 0$  or  $t = 0$  for all  $s, t \in [0, \infty)$ .

**Remark 2.** Note for some  $f$  we have that  $f(0, 0) = 0$ .

We denote  $C$ -class functions as  $\mathcal{C}$ .

**Example 1.** The following functions  $f : [0, \infty)^2 \rightarrow \mathbb{R}$  are elements of  $\mathcal{C}$ , for all  $s, t \in [0, \infty)$ :

- (1)  $f(s, t) = s - t$ ,  $f(s, t) = s \Rightarrow t = 0$ ;
- (2)  $f(s, t) = ms$ ,  $0 < m < 1$ ,  $f(s, t) = s \Rightarrow s = 0$ ;

$$(3) f(s, t) = \frac{s}{(1+t)^r}; r \in (0, \infty), f(s, t) = s \Rightarrow s = 0 \text{ or } t = 0;$$

$$(4) f(s, t) = \log(t + a^s)/(1 + t), a > 1, f(s, t) = s \Rightarrow s = 0 \text{ or } t = 0;$$

$$(5) f(s, t) = \ln(1 + a^s)/2, a > e, f(s, t) = s \Rightarrow s = 0.$$

**Definition 8** ([10]). A function  $\psi : [0, \infty) \rightarrow [0, \infty)$  is called an altering distance function if the following properties are satisfied:

(i)  $\psi$  is non-decreasing and continuous,

(ii)  $\psi(t) = 0$  if and only if  $t = 0$ .

**Definition 9.** An ultra altering distance function is a continuous, nondecreasing mapping  $\varphi : [0, \infty) \rightarrow [0, \infty)$  such that  $\varphi(t) > 0, t > 0$  and  $\varphi(0) \geq 0$ .

We denote by  $\Phi_u$  the set of ultra altering distance functions.

**Definition 10** ([12]). Let  $(X, \vartheta)$  be a uniform space and let " $\preceq$ " be an order relation on  $X$  and let  $T : X \times X \rightarrow X$  be an operator. We say that  $T$  has the mixed monotone property if  $T(x, y)$  is monotone nondecreasing in  $x$  and is monotone nonincreasing in  $y$ , that is for any  $x, y \in X$ ,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow T(x_1, y) \preceq T(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow T(x, y_1) \succeq T(x, y_2).$$

**Definition 11** ([12]). Let  $(X, \vartheta)$  be a uniform space and let " $\preceq$ " be an order relation on  $X$  and let  $T : X \times X \rightarrow X, g : X \rightarrow X$  be operators. We say  $T$  has the mixed  $g$ -monotone property if  $T$  is monotone  $g$ -non-decreasing in its first argument and is monotone  $g$ -non-increasing in its second argument, that is, for any  $x, y \in X$ ,

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \text{ implies } T(x_1, y) \preceq T(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \text{ implies } T(x, y_1) \succeq T(x, y_2).$$

**Remark 3.** If  $g$  is the identity mapping, then Definition 11 reduces to Definition 10.

## 1 THE MAIN RESULTS

**Theorem 1.** Let  $(X, \vartheta)$  be a Hausdorff uniform space, " $\preceq$ " is an order on  $X$  and suppose there is an  $E$ -distance  $p$  on  $X$  such that  $(X, p)$  is a  $p$ -Cauchy complete uniform space. Assume there is a function  $F \in C, \varphi \in \Phi_u$  and also suppose  $T : X \times X \rightarrow X$  and  $g : X \rightarrow X$  are such that  $T$  has the mixed  $g$ -monotone property and

$$\begin{aligned} & p(T(x, y), T(u, v)) \\ & \leq F \left( \left( \frac{p(g(x), g(u)) + p(g(y), g(v))}{2} \right), \varphi \left( \frac{p(g(x), g(u)) + p(g(y), g(v))}{2} \right) \right) \end{aligned} \quad (1)$$

for all  $x, y, u, v \in X$  for which  $g(x), g(u)$  are comparable and  $g(y), g(v)$  are comparable. Suppose  $T(X \times X) \subseteq g(X), g$  is  $\tau(\vartheta)$ -continuous and monotone increasing and  $T$  and  $g$  be compatible mappings. Also suppose

