UDK 539.3

Y. Z. Povstenko✉

FROM CHEMICAL POTENTIAL TENSOR AND CONCENTRATION TENSOR TO NONLOCAL CONTINUUM THEORIES

The nontraditional thermodynamic pair (the chemical potential tensor and the concentration tensor) were introduced in the pioneering studies by Pidstryhach (spelled also Podstrigach). Eliminating the chemical potential tensor and the concentration tensor from the constitutive equations for the stress tensor, Pidstryhach obtained the space-time-nonlocal equation for the stress tensor. In this survey, we discuss the development of Pidstryhach' scientific ideas.

Key words: *chemical potential tensor, concentration tensor, nonlocal theory of elasticity, integrals and derivatives of fractional order.*

Introduction. The outstanding Ukrainian scientist Academician Yaroslav Stepanovych Pidstryhach (spelled also Podstrigach) had wide range of research interests. He was the author of fundamental works on nonequilibrium thermodynamics, thermomechanics, thermodiffusion in elastic solids, surface effects, mechanics of bodies with coatings, nonlocal theory of elasticity and so forth. His ideas on interaction of fields of different physical natures were used in solving many practical problems of modern engineering.

The classical theory of heat conduction dates back to 1822 when Fourier [67] postulated the famous Fourier law. A few years later, Fourier's disciple Duhamel coupled the temperature field and the solid deformation [54] and pioneered studies on thermoelasticity. The Duhamel–Neumann equation [54, 88] states that stresses in a solid depend not only on strains but also on the temperature field. Coupling of temperature and strains in the heat conduction equation was set up by Biot [47] using irreversible thermodynamics. The Fourier law is a phenomenological law which states the proportionality of the flux to the gradient of the transported quantity. Similarly, the Fick law [66] in the theory of diffusion states the linear dependence between the matter flux and the concentration gradient. The idea of matter redistribution during deformation was proposed by Gorsky [70]. This idea was used as the basis for the diffusion theory of elastic aftereffect and was developed by Konobeevsky [11] and Fastov [44]. The modern theory of thermodiffusion in a deformable solid based on the methods of continuum mechanics and irreversible thermodynamics was formulated by Pidstryhach [18–21, 36] and was developed by Nowacki [91, 92].

1. The chemical potential tensor and the concentration tensor. New nontraditional thermodynamic pair (the chemical potential tensor $\boldsymbol{\varphi}$ and the concentration tensor $\mathbf{c}$) was introduced in the pioneering studies of Pidstryhach [18, 20, 33, 34] (see also [37, 39, 121]). The corresponding Gibbs equation was written as

$$df = -sdT + \frac{1}{\rho}\boldsymbol{\sigma} : d\mathbf{e} + \boldsymbol{\varphi} : d\mathbf{c}\,. \tag{1}$$

Here f is the free energy density, s is the entropy density, T denotes temperature, ρ stands for the mass density, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{e}$ is the strain tensor, the colon symbol denotes the double inner product.

The chemical potential tensor appears naturally in the theory of elastic mixtures proposed by Bowen [48–50] and developed in many subsequent studies (see, for example, [45, 51, 133]):

✉ j.povstenko@ajd.czest.pl

$$\boldsymbol{\varphi}_{(k)} = f_{(k)}\mathbf{I} - \frac{\boldsymbol{\sigma}_{(k)}}{\rho_{(k)}}, \tag{2}$$

where the index in parentheses refers to the kth component of a mixture, $\boldsymbol{\sigma}_{(k)}$ is the partial stress tensor, $\mathbf{I}$ stands for the unit tensor.

When $\boldsymbol{\sigma}_{(k)} = -p_{(k)}\mathbf{I}$, where $p_{(k)}$ is the pressure, the expression (2) reduces to the usual scalar chemical potential

$$\varphi_{\{k)} = f_{(k)} + \frac{p_{(k)}}{\rho_{(k)}}. \tag{3}$$

The tensorial character of the chemical potential means that for solids the work of bringing the substance into volume depends on the direction. Similarly, the tensorial nature of the chemical potential can be associated with the crystal growth direction [129].

The concentration tensor $\mathbf{c}$ was connected with the anisotropic mass distribution [34] and with microscopic deformation [23, 28], was interpreted as the mass displacement tensor [116, 117, 119, 120]. A point defect in elastic continuum is described by the volume force

$$\mathbf{F}(\mathbf{r}) = -\mathbf{a}\cdot\nabla\delta(\mathbf{r}), \tag{4}$$

where $\mathbf{a}$ is a force dipole tensor [65, 81], $\delta(\mathbf{r})$ is the Dirac delta function, ∇ denotes the gradient operator. Such elastic dipoles give rise to dia- and paraelasticity and after averaging can be treated as the concentration tensor $\mathbf{c}$ [41]). For isotropic point defects $\mathbf{a} = a\mathbf{I}$ the averaging procedure resulting in the scalar concentration c was described by Kosevich and Kovalev [12]. The averaged tensor of elastic dipoles [81], in the mechanics of continuum medium can be viewed as a tensor state parameter − the analogue of the usual scalar concentration of dissolved substance.

