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NOTE ON BASES IN ALGEBRAS OF ANALYTIC FUNCTIONS ON BANACH SPACES

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of continuous algebraically independent homogeneous polynomials on a complex Banach space X. We consider the following question: Under which conditions polynomials $\{P_1^{k_1}\cdots P_n^{k_n}\}$ form a Schauder (perhaps absolute) basis in the minimal subalgebra of entire functions of bounded type on X which contains the sequence $\{P_n\}_{n=0}^{\infty}$? In the paper we study the following examples: when P_n are coordinate functionals on c_0, and when P_n are symmetric polynomials on ℓ_1 and on $L_\infty[0,1]$. We can see that for some cases $\{P_1^{k_1}\cdots P_n^{k_n}\}$ is a Schauder basis which is not absolute but for some cases it is absolute.

Key words and phrases: Schauder bases, analytic functions on Banach spaces, symmetric analytic functions.

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INTRODUCTION AND PRELIMINARIES

Let X be a complex Banach space. We recall that $H_b(X)$ is the algebra of all entire analytic functions on X which are bounded on bounded subsets. It is well known that $H_b(X)$ endowed with the metrisable topology generated by the countable family of norms

$$\|f\|_r = \sup_{\|x\|\le r} |f(x)|, \quad r \in \mathbb{Q}_+, f \in H_b(X),$$

is a Fréchet algebra and the space $\mathcal{P}(X)$ of all continuous polynomials on X is a dense subalgebra in $H_b(X)$.

Let $\mathbb{P} = \{P_n\}_{n=0}^{\infty}$ be a sequence of continuous algebraically independent homogeneous polynomials on X with $\|P_n\| = 1$ and $P_0 = 1$. We denote by $\mathcal{P}_{\mathbb{P}}(X)$ the algebra of all polynomials generated by the sequence $\mathbb{P}$ and by $H_{b\mathbb{P}}(X)$ its closure in $H_b(X)$.

Clearly,

$$\{P^{(k)} = P_1^{k_1}\cdots P_n^{k_n} : (k) = (k_1,\ldots,k_n), \quad n = 0,1,2,\ldots\}$$

is a linear basis in $\mathcal{P}_{\mathbb{P}}(X)$, and so the span of $P^{(k)}$ is dense in $H_{b\mathbb{P}}(X)$. Here we set $P_0 = 1$. This work is motivated by the following natural question: *Under which conditions $\{P^{(k)}\}$ is a*

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Schauder (perhaps absolute) basis in $H_{b\mathbb{P}}(X)$? The main result of this paper is that depending on the sequence $\mathbb{P}$ we can have different answers on this question. In the paper we study the following examples: when P_n are coordinate functionals on c_0, and when P_n are symmetric polynomials on ℓ_1 and on $L_\infty[0,1]$.

Let us recall some definitions in the theory of locally convex spaces (see e.g. [14]).

A sequence of subspaces $\{E_n\}_n$ of a locally convex space E is a *Schauder decomposition* of E if for each x in E there exists a unique sequence of vectors $(x_n)_n$, $x_n \in E_n$, such that

$$x = \sum_{n=1}^{\infty} x_n := \lim_{m\to\infty} \sum_{n=1}^{m} x_n$$

and the projections $(u_m)_{m=1}^{\infty}$ defined by

$$u_m\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{m} x_n$$

are continuous. A Schauder decomposition $\{E_n\}_n$ of a locally convex space E is *absolute* if for each semi-norm $p \in cs(E)$,

$$q\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} p(x_n)$$

defines a continuous semi-norm on E. Finally, a Schauder decomposition $\{E_n\}_n$ of a locally convex space E is *global* if for all $r > 0$, all $x = \sum_{n=1}^{\infty} x_n \in E$ with all $x_n \in E_n$

$$\sum_{n=1}^{\infty} r^n x_n \in E$$

and for each $p \in cs(E)$,

$$p_r\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} r^n p(x_n)$$

defines a continuous semi-norm on E.

If each E_n is a finite dimensional subspace, then the decomposition is called *finite dimensional*. If each E_n is one dimensional and e_n spans E_n, then $(e_n)_{n=1}^{\infty}$ is a Schauder basis.