(a)  $T$  is  $\tau(\vartheta)$ -continuous

or

(b)  $X$  has the following property :

(i) if a non-decreasing sequence

$$\{x_n\} \rightarrow x, \text{ then } x_n \preceq x \text{ for all } n, \quad (2)$$

(ii) if a non-increasing sequence

$$\{y_n\} \rightarrow y, \text{ then } y \preceq y_n \text{ for all } n. \quad (3)$$

If there exist  $x_0, y_0 \in X$  such that  $g(x_0) \preceq T(x_0, y_0)$  and  $g(y_0) \succeq T(y_0, x_0)$ , then there exist  $x, y \in X$  such that  $g(x) = T(x, y)$  and  $g(y) = T(y, x)$ , that is,  $T$  and  $g$  have a coupled coincidence point in  $X$ .

*Proof.* Let  $x_0, y_0 \in X$  be such that  $g(x_0) \preceq T(x_0, y_0)$  and  $g(y_0) \succeq T(y_0, x_0)$ . Since  $T(X \times X) \subseteq g(X)$ , we can define  $x_1, y_1 \in X$  such that  $g(x_1) = T(x_0, y_0)$  and  $g(y_1) = T(y_0, x_0)$ .

In the same way we construct,  $g(x_2) = T(x_1, y_1)$  and  $g(y_2) = T(y_1, x_1)$ . Continuing in this way we construct two sequences  $\{g(x_n)\}$  and  $\{g(y_n)\}$  in  $X$  such that,

$$g(x_{n+1}) = T(x_n, y_n) \text{ and } g(y_{n+1}) = T(y_n, x_n) \text{ for all } n \geq 0. \quad (4)$$

Now we prove that for all  $n \geq 0$ ,

$$g(x_n) \preceq g(x_{n+1}) \quad (5)$$

and

$$g(y_n) \succeq g(y_{n+1}). \quad (6)$$

Since  $g(x_0) \preceq T(x_0, y_0)$  and  $g(y_0) \succeq T(y_0, x_0)$ , in view of  $g(x_1) = T(x_0, y_0)$  and  $g(y_1) = T(y_0, x_0)$ , we have  $g(x_0) \preceq g(x_1)$  and  $g(y_0) \succeq g(y_1)$ , that is, (5) and (6) hold for  $n = 0$ .

We presume that (5) and (6) hold for some  $n > 0$ . As  $T$  has the mixed  $g$ -monotone property and  $g(x_n) \preceq g(x_{n+1})$ ,  $g(y_n) \succeq g(y_{n+1})$ , from (4), we get

$$g(x_{n+1}) = T(x_n, y_n) \preceq T(x_{n+1}, y_n) \text{ and } T(y_{n+1}, x_n) \preceq T(y_n, x_n) = g(y_{n+1}). \quad (7)$$

Also for the same reason we have

$$g(x_{n+2}) = T(x_{n+1}, y_{n+1}) \succeq T(x_{n+1}, y_n) \text{ and } T(y_{n+1}, x_n) \succeq T(y_{n+1}, x_{n+1}) = g(y_{n+2}). \quad (8)$$

Then from (7) and (8),  $g(x_{n+1}) \preceq g(x_{n+2})$  and  $g(y_{n+1}) \succeq g(y_{n+2})$ . Then, by mathematical induction it follows that (5) and (6) hold for all  $n \geq 0$ .