Now we recall the main result of the paper [28]. Suppose that the principle of local equilibrium is fulfilled for each component of many-component deformable solid

$$df_{(k)} = -s_{(k)}dT_{(k)} + \frac{1}{\rho_{(k)}}\boldsymbol{\sigma}_{(k)} : d\mathbf{e}_{(k)}, \qquad k = 1,2,\ldots,N, \tag{5}$$

According to Eringen [62], in the linear approximation the partial strain tensor $\mathbf{e}_{(k)}$ can be splitted into two parts

$$\mathbf{e}_{(k)} = \mathbf{e} + \boldsymbol{\psi}_{(k)}, \tag{6}$$

where $\mathbf{e}$ is the strain of the mass center, $\boldsymbol{\psi}_{(k)}$ is the strain connected with microstructure. Hence,

$$\rho_{(k)}df_{(k)} = -\rho_{(k)}s_{(k)}dT_{(k)} + \boldsymbol{\sigma}_{(k)} : d\mathbf{e} + \boldsymbol{\sigma}_{(k)} : d\boldsymbol{\psi}_{(k)}. \tag{7}$$

By summing over k from 1 to N and assuming that

$$T_{(1)} = T_{(2)} = \ldots = T_{(N)} = T, \tag{8}$$

$$\rho f = \sum_{k=1}^{N}\rho_{(k)}f_{(k)}, \qquad \rho s = \sum_{k=1}^{N}\rho_{(k)}s_{(k)}, \qquad \boldsymbol{\sigma} = \sum_{k=1}^{N}\boldsymbol{\sigma}_{(k)}, \tag{9}$$

we get

$$df = -sdT + \frac{1}{\rho}\boldsymbol{\sigma} : d\mathbf{e} + \frac{1}{\rho}\sum_{k=1}^{N}\rho_{(k)}\left(\frac{\boldsymbol{\sigma}_{(k)}}{\rho_{(k)}} - f_{(k)}\mathbf{I}\right) : d\boldsymbol{\psi}_{(k)}. \tag{10}$$

If we introduce the chemical potential tensor $\boldsymbol{\varphi}_{(k)}$ according to the expression

(2) and define the concentration tensor $\mathbf{c}_{(k)}$ as

$$d\mathbf{c}_{(k)} = -\frac{\rho_{(k)}}{\rho} d\boldsymbol{\psi}_{(k)}, \tag{11}$$

we arrive at the Gibbs equation

$$df = -sdT + \frac{1}{\rho}\boldsymbol{\sigma} : d\mathbf{e} + \sum_{k=1}^{N} \boldsymbol{\varphi}_{(k)} : d\mathbf{c}_{(k)} \tag{12}$$

postulated by Pidstryhach.

In the nonlinear case, the chemical potential tensors connected with various stress tensors (Piola – Kirchhoff, Cauchy) were used by Grinfeld [3–5, 72, 73] as a foundation for investigation of phase equilibria in elastic media. Samohýl and Pabst [124] also employed the chemical potential tensor to study phase equilibrium of materials with arbitrary symmetry. The theory of chemoplasticity, based on invariants of the stress and chemical potential tensors, was proposed in [14–17]. The tensorial character of the chemical potential in the mechanochemistry was discussed by Rusanov [116, 117]. The interested reader is also referred to the review papers [118, 119], where historical background and additional references can be found. The role of the chemical potential can also be played by the Eshelby tensor, see [85, 93, 122–125, 139]. The chemical potential tensor was usefully employed in many subsequent studies (see [6–10, 27, 42, 43, 135], among others). The tensor character of the considered thermodynamic quantities is usually much more pronounced in surface thermodynamics [25, 37, 38, 116–120].

It should be noted that together with the chemical potential tensor $\boldsymbol{\varphi}$ and the concentration tensor $\mathbf{c}$ other tensorial termodynamic parameters can be considered, for example, the enthalpy tensor [28, 52, 139]. Povstenko [28] introduced the enthalpy tensor

$$\mathbf{h}_{(k)} = u_{(k)}\mathbf{I} - \frac{\boldsymbol{\sigma}_{(k)}}{\rho_{(k)}}, \tag{13}$$

where $u_{(k)}$ is the inner energy density. The enthalpy tensor is connected with the chemical potential tensor by the Legendre transformation

$$\mathbf{h}_{(k)} = \boldsymbol{\varphi}_{(k)} + s_{(k)}T_{(k)}\mathbf{I}. \tag{14}$$

In a Cosserat continuum, along with the chemical potential tensor defined by Eq. (2), the couple chemical potential tensor $\boldsymbol{\xi}_{(k)}$ can be introduced [23, 29]:

$$\boldsymbol{\xi}_{(k)} = -\frac{\boldsymbol{\tau}_{(k)}}{\rho_{(k)}}, \tag{15}$$

where $\boldsymbol{\tau}_{(k)}$ is the partial couple stress tensor. The partial bend-twist tensor $\boldsymbol{\varkappa}_{(k)}$ can also be splitted into the bend-twist tensor of the mass center $\boldsymbol{\varkappa}$ and the bend-twist tensor $\boldsymbol{\chi}_{(k)}$ connected with microstructure

$$\boldsymbol{\varkappa}_{(k)} = \boldsymbol{\varkappa} + \boldsymbol{\chi}_{(k)}. \tag{16}$$

The couple concentration tensor $\boldsymbol{\eta}_{(k)}$, similarly to (11), can be defined as [23, 29]

$$d\boldsymbol{\eta}_{(k)} = -\frac{\rho_{(k)}}{\rho} d\boldsymbol{\chi}_{(k)}. \tag{17}$$

The adopted terminology is induced by the analogy with the stress tensor and the couple stress tensor.

In parallel with Eq. (4), in a Cosserat continuum with defects modeled by volume couples

$$\mathbf{M}(\mathbf{r}) = -\mathbf{b} \cdot \nabla \delta(\mathbf{r}), \tag{18}$$

where $\mathbf{b}$ is a couple dipole tensor, the averaging procedure leads to the couple concentration tensor.

Investigation of heat conduction in media with complex internal structure can also lead to further generalizations of the theory. For Cosserat surfaces, Green and Naghdi [71] and Rubin [115] introduced the conventional temperature and the quantities representing the effects of temperature variations along the thickness of a shell-like body. Woźniak [137, 138] considered the microtemperature vector; for this quantity Grot [74] obtained the corresponding heat conduction equation. The theory of thermoelasticity with microtemperatures was intensively studied (see, e.g., [53, 77–79, 126–128, 132], where additional references can be found). Stuke [130, 131] emphasized that under extreme non-equilibrium conditions, for example, in shock waves, the temperature and entropy can be described by the second rank tensors.