1 MAIN RESULTS

Let $X = c_0$ and $P_n = e_n^*$ be the coordinate functionals on c_0. Then

$$P^{(k)}(x) = (e_1^*(x))^{k_1} \cdots (e_n^*(x))^{k_n} = x_1^{k_1} \cdots x_n^{k_n}, \quad n = 0, 1, 2, \ldots,$$

are so-called $k_1 + \ldots + k_n$-homogeneous monomials on c_0. Since every polynomial on c_0 can be approximated by polynomials of finite type and every polynomial of finite type belongs to linear span of monomials, we have that $H_{b\mathbb{P}}(c_0) = H_b(c_0)$. Moreover, in [8] it is proved that the monomials $\{P^{(k)}\}$ endowed with some special order form a Schauder basis for $H_b(c_0)$ which however is not absolute. Indeed, if it is absolute, then the subset of monomials $\{P^{(k)}\colon \deg P^{(k)} = m\}$ form an unconditional basis in the Banach space of all m-homogeneous polynomials $\mathcal{P}(^m c_0)$. But it is not so for $m > 1$, according to [6].

Let now $\deg P_n = n$. So if $P \in \mathcal{P}_{\mathbb{P}}(X)$ and $\deg P = m$, then

$$P(x) = \sum_{n=0}^{m} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1\ldots k_n} P_1^{k_1}(x) \cdots P_n^{k_n}(x), \qquad a_{k_1\ldots k_n} \in \mathbb{C}. \tag{1}$$

We denote $\mathcal{P}_{\mathbb{P}}(^nX)$ the linear space of all n-homogeneous polynomials in $\mathcal{P}_{\mathbb{P}}(X)$. From (1) it follows that $\mathcal{P}_{\mathbb{P}}(^nX)$ is finite dimensional, polynomials $\{P_1^{k_1} \cdots P_n^{k_n} : k_1 + 2k_2 + \ldots + nk_n = n\}$ form a linear basis in $\mathcal{P}_{\mathbb{P}}(^nX)$ and $\dim \mathcal{P}_{\mathbb{P}}(^nX) = \mathfrak{p}(n)$, where $\mathfrak{p}(n)$ is the number of partitions of n.

Proposition 1. *Let* $\deg P_n = n$. *Then the sequence of spaces* $\{\mathcal{P}_{\mathbb{P}}(^nX)\}_{n=0}^{\infty}$ *is a global finite dimensional Schauder decomposition for* $H_{b\mathbb{P}}(X)$. *Here* $\mathcal{P}_{\mathbb{P}}(^0X) = \mathbb{C}$.

Proof. In [14] it is proved that $\{\mathcal{P}(^nX)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $H_b(X)$. Since $H_{b\mathbb{P}}(X)$ is a closed subspace of $H_b(X)$, $\mathcal{P}_{\mathbb{P}}(^nX) = \mathcal{P}(^nX) \cap H_{b\mathbb{P}}(X)$ is a global Schauder decomposition for $H_{b\mathbb{P}}(X)$. ☐

Note that in the general case the existence of a finite dimensional Schauder decomposition does not imply the existence of a Schauder basis (see [13]).

Algebras of symmetric functions on ℓ_1 or $L_1[0,1]$ deliver us interesting examples of $H_{b\mathbb{P}}(X)$. By a symmetric function on ℓ_1 we mean a function which is invariant under any reordering of the basis in ℓ_1. We use the notations $\mathcal{H}_{bs}(\ell_1)$ for the algebra of all symmetric analytic functions on ℓ_1 that are bounded on bounded sets.

In [12] it is proved that the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \qquad k = 1, 2, \ldots,$$

form an algebraic basis in the algebra of all symmetric polynomials on ℓ_1. This means that the polynomials $\{F_k\}$ are algebraically independent and their algebraic combinations coincide with the space of all symmetric polynomials $\mathcal{P}_s(\ell_1)$ on ℓ_1. Thus, $\{F^{(k)} = F_1^{k_1} \cdots F_k^{k_n}\}$ forms a linear basis in $\mathcal{P}_s(\ell_1)$ or, in other words, $\mathcal{H}_{bs}(\ell_1) = H_{b\mathbb{F}}(\ell_1)$.

The algebras $\mathcal{H}_{bs}(\ell_p)$ and their spectrum were investigated in [2–4, 10].

In [5] was constructed an example of a symmetric analytic function on ℓ_1 which is not of bounded type.