Let

$$\delta_n = p(g(x_n), g(x_{n+1})) + p(g(y_n), g(y_{n+1}))$$

and

$$\delta_n^l = p(g(x_{n+1}), g(x_n)) + p(g(y_{n+1}), g(y_n)).$$

Next we prove that

$$\delta_n \leq 2F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right) \text{ and } \delta_n^l \leq 2F\left(\frac{\delta_{n-1}^l}{2}, \varphi\left(\frac{\delta_{n-1}^l}{2}\right)\right). \quad (9)$$

Since for all  $n \geq 0$ ,  $g(x_{n-1}) \preceq g(x_n)$  and  $g(y_{n-1}) \succeq g(y_n)$ , we have from (1) and (4),

$$\begin{aligned} p(g(x_n), g(x_{n+1})) &= p(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\leq F\left(\left(\frac{p(g(x_{n-1}), g(x_n)) + p(g(y_{n-1}), g(y_n))}{2}\right), \varphi\left(\frac{p(g(x_{n-1}), g(x_n)) + p(g(y_{n-1}), g(y_n))}{2}\right)\right) \\ &= F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right) \end{aligned}$$

and

$$\begin{aligned} p(g(x_{n+1}), g(x_n)) &= p(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\ &\leq F\left(\left(\frac{p(g(x_n), g(x_{n-1})) + p(g(y_n), g(y_{n-1}))}{2}\right), \varphi\left(\frac{p(g(x_n), g(x_{n-1})) + p(g(y_n), g(y_{n-1}))}{2}\right)\right) \\ &= F\left(\frac{\delta_{n-1}^l}{2}, \varphi\left(\frac{\delta_{n-1}^l}{2}\right)\right). \end{aligned} \quad (10)$$

Similarly from (1) and (4), we have for all  $n \geq 0$ ,

$$\begin{aligned} p(g(y_n), g(y_{n+1})) &= p(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\ &\leq \varphi\left(\frac{p(g(y_{n-1}), g(y_n)) + p(g(x_{n-1}), g(x_n))}{2}\right) = F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right) \end{aligned}$$

and

$$\begin{aligned} p(g(y_{n+1}), g(y_n)) &= p(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \\ &\leq \varphi\left(\frac{p(g(y_n), g(y_{n-1})) + p(g(x_n), g(x_{n-1}))}{2}\right) = F\left(\frac{\delta_{n-1}^l}{2}, \varphi\left(\frac{\delta_{n-1}^l}{2}\right)\right). \end{aligned} \quad (11)$$

Combining (10) and (11) we obtain (9). Since  $\varphi(t) > 0$  for  $t > 0$ , it follows from (9) that the sequences  $\{\delta_n\}$  and  $\{\delta_n^l\}$  are monotone decreasing sequence of non-negative real numbers. Hence there exist  $\delta \geq 0$  and  $\delta^l \geq 0$  such that  $\lim_{n \rightarrow \infty} \delta_n = \delta$  and  $\lim_{n \rightarrow \infty} \delta_n^l = \delta^l$ . Taking the limit as  $n \rightarrow \infty$  in (9), we obtain  $\delta = \lim_{n \rightarrow \infty} \delta_n \leq 2 \lim_{n \rightarrow \infty} F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right) = 2F\left(\left(\frac{\delta}{2}\right), \varphi\left(\frac{\delta}{2}\right)\right)$ . So,  $\frac{\delta}{2} = 0$  or  $\varphi\left(\frac{\delta}{2}\right) = 0$ . Thus  $\delta = 0$ . Hence we have

$$\lim_{n \rightarrow \infty} [p(g(x_n), g(x_{n+1})) + p(g(y_n), g(y_{n+1})))] = \lim_{n \rightarrow \infty} \delta_n = 0$$

and similarly  $\delta^l = 0$  that is

$$\lim_{n \rightarrow \infty} [p(g(x_{n+1}), g(x_n)) + p(g(y_{n+1}), g(y_n))] = \lim_{n \rightarrow \infty} \delta_n^l = 0. \quad (12)$$