2. The nonlocal theory of elasticity. The local dependence of one physical quantity (an effect E) at a point $\mathbf{x}$ at time t on another physical quantity (a cause C) at the same point and at the same time can be written as

$$E(x,y,z,t) = F\left[C(x,y,z,t)\right]. \tag{19}$$

For materials with time-nonlocality the effect E at a point $\mathbf{x}$ at time t depends on the history of causes at a point $\mathbf{x}$ at all past and present times:

$$E(x,y,z,t) = \int_0^t \gamma_T(t-\tau) F[C(x,y,z,\tau)]\, d\tau, \tag{20}$$

where $\gamma_T(t-\tau)$ is the time-nonlocality kernel.

In the case of space-nonlocality the effect E at a point $\mathbf{x}$ at time t depends on causes at all the points $\mathbf{x}$ at time t:

$$E(x,y,z,t) = \int_V \gamma_S(|\mathbf{x}-\mathbf{x}'|)\, F[C(x',y',z',t)]\, dx'\, dy'\, dz' \tag{21}$$

where $\gamma_S(|\mathbf{x}-\mathbf{x}'|)$ is the space-nonlocality kernel.

Space-time-nonlocality means that the effect E at a point $\mathbf{x}$ at time t depends on causes at all the points $\mathbf{x}'$ and at all the times prior to and at time t:

$$E(x,y,z,t) = \int_0^t \int_V \gamma_{ST}(|\mathbf{x}-\mathbf{x}'|, t-\tau)\, F[C(x',y',z',\tau)]\, dx'\, dy'\, dz\, d\tau. \tag{22}$$

Here $\gamma_{ST}(|\mathbf{x}-\mathbf{x}'|, t-\tau)$ denotes the space-time-nonlocality kernel.

Time nonlocality describes memory (history) effects, space nonlocality deals with the long-range interaction, represents attempts to extend the continuum approach to smaller length scales and to link some aspects of lattice mechanics to continuum theories. In the second half of the twentieth century, considerable research efforts has been expended to develop the nonlocal theory of elasticity and to solve various problems of continuum mechanics using this theory. In the case of the theory of elasticity, the strain tensor $\mathbf{e}$ is a cause and the stress tensor $\boldsymbol{\sigma}$ is an effect. Assuming the linear Hooke law with λ and μ being the Lamé constants, the equations above are rewritten as

$$\boldsymbol{\sigma}(x,y,z,t) = 2\mu\, \mathbf{e}(x,y,z,t) + \lambda\, \mathrm{tr}\, \mathbf{e}(x,y,z,t)\, \mathbf{I}, \tag{23}$$

$$\boldsymbol{\sigma}(x,y,z,t) = \int_0^t \gamma_T(t-\tau)[2\mu\,\mathbf{e}(x,y,z,t) + \lambda\,\mathrm{tr}\,\mathbf{e}(x,y,z,t)\,\mathbf{I}]\,d\tau\,, \tag{24}$$

$$\boldsymbol{\sigma}(x,y,z,t) = \int_V \gamma_S(|\mathbf{x}-\mathbf{x}'|)[2\mu\,\mathbf{e}(x',y',z',t) + \lambda\,\mathrm{tr}\,\mathbf{e}(x',y',z',t)\,\mathbf{I}]\,dx'\,dy'\,dz'\,, \tag{25}$$

$$\boldsymbol{\sigma}(x,y,z,t) = \int_0^t \int_V \gamma_{ST}(|\mathbf{x}-\mathbf{x}'|, t-\tau)[2\mu\,\mathbf{e}(x',y',z',\tau) + \lambda\,\mathrm{tr}\,\mathbf{e}(x',y',z',\tau)\,\mathbf{I}]\,dx'\,dy'\,dz'\,d\tau\,. \tag{26}$$

The nonlocal moduli $\gamma_T(t-\tau)$, $\gamma_S(|\mathbf{x}-\mathbf{x}'|)$, and $\gamma_{ST}(|\mathbf{x}-\mathbf{x}'|, t-\tau)$ appearing in (24)–(26) depend on parameters proportional to a characteristic time ratio and a characteristic length ratio [64]:

$$\delta = \frac{t}{T} = \frac{\text{Internal characteristic time}}{\text{External characteristic time}}\,, \tag{27}$$

$$\varepsilon = \frac{\ell}{L} = \frac{\text{Internal characteristic length}}{\text{External characteristic length}}\,. \tag{28}$$

In Eq. (27) the *internal characteristic time* t may be the relaxation time or signal travel time between molecules and the *external characteristic time* T may be the duration of applied loads or period of oscillations. In Eq. (28) the *internal characteristic length* ℓ could be the lattice parameter, granular size or molecular diameter and the *external characteristic length* L could be the wave-length, crack length, body thickness, etc. When $\delta \to 0$, the memory disappears. When $\varepsilon \to 0$, the space-nonlocality effects are eliminated [64]. Eringen ascertained the properties of space-nonlocal kernels $\gamma_S(|\mathbf{x}-\mathbf{x}'|)$ and found several different forms giving a perfect match with the Born – Kármán model of the atomic lattice dynamics and the atomic dispersion curves. In particular, Eringen [60] proposed the following nonlocal modulus:

$$\gamma_S(|\mathbf{x}-\mathbf{x}'|) = \frac{1}{8(\pi\zeta)^{3/2}} \exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\zeta}\right), \tag{29}$$

where ζ is an appropriate constant connected with a characteristic length ratio.