The algebra $\mathcal{P}_{bs}(\ell_1)$ has other natural algebraic bases. For us it is important the basis $\{G_n\}$:

$$G_n(x) = \sum_{k_1<\cdots<k_n}^{\infty} x_{k_1} \cdots x_{k_n}$$

and $G_0 := 1$. It is known [3] that $\|G_n\| = 1/n!$. By the Waring's formula we have

$$G_k = \sum_{\lambda_1+2\lambda_2+\ldots+k\lambda_k=k} (-1)^{k-(\lambda_1+\lambda_2+\ldots+\lambda_k)} \frac{1}{\lambda_1! 1^{\lambda_1} \cdots \lambda_k! k^{\lambda_k}} F_1^{\lambda_1} \cdots F_k^{\lambda_k}.$$

Note that in the general case, algebra $\mathcal{P}_{\mathbb{P}}(X)$ admits a lot of algebraic bases of homogeneous polynomials and linear bases as well. Indeed, if $\deg P_n = n$, then we can set $Q_1 = a_{11}P_1$ and

$$Q_n = a_{n1}Q_{n-1}P_1 + a_{n2}Q_{n-2}P_1 + \cdots + a_{nn}P_n$$

for some complex numbers a_{ij} such that $a_{ii} \neq 0$. Then polynomials Q_n form an algebraic basis and $Q^{(k)} = Q_1^{k_1} \cdots Q_k^{k_n}$ form a linear basis in $\mathcal{P}_{\mathbb{P}}(X)$. Note that there is a linear basis of $\mathcal{P}_s(^n\ell_1)$ which is not generated by an algebraic basis. For a given partition $(k) = (k_1, \ldots, k_n)$ such that $|(k)| = k_1 + \ldots + k_n = n$ we denote by $M^{(k)}(x) = \sum_{i_1 \neq \ldots \neq i_n} x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}$. Then $\{M^{(k)}\}_{|k|=0}^{\infty}$ is a linear basis in $\mathcal{P}_s(^n\ell_1)$.

We need the following simple lemma which probably is well known (c.f. [1, Theorem 2.1]).

Lemma 1. *Let $P_1, \ldots, P_N$ be algebraically independent polynomials from a Banach space X to $\mathbb{C}$ such that the map*

$$X \ni x \mapsto (P_1(x), \ldots, P_N(x)) \in \mathbb{C}^N$$

is onto. Then there is an isomorphism I_N from the minimal subalgebra of entire functions generated by $P_1, \ldots, P_N$ onto the algebra of all entire functions on $\mathbb{C}^N$, $H(\mathbb{C}^N)$ such that $I_N(P_k) = t_k$, $k = 1, \ldots, N$, $(t_1, \ldots, t_N) \in \mathbb{C}^N$.

Theorem 1. *Let $P_n = n!G_n$. Then $\{P^{(k)} = P_1^{k_1} \cdots P_k^{k_n}\}$ is a Schauder basis in $\mathcal{H}_{bs}(\ell_1)$.*

Proof. Let r_N be the operator of restriction onto subspace $V_N \subset \ell_1$ spanned on the standard basis vectors $e_1, \ldots, e_N$. Clearly that $r_N(G_k) = 0$ if $N < k$. Also, we know that $r_N(P_1), \ldots, r_N(P_N)$ are algebraically independent and the map

$$\ell_1 \ni x \mapsto (r_N(P_1), \ldots, r_N(P_N)) \in \mathbb{C}^N$$

is onto. So from Lemma 1 we have the isomorphism I_N from the minimal subalgebra of entire functions $H_s(V_N)$ on V_N, generated by $r_N(P_1), \ldots, r_N(P_N)$ to $H(\mathbb{C}^N)$. By the same reason, we have the isomorphism $\mathcal{I}_N$ from the minimal subalgebra of entire functions $H_s^N(\ell_1)$ on ℓ_1, generated by $P_1, \ldots, P_N$ to $H(\mathbb{C}^N)$. From here we have that the operator of restriction $r_N \colon \mathcal{H}_{bs}(\ell_1) \to H_s(V_N)$ is onto and $\mathcal{I}_N^{-1} \circ I_N$ is the "extension" isomorphism from $H_s(V_n)$ to $H_s^N(\ell_1)$. Also, we know [7, p. 240] that monomials on $t_1, \ldots, t_n$ form an absolute basis in $H(\mathbb{C}^N)$. Thus $P_1^{k_1} \cdots P_k^{k_n}$ for $k \leq N$ form an absolute basis in $H_s^N(\ell_1)$ and so all projections T_m to finite dimensional subspaces W_m generated by these basis vectors are continuous. Thus any projection u_m from $\mathcal{H}_{bs}(\ell_1)$ to W_m can be represented by

$$u_k = T_n \circ \mathcal{I}_N^{-1} \circ I_N \circ r_N$$

and so is continuous. □

Let us denote $\mathcal{A}_{us}(B_{\ell_1})$ the completion of $\mathcal{H}_{bs}(\ell_1)$ by the norm $\| \cdot \|_1$ that is, the sup-norm on the unit ball B_{ℓ_1} of ℓ_1. Such algebra consists of analytic and uniformly continuous functions on B_{ℓ_1} and was considered in [1].