Next we show that  $\{g(x_n)\}$  and  $\{g(y_n)\}$  are  $p$ -Cauchy sequences. Let at least one of  $\{g(x_n)\}$  and  $\{g(y_n)\}$  be not a  $p$ -Cauchy sequence. Then there exists  $\varepsilon > 0$  and sequences of natural numbers  $\{m(k)\}$  and  $\{l(k)\}$  such that for every natural number  $k$ ,  $m(k) > l(k) \geq k$  and

$$p_k = p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right) \geq \varepsilon. \quad (13)$$

Now corresponding to  $l(k)$  we can choose  $m(k)$  to be the smallest positive integer for which (13) holds. Then,

$$p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)-1}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right)\right) < \varepsilon. \quad (14)$$

Further from (13) and (14), for all  $k \geq 0$ , we have

$$\begin{aligned} \varepsilon \leq p_k &\leq p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)-1}\right)\right) \\ &+ p\left(g\left(x_{m(k)-1}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right)\right) + p\left(g\left(y_{m(k)-1}\right), g\left(y_{m(k)}\right)\right) \\ &= p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)-1}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right)\right) + \delta_{m(k)-1} < \varepsilon + \delta_{m(k)-1}. \end{aligned}$$

Taking the limit as  $k \rightarrow \infty$ , we have by (12),

$$\lim_{k \rightarrow \infty} p_k = \varepsilon. \quad (15)$$

Again, for all  $k \geq 0$ , we have,

$$\begin{aligned} p_k &= p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right) \\ &\leq p\left(g\left(x_{l(k)}\right), g\left(x_{l(k)+1}\right)\right) + p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) + p\left(g\left(x_{m(k)+1}\right), g\left(x_{m(k)}\right)\right) \\ &+ p\left(g\left(y_{l(k)}\right), g\left(y_{l(k)+1}\right)\right) + p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right) + p\left(g\left(y_{m(k)+1}\right), g\left(y_{m(k)}\right)\right) \\ &= p\left(g\left(x_{l(k)}\right), g\left(x_{l(k)+1}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{l(k)+1}\right)\right) + p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) \\ &+ p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right) + p\left(g\left(x_{m(k)+1}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{m(k)+1}\right), g\left(y_{m(k)}\right)\right). \end{aligned}$$

Hence, for all  $k \geq 0$

$$p_k \leq \delta_{l(k)} + \delta_{m(k)}^l + p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) + p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right). \quad (16)$$

From (1), (4), (5), (6) and (13), for all  $k \geq 0$ , we obtain

$$\begin{aligned} p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) &= p\left(T\left(x_{l(k)}, y_{l(k)}\right), T\left(x_{m(k)}, y_{m(k)}\right)\right) \\ &\leq F\left(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}, \right. \\ &\quad \left. \varphi\left(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}\right)\right) = F\left(\frac{p_k}{2}, \varphi\left(\frac{p_k}{2}\right)\right). \end{aligned} \quad (17)$$

Also by (1), (4), (5), (6) and (13), for all  $k \geq 0$ , we have,

$$\begin{aligned} p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right) &= p\left(T\left(y_{l(k)}, x_{l(k)}\right), T\left(y_{m(k)}, x_{m(k)}\right)\right) \\ &\leq F\left(\left(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}\right), \right. \\ &\quad \left. \varphi\left(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}\right)\right) = F\left(\frac{p_k}{2}, \varphi\left(\frac{p_k}{2}\right)\right). \end{aligned} \quad (18)$$

Putting (17) and (18) in (16), for all  $k \geq 0$ , we obtain,  $p_k \leq \delta_{l(k)} + \delta_{m(k)}^l + 2F\left(\frac{p_k}{2}, \varphi\left(\frac{p_k}{2}\right)\right)$ .