The essentials of the nonlocal theory were established by Podstrigach [35], Kröner [82], Eringen [58, 59, 64], Edelen [55] and Kunin [13, 83, 84]. The space-nonlocal theory reduces to the classical theory of elasticity in the long-wavelength limit and to the atomic lattice theory in the short-wavelength limit. The nonlocal theory is effective in removing nonphysical singularities occurring at point defects [30], dislocations [22, 26, 31, 32, 56, 61, 96], disclinations [26, 101, 107], points of applications of singular forces [26, 95, 106], cracks [46, 57], etc. [24, 99, 100, 103].

3. The nonlocal theories of heat conduction and diffusion. The classical theory of heat conduction is based on the local Fourier law [67], which relates the heat flux vector $\mathbf{q}$ to the temperature gradient

$$\mathbf{q} = -k_T \nabla T\,, \tag{30}$$

where k_T is the thermal conductivity of a solid. From a mathematical viewpoint, the Fourier law [67] in the theory of heat conduction and the Fick law [66] in the theory of diffusion,

$$\mathbf{J} = -k\nabla c\,, \tag{31}$$

where $\mathbf{J}$ is the matter flux, c is concentration, and k denotes the diffusion conductivity, are identical.

The classical heat conduction and diffusion equations based on the Fourier and Fick laws, respectively, are quite acceptable for different physical situations. However, many theoretical and experimental studies testify that in media with complex internal structure (porous, random and granular materials, semiconductors, polymers, glasses, etc.) the standard parabolic equations are no longer sufficiently accurate. Nonclassical theories, in which the Fourier law and the Fick law as well as the standard heat conduction and diffusion equations are replaced by more general equations, constantly attract the attention of researchers. Some of these generalizations were formulated in terms of heat conduction, others in terms of diffusion. In this paper, we will discuss diffusion, but it is obvious that the discussion also concerns heat conduction.

The general time-nonlocal generalization of the Fick law (the Fourier law) was considered in [75, 89, 90]. This constitutive equation can be written as

$$\mathbf{J}(x,y,z,t) = -k\int_0^t \gamma_T(t-\tau)\nabla c(x,y,z,\tau)\,d\tau\,. \tag{32}$$

Similarly,

$$\mathbf{J}(x,y,z,t) = -k\int_V \gamma_S(|\mathbf{x}-\mathbf{x}'|)\nabla' c(x',y',z',t)\,dx'\,dy'\,dz'\,, \tag{33}$$

$$\mathbf{J}(x,y,z,t) = -k\int_0^t\int_V \gamma_{ST}(|\mathbf{x}-\mathbf{x}'|,t-\tau)\nabla' c(x',y',z',\tau)\,dx'\,dy'\,dz'\,d\tau\,. \tag{34}$$

Eringen [63] considered the space-nonlocal constitutive equation (33) with the exponential kernel $\gamma_S(|\mathbf{x}-\mathbf{x}'|)$. The particular case of the time-nonlocal constitutive equation (32) with the "long-tail" power kernel [97, 98, 111, 113]

$$\mathbf{J}(x,y,z,t) = -\frac{k}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\nabla c(x,y,z,\tau)\,d\tau, \qquad 0<\alpha\le 1\,, \tag{35}$$

$$\mathbf{J}(x,y,z,t) = -\frac{k}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}\nabla c(x,y,z,\tau)\,d\tau, \quad 1<\alpha\le 2\,, \tag{36}$$

where $\Gamma(\alpha)$ is the gamma function, can be interpreted in terms of fractional integrals and derivatives

$$\mathbf{J}(x,y,z,t) = -kD_{\mathrm{RL}}^{1-\alpha}\,\nabla c(x,y,z,t), \qquad 0<\alpha\le 1\,, \tag{37}$$

$$\mathbf{J}(x,y,z,t) = -kI^{\alpha-1}\,\nabla c(x,y,z,t), \qquad 1<\alpha\le 2\,, \tag{38}$$

and results in the diffusion equation with Caputo fractional derivative

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\Delta c, \qquad 0<\alpha\le 2\,. \tag{39}$$

Recall the basic notions of the fractional calculus [2, 40, 69, 80, 94]. The Riemann – Liouville fractional integral is introduced as a natural generalization of the repeated integral written in a convolution type form:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \qquad \alpha>0\,, \tag{40}$$

The Riemann – Liouville derivative of the fractional order α is defined

as left-inverse to the fractional integral $I^{\alpha} f(t)$:

$$D^{\alpha}_{\mathrm{RL}} f(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right], \qquad n-1<\alpha<n. \quad (41)$$

The Caputo fractional derivative is introdused as

$$D^{\alpha}_{\mathrm{C}} f(t) \equiv \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}\,d\tau, \quad n-1<\alpha<n. \quad (42)$$

Fractional calculus has many applications in physics, geophysics, geology, chemistry, rheology, engineering, bioengineering, medicine and finance (see, for example, [76, 86, 87, 94, 134, 136], among others). Theories of thermo-elasticity based on time-, spase- and space-time-fractional heat conduction equations were proposed in [97, 98, 112–114]; theory of diffusive stresses based on fractional diffusion equation was suggested in [102]. Many problems in the framework of these theories were solved by the author (see [104, 105] and the references therein). The author research in the field of fractional heat conduction (diffusion) and fractional thermoelasticity was summed up in monographs [108] and [110].