Theorem 2. *$\{F^{(k)} = F_1^{k_1} \cdots F_k^{k_n}\}$ cannot be an absolute Schauder basis in $\mathcal{H}_{bs}(\ell_1)$ and cannot be an unconditional basis in $\mathcal{A}_{us}(\ell_1)$.*

Proof. Let us remind that a sequence $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis of a Banach space, if there exists a constant M such that for every $\sum_{n=1}^{m} a_n e_n$ and for every $\varepsilon_1, \ldots, \varepsilon_n$, $|\varepsilon_k| = 1$, we have

$$M \Big\| \sum_{n=1}^{m} a_n e_n \Big\| \geq \Big\| \sum_{n=1}^{m} \varepsilon_n a_n e_n \Big\|. \tag{2}$$

It is well known in combinatorics that

$$\sum_{\lambda_1+2\lambda_2+\ldots+k\lambda_k=k} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} = 1. \tag{3}$$

Let $g(x) = \sum_{n=0}^{\infty} G_n(x)$. Since $\|G_n\| = \frac{1}{n!}$, $g(x) \in \mathcal{H}_{bs}(\ell_1) \subset \mathcal{A}_{us}(\ell_1)$. According to the Waring's formula,

$$g(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} (-1)^{n-(k_1+k_2+\ldots+k_n)} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}} F_1^{k_1}\cdots F_n^{k_n}.$$

We set $\varepsilon_{(k)} = \varepsilon_{(k_1,\ldots,k_n)} = (-1)^{(k_1+k_2+\ldots+k_n+n)}$. According to (3) and $\|F_1^{k_1}\cdots F_n^{k_n}\|_1 = 1$ the series

$$\sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}} F_1^{k_1}\cdots F_n^{k_n}$$

diverges. It contradicts (2). Also, if $\{F^{(k)}\}$ is an absolute basis in $\mathcal{H}_{bs}(\ell_1)$, then the series

$$\sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \left\|(-1)^{n-(k_1+k_2+\ldots+k_n)} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}} F_1^{k_1}\cdots F_n^{k_n}\right\|_1 = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}}$$

should be convergent. But it is not so. □

Algebra of symmetric analytic functions $H_{bs}(L_\infty[0,1])$ on $L_\infty[0,1]$ consists of analytic functions which are invariant with respect to all measurable automorphisms of $[0,1]$.

According to [9] polynomials $P_n = R_n$, where

$$R_n(x) = \int_{[0,1]} (x(t))^n dt, \quad n \in \mathbb{N},$$

form an algebraic basis in the algebra of all symmetric polynomials on $L_\infty[0,1]$. In [11] it is proved that $\{R^{(k)} = R_1^{k_1}\cdots R_k^{k_n}\}$ is an absolute basis in $H_{bs}(L_\infty[0,1])$.

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Чернега І., Загороднюк А. *Про базиси в алгебрах аналітичних функцій на банахових просторах* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 42–47.

Нехай $\{P_n\}_{n=0}^{\infty}$ — послідовність неперервних алгебраїчно незалежних однорідних поліномів на комплексному банаховому просторі X. Розглянемо наступне питання: За яких умов поліноми $\{P_1^{k_1} \cdots P_n^{k_n}\}$ утворюють базис Шаудера (можливо абсолютний) в мінімальній підалгебрі цілих функцій обмеженого типу на X, які містять послідовність $\{P_n\}_{n=0}^{\infty}$? У роботі досліджуються наступні приклади: коли P_n є координатними функціоналами c_0, і коли P_n є симетричними поліномами на ℓ_1 і на $L_\infty[0,1]$. Ми бачимо, що у деяких випадках $\{P_1^{k_1} \cdots P_n^{k_n}\}$ є базисом Шаудера який не є абсолютним, але в деяких випадках є абсолютним.

Ключові слова і фрази: базис Шаудера, аналітичні функції на банахових просторах, симетричні аналітичні функції.