Letting  $n \rightarrow \infty$  in the above inequality and using (12), (13) and (15) we obtain,

$$\varepsilon \leq 2 \lim_{k \rightarrow \infty} F \left( \frac{p_k}{2}, \varphi \left( \frac{p_k}{2} \right) \right) = 2F \left( \frac{\varepsilon}{2}, \varphi \left( \frac{\varepsilon}{2} \right) \right).$$

So,  $\frac{\varepsilon}{2} = 0$ , or  $\varphi \left( \frac{\varepsilon}{2} \right) = 0$  which is a contradiction. Therefore,  $\{g(x_n)\}$  and  $\{g(y_n)\}$  are  $p$ -Cauchy sequences in  $X$  and hence they are convergent in the  $p$ -Cauchy complete uniform space  $(X, \vartheta)$ . Let

$$\lim_{n \rightarrow \infty} T(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x \quad \text{and} \quad \lim_{n \rightarrow \infty} T(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y. \quad (19)$$

Since  $T$  and  $g$  are compatible mappings, we have by (19),

$$\lim_{n \rightarrow \infty} p(g(T(x_n, y_n)), T(g(x_n), g(y_n))) = 0 \quad (20)$$

and

$$\lim_{n \rightarrow \infty} p(g(T(y_n, x_n)), T(g(y_n), g(x_n))) = 0. \quad (21)$$

Next we prove that  $g(x) = T(x, y)$  and  $g(y) = T(y, x)$ . Let (a) hold. For all  $n \geq 0$ , we have,

$$p(g(x_n), T(g(x_n), g(y_n))) \leq p(g(x_n), g(T(x_n, y_n))) + p(g(T(x_n, y_n)), T(g(x_n), g(y_n)))$$

Taking the limit as  $n \rightarrow \infty$ , using (4), (19), (20) and the fact that  $T$  and  $g$  are continuous, we have  $p(g(x_n), T(x, y)) = 0$ .

Similarly, from (4), (19), (21) and the continuities of  $T$  and  $g$ , we have  $p(g(y_n), T(y, x)) = 0$ .

Combining the above two results we get  $g(x) = T(x, y)$  and  $g(y) = T(y, x)$ .

Next we suppose that (b) holds. By (5), (6) and (19) we have  $\{g(x_n)\}$  is non-decreasing sequence,  $g(x_n) \rightarrow x$  and  $\{g(y_n)\}$  is non-increasing sequence,  $g(y_n) \rightarrow y$  as  $n \rightarrow \infty$ . Then by (2) and (3) we have for all  $n \geq 0$ ,

$$g(x_n) \preceq x \quad \text{and} \quad g(y_n) \succeq y. \quad (22)$$

Since,  $T$  and  $g$  are compatible mappings and  $g$  is continuous, by (20) and (21) we have,

$$\lim_{n \rightarrow \infty} g(g(x_n)) = g(x) = \lim_{n \rightarrow \infty} g(T(x_n, y_n)) = \lim_{n \rightarrow \infty} T(g(x_n), g(y_n)) \quad (23)$$

and

$$\lim_{n \rightarrow \infty} g(g(y_n)) = g(y) = \lim_{n \rightarrow \infty} g(T(y_n, x_n)) = \lim_{n \rightarrow \infty} T(g(y_n), g(x_n)). \quad (24)$$

Now we have  $p(g(x), T(x, y)) \leq p(g(x), g(g(x_{n+1}))) + p(g(g(x_{n+1})), T(x, y))$ . Taking the limit as  $n \rightarrow \infty$  in the above inequality, using (4) and (23) we have,

$$\begin{aligned} p(g(x), T(x, y)) &\leq \lim_{n \rightarrow \infty} p(g(x), g(g(x_{n+1}))) + \lim_{n \rightarrow \infty} p(g(T(x_n, y_n)), T(x, y)) \\ &\leq \lim_{n \rightarrow \infty} p(T(g(x_n), g(y_n)), T(x, y)). \end{aligned}$$

Since the mapping  $g$  is monotone increasing, by (1), (22) and the above inequality, we have for all  $n \geq 0$ , Using (19)

$$\begin{aligned} p(g(x), T(x, y)) &\leq \lim_{n \rightarrow \infty} F \left( \frac{p(g(g(x_n)), g(x)) + p(g(g(y_n)), g(y))}{2}, \right. \\ &\left. \varphi \left( \frac{p(g(g(x_n)), g(x)) + p(g(g(y_n)), g(y))}{2} \right) \right) = F(p(g(x), T(x, y)), \varphi(p(g(x), T(x, y))). \end{aligned}$$



So,  $p(g(x), T(x, y)) = 0$ , or  $\varphi(p(g(x), T(x, y))) = 0$ .