4. From the chemical potential and concentration tensors to nonlocal theory of elasticity. In the pioneering paper [35], Pidstryhach started from the diffusion equation arising in his theory

$$\rho\frac{\partial \mathbf{c}}{\partial t} = 2a_1\Delta\boldsymbol{\varphi} + a_2\Delta(\mathrm{tr}\boldsymbol{\varphi})\mathbf{I} \quad (43)$$

with the corresponding diffusion coefficients a_1 and a_2 and splits it into the mean and deviatoric parts

$$\rho\frac{\partial(\mathrm{tr}\,\mathbf{c})}{\partial t} = 3a\Delta(\mathrm{tr}\,\boldsymbol{\varphi}), \qquad a = \frac{2}{3}a_1 + a_2, \quad (44)$$

$$\rho\frac{\partial(\mathrm{dev}\,\mathbf{c})}{\partial t} = 2a_1\Delta(\mathrm{dev}\,\boldsymbol{\varphi}). \quad (45)$$

Using the new nontraditional thermodynamic pare (the chemical potential tensor $\boldsymbol{\varphi}$ and the concentration tensor $\mathbf{c}$), Pidstryhach obtained the state equations for the stress tensor (written in terms of the mean stress and the deviatoric stress):

$$\frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} = K_{\varphi}\mathrm{tr}\,\mathbf{e} - æ_{\varphi}\mathrm{tr}\,\boldsymbol{\varphi}, \quad (46)$$

$$\mathrm{dev}\,\boldsymbol{\sigma} = 2\mu_{\varphi}\mathrm{dev}\,\mathbf{e} - 2\gamma_{\varphi}\mathrm{dev}\,\boldsymbol{\varphi} \quad (47)$$

and

$$\frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} = K_{c}\mathrm{tr}\,\mathbf{e} - æ_{c}\mathrm{tr}\,\mathbf{c}, \quad (48)$$

$$\mathrm{dev}\,\boldsymbol{\sigma} = 2\mu_{c}\mathrm{dev}\,\mathbf{e} - 2\gamma_{c}\mathrm{dev}\,\mathbf{c} \quad (49)$$

with corresponding material constants K_{φ}, μ_{φ}, K_c, μ_c, κ_{φ}, γ_{φ}, κ_c, and γ_c.

If we apply the Laplace operator to the both sides of (46) and (47), differentiate (48) and (49) with respect to time and insert the obtained results into the diffusion equations (44) and (45), we get

$$\Delta(\mathrm{tr}\,\boldsymbol{\sigma}) - p\frac{\partial(\mathrm{tr}\,\boldsymbol{\sigma})}{\partial t} = 3\left[K_{\varphi}\Delta(\mathrm{tr}\,\mathbf{e}) - pK_c\frac{\partial(\mathrm{tr}\,\mathbf{e})}{\partial t}\right], \quad (50)$$

$$\Delta(\mathrm{dev}\boldsymbol{\sigma}) - q\frac{\partial(\mathrm{dev}\,\boldsymbol{\sigma})}{\partial t} = 2\left[\mu_{\varphi}\Delta(\mathrm{dev}\,\mathbf{e}) - q\mu_c\frac{\partial(\mathrm{dev}\;\mathbf{e})}{\partial t}\right], \quad (51)$$

where

$$p = \frac{\rho \varkappa_{\varphi}}{3a\varkappa_c}, \qquad q = \frac{\rho \gamma_{\varphi}}{2a_1 \gamma_c}. \tag{52}$$

Pidstryhach solved the diffusion equations (50) and (51) in infinite solid under appropriate initial conditions using the Laplace transform with respect to time and the triple Fourier transform with respect to the spatial coordinates and obtained the space-time-nonlocal dependence between the stress tensor and the strain tensor

$$\operatorname{tr}\boldsymbol{\sigma} = 3K_c \operatorname{tr}\mathbf{e} + \frac{3(K_{\varphi} - K_c)}{p} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{(p)}(|x-x'|, |y-y'|, |z-z'|, t-\tau) \times$$

$$\times \operatorname{tr}\mathbf{e}(x', y', z', \tau)\, dx'\, dy'\, dz'\, d\tau, \tag{53}$$

$$\operatorname{dev}\boldsymbol{\sigma} = 2\mu_c \operatorname{dev}\mathbf{e} + \frac{2(\mu_{\varphi} - \mu_c)}{q} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{(q)}(|x-x'|, |y-y'|, |z-z'|, t-\tau) \times$$

$$\times \operatorname{dev}\mathbf{e}(x', y', z', \tau)\, dx'\, dy'\, dz'\, d\tau, \tag{54}$$

where the space-time-nonlocality kernel $K_{(p)}(x, y, z, t)$ had the following form:

$$K_{(p)}(x, y, z, t) = -\left(\frac{p}{2t}\right)^{5/2} \left(p\frac{x^2 + y^2 + z^2}{2t} - 3\right) \exp\left(-p\frac{x^2 + y^2 + z^2}{4t}\right). \tag{55}$$

The kernel $K_{(q)}(x, y, z, t)$ is obtained from $K_{(p)}(x, y, z, t)$ by substituting p by q. It should be emphasized that Eringen [60] postulated the exponential space-nonlocal kernel (29), whereas Pidstryhach [35] derived the exponential space-time-nonlocal kernel (55) from his theory of diffusive stresses.

In the case of time-fractional counterpart of the diffusion equations (44) and (45) for the mean and deviatoric parts,

$$\rho \frac{\partial^{\alpha}(\operatorname{tr}\mathbf{c})}{\partial t^{\alpha}} = 3a\Delta(\operatorname{tr}\boldsymbol{\varphi}), \tag{56}$$

$$\rho \frac{\partial^{\alpha}(\operatorname{dev}\mathbf{c})}{\partial t^{\alpha}} = 2a_1\Delta(\operatorname{dev}\boldsymbol{\varphi}), \tag{57}$$

the Pidstryhach result was generalized in [109]. For details the interested reader is referred to this paper. Here, we present only the final expression for the kernel $K_{(p)}(x, y, z, t)$:

$$K_{(p)}(x, y, z, t) = -\frac{\sqrt{\pi}\, p^2}{\sqrt{2}t^{\alpha+1}\sqrt{x^2 + y^2 + z^2}} \times$$

$$\times W\left(-\frac{\alpha}{2}, -\alpha; -\sqrt{p}\frac{\sqrt{x^2 + y^2 + z^2}}{t^{\alpha/2}}\right). \tag{58}$$