That is  $g(x) = T(x, y)$  and similarly, by virtue of (4), (19) and (24) we obtain  $g(y) = T(y, x)$ . Thus we have proved that  $T$  and  $g$  have coupled coincidence point in  $X$ . This completes the proof.  $\square$

**Remark 4.** If we take  $F(s, t) = \varphi(s)$  where  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is a continuous function such that  $\varphi(0) = 0$  and  $\varphi(t) < t$  for  $t > 0$  in the above theorem then we obtain a corollary in [12].

**Corollary 1.** Let  $(X, \vartheta)$  be a Hausdorff uniform space, " $\preceq$ " is an order on  $X$  and suppose there is an  $E$ -distance  $p$  on  $X$  such that  $(X, p)$  is a  $p$ -Cauchy complete uniform space. Assume there is a function  $F \in C, \varphi \in \Phi_u$  and also suppose  $T : X \times X \rightarrow X$  and  $g : X \rightarrow X$  are such that  $T$  has the mixed  $g$ -monotone property and

$$p(T(x, y), T(u, v)) \leq F\left(\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right), \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right)\right)$$

for all  $x, y, u, v \in X$  for which comparable  $g(x), g(u)$  and comparable  $g(y), g(v)$ . Suppose  $T(X \times X) \subseteq g(X)$ ,  $g$  is  $\tau(\vartheta)$ -continuous and commutes with  $T$  and also suppose either (a)  $T$  is  $\tau(\vartheta)$ -continuous or (b)  $X$  has the following property:

- (i) if a non-decreasing sequence  $\{x_n\} \rightarrow x$ , then  $x_n \preceq x$  for all  $n$ ,
- (ii) if a non-increasing sequence  $\{y_n\} \rightarrow y$ , then  $y \preceq y_n$  for all  $n$ .

If there exist  $x_0, y_0 \in X$  such that  $g(x_0) \preceq T(x_0, y_0)$  and  $g(y_0) \succeq T(y_0, x_0)$ , then there exist  $x, y \in X$  such that  $g(x) = T(x, y)$  and  $g(y) = T(y, x)$ , that is,  $T$  and  $g$  have a coupled coincidence.

**Example 2.** Let  $X = [0, 1], p(x, y) = |x - y|$ . Then for  $x, y \in X$  and " $\preceq$ " is a partially ordered with the natural ordering of real numbers. Then  $(X, \preceq)$  is an ordered uniform space and  $(X, p)$  is a  $p$ -Cauchy complete uniform space. Let  $g : X \rightarrow X$  be defined as  $g(x) = x$  for all  $x \in X$ .

Let  $T : X \times X \rightarrow X$  be defined as  $T(x, y) = \begin{cases} \frac{x-y}{2}, & x, y \in X, x \succeq y \\ 0, & x \prec y \end{cases}$ .  $T$  obeys the mixed

$g$ -monotone property.

Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be defined as  $\varphi(s) = s$ , for  $s \in [0, \infty)$  and  $F(s, \varphi(s)) = \varphi(s)$ . Therefore  $F(s, \varphi(s)) = \varphi(s) = s \leq s$  and  $F(s, \varphi(s)) = s \Rightarrow s = 0$  or  $\varphi(s) = 0$  and  $\varphi(s) = 0 \Rightarrow s = 0$ . So  $F \in C, \varphi \in \Phi_u$ . Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in  $X$  such that,  $\lim_{n \rightarrow \infty} T(x_n, y_n) = a$ ,  $\lim_{n \rightarrow \infty} g(x_n) = a$ , and  $\lim_{n \rightarrow \infty} T(y_n, x_n) = b$ ,  $\lim_{n \rightarrow \infty} g(y_n) = b$ . Then obviously,  $a = 0$  and  $b = 0$ .

Now, for all  $n \geq 0; g(x_n) = x_n, x_n \in X$  and  $g(y_n) = y_n, y_n \in X$ ,

$$T(x_n, y_n) = \begin{cases} \frac{x_n - y_n}{2}, & \text{if } x_n \succeq y_n, \\ 0, & \text{if } x_n \prec y_n, \end{cases} \quad \text{and} \quad T(y_n, x_n) = \begin{cases} \frac{y_n - x_n}{2}, & \text{if } y_n \succeq x_n, \\ 0, & \text{if } y_n \prec x_n. \end{cases}$$

Then, it follows that

$$\lim_{n \rightarrow \infty} p(g(T(x_n, y_n)), T(g(x_n), g(y_n))) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$\lim_{n \rightarrow \infty} p(g(T(y_n, x_n)), T(g(y_n), g(x_n))) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, the mappings  $T$  and  $g$  are compatible in  $X$ . Also,  $x_0 = 0$  and for a positive number  $m$ ,  $y_0 = m$  are two points in  $X$  such that  $g(x_0) = g(0) = 0 = T(0, m) = T(x_0, y_0)$  and  $g(y_0) = g(m) = m \succeq \frac{m}{2} = T(m, 0) = T(y_0, x_0)$ . We next verify inequality (1) of Theorem 1. We take  $x, y, u, v \in X$ , such that  $g(x) \preceq g(u)$  and  $g(y) \succeq g(v)$ , that is,  $x \preceq u$  and  $y \succeq v$ .

We consider the following cases:

**Case 1:**  $x \succeq y$  and  $u \succeq v$ .

Then

$$\begin{aligned} p(T(x, y), T(u, v)) &= p\left(\frac{x-y}{2}, \frac{u-v}{2}\right) = \left|\frac{x-y}{2} - \frac{u-v}{2}\right| = \left|\frac{x-u}{2} - \frac{y-v}{2}\right| \\ &\preceq \left|\frac{x-u}{2}\right| + \left|\frac{y-v}{2}\right| = \varphi\left(\frac{|x-u|}{2} + \frac{|y-v|}{2}\right) = \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right) \\ &= F\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}, \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right)\right). \end{aligned}$$

**Case 2:**  $x \succeq y$  and  $u \prec v$ .

Then

$$\begin{aligned} p(T(x, y), T(u, v)) &= p\left(\frac{x-y}{2}, 0\right) = \left|\frac{x-y}{2}\right| = \frac{x-y}{2} = \frac{u+x-y-u}{2} \\ &= \frac{(u-y) - (u-x)}{2} \text{ (since } v \succ u) \preceq \frac{|u-x|}{2} + \frac{|v-y|}{2} \\ &= \varphi\left(\frac{|x-u|}{2} + \frac{|y-v|}{2}\right) = \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right) \\ &= F\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}, \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right)\right). \end{aligned}$$

**Case 3:**  $x \prec y$  and  $u \succeq v$ .

Then

$$\begin{aligned} p(T(x, y), T(u, v)) &= p\left(0, \frac{u-v}{2}\right) = \left|\frac{u-v}{2}\right| = \frac{u-v}{2} = \frac{u+x-v-x}{2} \\ &= \frac{(u-x) - (v-x)}{2} \text{ (since } y \succ x) \preceq \frac{|u-x|}{2} + \frac{|v-y|}{2} = \varphi\left(\frac{|x-u|}{2} + \frac{|y-v|}{2}\right) \\ &= \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right) \\ &= F\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}, \varphi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right)\right). \end{aligned}$$

**Case 4:**  $x \prec y$  and  $u \prec v$ .

Then  $T(x, y) = 0$  and  $T(u, v) = 0$ , that is  $p(T(x, y), T(u, v)) = 0$ . Obviously (1) is satisfied.