The Wright function $W(\alpha, \beta; z)$, which plays the important role in fractional calculus, is defined by the series representation [1, 68, 80, 94]

$$W(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}, \qquad \alpha > -1, \qquad z \in \mathbb{C}, \tag{59}$$

and is a generalization of the exponential function and the Bessel functions. In particular, we have [94]:

$$W(0,1;z) = e^z, \tag{60}$$

$$W\left(-\frac{1}{2},\frac{1}{2};-z\right) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right), \tag{61}$$

$$W\left(1,1;-\frac{z^2}{4}\right) = J_0(z), \tag{62}$$

$$W\left(1,1;\frac{z^2}{4}\right) = I_0(z). \tag{63}$$

Using (61) and the formula for derivative of the Wright function [1]

$$\frac{dW(\alpha,\beta;z)}{dz} = W(\alpha,\alpha+\beta;z), \tag{64}$$

we obtain that for the parabolic diffusion equation ($\alpha = 1$) the kernel (58) coincides with the Pidstryhach kernel (55).

1. *Бейтмен Г., Эрдейи А.* Высшие трансцендентные функции: В 3 т. – Т. 3. Эллиптические и автоморфные функции. Функции Ламе и Матье. – Москва: Наука, 1967. – 299 с.
   Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. Higher transcendental functions. – Vol. 3. – New York: McGraw-Hill, 1955. – 292 p.
2. *Вірченко Н. О., Рибак В. Я.* Основи дробового інтегро-диференціювання. – Київ: Задруга, 2007. – 364 с.
3. *Гринфельд М. А.* Методы механики сплошных сред в теории фазовых превращений. – Москва: Наука, 1990. – 312 с.
4. *Гринфельд М. А.* О двух типах гетерогенных фазовых равновесий // Докл. АН СССР. – 1981. – **258**, № 3. – С. 567–569.
5. *Гринфельд М. А.* Устойчивость гетерогенного равновесия в системах, содержащих твердые упругие фазы // Докл. АН СССР. – 1982. – **265**, № 4. – С. 836–840.
6. *Грицина О., Кондрат В.* Термомеханіка конденсованих систем за урахування локального зміщення маси. І. Основи теорії. – Львів: Растр-7, 2017. – 208 с.
7. *Гузев М. А.* Структура тензора химического потенциала для двухфазной упругой среды в динамических условиях // Журн. физ. химии. – 2005. – **79**, № 9. – С. 1639–1643.
   Guzev M. A. Structure of the chemical potential tensor for a two-phase elastic medium under dynamic conditions // Russian J. Phys. Chem. A. – 2005. – **79**, № 9. – P. 1451–1454.
8. *Гузев М. А.* Тензор химического потенциала для модели двухфазной сплошной среды // Прикл. механика и техн. физика. – 2005. – № 3. – С. 12–22.
   Guzev M. A. Chemical potential tensor for a two-phase continuous medium model // J. Appl. Mech. Techn. Phys. – 2005. – **46**, No. 3. – P. 315–323.
9. *Гузев М. А.* Условия на границе раздела фаз нелинейно-упругого материала в динамическом случае // Докл. АН СССР. – 2007. – **416**, № 6. – С. 763–765.
10. *Кондауров В. И.* Уравнения Клапейрона–Клаузиуса для фазовых переходов первого рода в термоупругом материале // Прикл. математика и механика. – 2004. – **68**, № 1. – С. 73–90.
    Kondaurov V. I. The Clausius–Clapeyron equations for phase transitions of the first kind in a thermoelastic material // J. Appl. Math. Mech. – 2004. – **68**, No. 1. – P. 65–79.
11. *Конобеевский С. Т.* К теории фазовых превращений. II. Диффузия в твердых растворах под влиянием распределения напряжений // Журн. эксперим. и теорет. физики. – 1943. – **13**, № 6. – С. 200–214.
12. *Косевич А. М., Ковалев А. С.* Усредненные уравнения равновесия и движения упругой среды с точечными дефектами // Физика твердого тела. – 1971. – **13**, № 1. – С. 218–224.
13. *Кунин И. А.* Теория упругих сред с микроструктурой. Нелокальная теория упругости – Москва: Наука, 1975. – 415 с.
14. *Макаров Э. С.* Вариант построения и некоторые плоские задачи теории хемопластичности // Изв. АН СССР. Механика твердого тела. – 1989. – **13**, № 5. – С. 70–72.

15. *Макаров Э. С.* Применение теории хемопластичности к расчету предельных состояний корродирующих труб // Физ.-хим. механика материалов. – 1989. – **25**, № 2. – С. 115–117.
16. *Макаров Э. С., Агуреев И. Е.* Введение в теорию хемопластичности // Вопросы чистой и прикладной математики / А. Л. Толоконников (ред.). – Тула: Приокское кн. изд-во, 1988. – С. 196–212.
17. *Макаров Э. С., Агуреев И. Е.* Применение теории хемопластичности к расчету процессов обработки давлением // Изв. вузов. Машиностроение. – 1988. – № 3. – С. 110–114.
18. *Підстригач Я. С.* Диференціальні рівняння дифузійної теорії деформації твердого тіла // Доп. АН УРСР. – 1963. – № 3. – С. 336–339.
19. *Підстригач Я. С.* Диференціальні рівняння задачі термодифузії в твердому деформованому тілі // Доп. АН УРСР. – 1961. – № 2. – С. 169–172.
20. *Підстригач Я. С.* Про одне узагальнення теоретичної моделі твердого тіла // Доп. АН УРСР. – 1963. – № 8 – С. 1015–1017.
21. *Підстригач Я. С., Павлина В. С.* Загальні співвідношення термодинаміки твердих розчинів // Укр. фіз. журн. – 1961. – **6**, № 5. – С. 655–663.
22. *Повстенко Ю. З.* Кругова дислокаційна петля обертання в нелокально-пружному середовищі // Мат. методи та фіз.-мех. поля. – 1995. – Вип. 38. – С. 95–98.
Povstenko Yu. Z. A circular rotational dislocation loop in a nonlocally elastic medium // J. Math. Sci. – 1996. – **81**, No. 6. – P. 3080–3083.
23. *Повстенко Ю. З.* Математическая теория дефектов в континууме Коссера // Мат. методы и физ.-мех. поля. – 1988. – Вып. 27. – С. 34–40.
Povstenko Yu. Z. The mathematical theory of defects in a Cosserat continuum // J. Soviet Math. – 1992. – **62**, No. 1. – P. 2524–2530.
24. *Повстенко Ю. З.* Нелокальна і градієнтна теорії пружності та їх застосування до опису недосконалостей у твердих тілах // Мат. методи та фіз.-мех. поля. – 2003. – **46,** № 2. – С. 136–146.
25. *Повстенко Ю. З.* Описание поверхностных явлений в упруго полязизующихся твердых телах // Журн. прикл. механики и техн. физики. – 1983. – № 1. – С. 117–121.
Povstenko Yu. Z. Description of surface phenomena in elastically polarized solids // J. Appl. Mech. Techn. Phys. – 1983. – **24**, No. 1. – P. 102–105.
26. *Повстенко Ю. З.* Прямолінійні дислокації, дисклінації і зосереджені сили в нелокально пружному середовищі // Мат. методи та фіз.-мех. поля. – 2001. – **44,** № 1. – С. 124–129.
27. *Повстенко Ю. З.* Рівняння дифузії для тензора концентрації // Мат. методы и физ.-мех. поля. – 1991. – Вып. 33. – С. 36–39.
Povstenko Yu. Z. The diffusion equation for the concentration tensor // J. Soviet Math. – 1993. – **65**, No. 5. – P. 1841–1844.
28. *Повстенко Ю. З.* Тензорные термодинамические функции для деформируемых твердых тел // Мат. методы и физ.-мех. поля. – 1983. – Вып. 18. – С. 41–43.
29. *Повстенко Ю. З.* Термодинамика процессов диффузии и теплопроводности в континууме Коссера // Термодинамика необратимых процессов / А. И. Лопушанская (ред.). – Москва: Наука, 1992. – С. 150–156.
30. *Повстенко Ю. З.* Точковий дефект в нелокально-пружному середовищі // Мат. методи та фіз.-мех. поля. – 1998. – **41,** № 3. – С. 85–89.
Povstenko Yu. Z. Point defect in a nonlocal elastic medium // J. Math. Sci. – 2001. – **104**, No. 5. – P. 1501–1505.
31. *Повстенко Ю. З., Матковський О. А.* Гвинтова дислокація в нелокально пружному середовищі з моментними напруженнями // Доп. НАН України. – 1995. – № 10. – С. 57–60.
32. *Повстенко Ю. З., Матковський О. А.* Крайова дислокація в нелокально-пружному середовищі з моментними напруженнями // Мат. методи та фіз.-мех. поля. – 1997. – **40,** № 3. – С. 98–102.
Povstenko Yu. Z., Matkovskii O. A. Boundary dislocation in a nonlocally-elastic medium with moment stresses // J. Math. Sci. – 1999. – **96**, No. 1. – P. 2883–2886.
33. *Подстригач Я. С.* Диффузионная теория деформации изотропной сплошной среды // Вопр. механики реального твердого тела. – 1964. – Вып. 2. – С. 71–99.
34. *Подстригач Я. С.* Диффузионная теория неупругости металлов // Журн. прикл. механики и техн. физики. – 1965. – № 2. – С. 67–72.

Podstrigach Ya. S. Diffusion theory of the anelasticity of metals // J. Appl. Mech. Tech. Phys. – 1965. – **6**, No. 2. – P. 56–60.