Thus it is verified that the functions  $T, g, \varphi$  satisfy all the conditions of Theorem 1. Here  $(0, 0)$  is the coupled coincidence point of  $T$  and  $g$  in  $X$ .

## REFERENCES

- [1] Aamri M., El Moutawakil D. *Common fixed point theorems for E-contractive or E-expansive maps in uniform spaces*. Acta Math. Acad. Paedagog. Nyházi. ( N.S. ) 2004, **20** (1), 83–91.
- [2] Aamri M., El Moutawakil D. *Weak compatibility and common fixed point theorems for A-contractive and E-expansive maps in uniform spaces*. Serdica Math. J. 2005, **31**, 75–86.
- [3] Altun I., Imdad M. *Some fixed point theorems on ordered uniform spaces*. Filomat 2009, **23** (3), 15–22.
- [4] Ansari A.H. *Note on "  $\varphi$ - $\psi$ -contractive type mappings and related fixed point"*. In: The 2nd Regional Conference on Mathematics And Applications, PNU, 2014, 377-380.
- [5] Ansari A.H., Chandok S., Ionescu C. *Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions*. J. Inequal. Appl. 2014 **2014**, 429–445. doi:10.1186/1029-242X-2014-429
- [6] Bhaskar T., Lakshmikantham V. *Fixed point theorems in partially ordered metric spaces and applications*. Nonlinear Anal., 2006, **65**, 1379–1393.
- [7] Choudhury B.S., Kundu A. *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*. Nonlinear Anal. 2010, **73**, 2524–2531.
- [8] Fadail Z.M., Ahmad A.G., Ansari A.H., Radenović S., Rajović R. *Some Common Fixed Point Results of Mappings in 0-Complete Metric-like Spaces via New Function*. Appl. Math. Sci. 2015, **9** (83), 4109–4127. doi:10.12988/ams.2015.53192
- [9] Hoxha E., Ansari A.H., Zoto K. *Some common fixed point results through generalized altering distances on dislocated metric spaces*. Proceedings of EIIC, 2014, September 1-5, 403–409.
- [10] Khan M.S., Swaleh M., Sessa S. *Fixed point theorems by altering distances between the points*. Bull. Austral. Math. Soc. 1984 **30**, (1) 1–9. doi:10.1017/S0004972700001659
- [11] Lakshmikantham V., Ćirić L. *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*. Nonlinear Anal. 2009, **70**, 4341–4349.
- [12] Turkoglu D., Binbasioglu D. *Coupled coincidence point theorems for compatible mappings in ordered uniform space*. Miskolc Math. Notes 2005, **16** (1), 527–541.

Received 17.02.2018

---

Ансарі А.Г., Бінбасіоглу Д., Туркоглу Д. *Результати про зв'язану точку збігу для стискуючих відображень класу С у впорядкованих рівномірних просторах* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 3–13.

У літературі існує багато робіт, пов'язаних з теорією нерухомої точки. Ця теорія має багато застосувань, тому деякі автори зацікавлені в цих застосуваннях в різних просторах. У 2009 р. Алтун І. та Імдад М. визначили відношення порядку на рівномірних просторах і поняття сумісності відображень. Ансарі А. ввів концепцію функцій С-класу. У цій статі ми вибираємо функції С-класу, що ультра змінюють відстань, та доводимо деякі теореми про зв'язану точку збігу для відображень, що задовольняють властивість змішаної g-монотонності у впорядкованих рівномірних просторах. Ми також наводимо відповідні приклади.

*Ключові слова і фрази:* зв'язана точка збігу, С-клас відображень, впорядкований рівномірний простір.