35. *Подстригач Я. С.* Об одной нелокальной теории деформирования твердых тел // Прикл. механика. – 1967. – **3**, № 2. – С. 71–76.
Podstrigach Ya. S. On a nonlocal theory of solid body deformation // Soviet Appl. Mech. – 1967. – **3**, No. 2. – P. 44–46.
36. *Подстригач Я. С., Павлина В. С.* Дифференциальные уравнения термодинамических процессов в n-компонентном твердом растворе // Физ.-хим. механика материалов. – 1965. – **1**, № 4. – С. 383–389.
Podstrigach Ya. S., Pavlina V. S. Differential equations of thermodynamic processes in n-component solid solutions // Soviet Mater. Sci. – 1966. – **1**, No. 4. – P. 259–264.
37. *Подстригач Я. С., Повстенко Ю. З.* Введение в механику поверхностных явдений в деформируемых твердых телах. – Киев: Наук. думка, 1985. – 200 с.
38. *Русанов А. И.* Развитие фундаментальных положений термодинамики поверхностей // Коллоидный журн. – 2012. – **74**, № 2. – С. 148–166.
Rusanov A. I. The development of the fundamental concepts of surface termodynamics // Colloid J. – 2012. – **74**, No. 2. – P. 136–153.
39. *Рущицький Я. Я.* Нетрадиційні впорядковані пари термодинамічних параметрів: від теорії дифузійної пружності Підстригача до теорії сумішей Бедфорда – Драмхеллера // Мат. методи та фіз.-мех. поля. – 1998. – **41,** № 3. – С. 117–120..
Rushchyts'kyi Ya. Ya. Nontraditional ordered pairs of thermodynamic parameters: from the Pidstryhach theory of diffusion elasticity to the Bedford–Drumheller theory of mixtures // J. Math. Sci. – 2001. – **104**, No. 5. – P. 1538–1541.
40. *Самко С. Г., Килбас А. А., Маричев О. И.* Интегралы и производные дробного порядка и некоторые их приложения. – Минск: Наука и техника, 1987. – 688 с.
Samko S. G., Kilbas A. A., Marichev O. I. Fractional integrals and derivatives: Theory and applications. – Newark etc.: Gordon & Breach, 1993. – xxxvi+976 p.
41. *Темнов О. В.* Диа- и параупругая поляризация в твердых телах // Исслед. по теории упругости и пластичности. – 1974. – № 10. – С. 83–102.
42. *Трускиновский Л. М.* Равновесные межфазные границы // Докл. АН СССР. – 1982. – **265**, № 2. – С. 306–310.
Truskinovskii L. M. Equilibrium phase interfaces // Soviet Physics Doklady. – 1982. – **27**, No. 7. – P. 551–552.
43. *Трускиновский Л. М.* О тензоре химического потенциала // Геохимия. – 1983. – № 12. – С. 1730–1744.
44. *Фастов Н. С.* К термодинамике необратимых процессов в упруго деформированных телах // Проблемы металловедения и физики металлов. – 1958. – № 5. – С. 550–576.
45. *Araujo R. P., McElwain D. L. S.* A mixture theory for the genesis of residual stresses in growing tissues I: A general formulation // SIAM J. Appl. Math. – 2005. – **65**, No. 4. – P. 1261–1284.
46. *Ari N., Eringen A. C.* Nonlocal stress field at Griffith crack // Cryst. Lattice Defects Amorphous Mater. – 1983. – **10**. – P. 937–945.
47. *Biot M. A.* Thermoelasticity and irreversible thermodynamics // J. Appl. Phys. – 1956. – **27**, No. 3. – P. 240–253.
48. *Bowen R. M.* The thermochemistry of reacting mixture of elastic materials with diffusion // Arch. Rational Mech. Anal. – 1969. – **34**, No. 2. – P. 97–127.
49. *Bowen R. M.* Theory of mixtures // In: Continuum Physics, Vol. 3: Mixtures and EM field theories / A. C. Eringen (Ed.). – New York: Acad. Press, 1976. – P. 1–127.
50. *Bowen R. M.* Toward a thermodynamics and mechanics of mixtures // Arch. Rational Mech. Anal. – 1967. – **24**, No. 5. – P. 370–403.
51. *Bowen R. M., Wiese J. C.* Diffusion in mixtures of elastic materials // Int. J. Eng. Sci. – 1969. – **7**, No. 7. – P. 689–722.
52. *Buratti F., Huo Y., Müller I.* Eshelby tensor as a tensor of free enthalpy // J. Elast. – 2003. – **72**, No. 1-3. – P. 31–42.
53. *Ciarletta M., Chiriţă S.* Some non-standard problems related with the mathematical model of thermoelasticity with microtemperatures // J. Therm. Stresses. – 2013. – **36**, No. 6. – P. 517–536.
54. *Duhamel J.-M.-C.* Second mémoire sur les phénomines thermo-mécanique // J. Ecole Polytechn. – 1837. – **15**. – P. 1–57.

55. *Edelen D. G. B.* Nonlocal field theories // In: Continuum Physics, Vol. 4: Polar and nonlocal theories / A. C. Eringen (Ed.). – New York: Acad. Press, 1976. – P. 75–204.
56. *Eringen A. C.* Edge dislocation in nonlocal elasticity // Int. J. Eng. Sci. – 1977. – **15**, No. 3. – P. 177–183.
57. *Eringen A. C.* Line crack subject to shear // Int. J. Fracture. – 1978. – **14**, No. 4. – P. 367–379.
58. *Eringen A. C.* Linear theory of nonlocal elasticity and dispersion of plane waves // Int. J. Eng. Sci. – 1972. – **10**, No. 5. – P. 425–435.
59. *Eringen A. C.* Nonlocal continuum field theories. – New York: Springer, 2002. – 376 p.
60. *Eringen A. C.* On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves // J. Appl. Phys. – 1983. – **54**, No. 9. – P. 4703–4710.
61. *Eringen A. C.* Screw dislocation in nonlocal elasticity // J. Phys. D: Appl. Phys. – 1977. – **10**, No. 5. – P. 671–678.
62. *Eringen A. C.* Theory of micropolar elasticity // In: Fracture, Vol. 2 / H. Liebowitz (Ed.). – New York: Acad. Press, 1968. – P. 621–729.
63. *Eringen A. C.* Theory of nonlocal thermoelasticity // Int. J. Eng. Sci. – 1974. – **12**, No. 12. – P. 1063–1077.
64. *Eringen A. C.* Vistas of nonlocal continuum physics // Int. J. Eng. Sci. – 1992. – **30**, No. 10. – P. 1551–1565.
65. *Eshelby J. D.* The continuum theory of lattice defects // In: Solid State Physics, Vol. 3 / F. Seitz, D. Turnbull (Eds.). – New York: Acad. Press, 1956. – P. 79–144.
66. *Fick A.* Über Diffusion // Ann. Phys. – 1855. – **94**. – S. 59–86.
67. *Fourier J. B. J.* Théorie analytique de la chaleur. – Paris: Firmin Didot, 1822. – 639 p.
68. *Gorenflo R., Luchko Yu., Mainardi F.* Analytical propertyies and applications of the Wright function // Fract. Calc. Appl. Anal. – 1999. – **2**, No. 4. – P. 383–414.
69. *Gorenflo R., Mainardi F.* Fractional calculus: integral and differential equations of fractional order // Fractals and fractional calculus in continuum mechanics / Eds. A. Carpinteri, F. Mainardi. – Wien: Springer, 1997. – P. 223–276.
70. *Gorsky W. S.* Theorie der elastischen Nachwirkung in ungeordneten Misch-Kristallen (elastische Nachwirkung zweiter Art) // Phys. Z. Sowjetunion. – 1935. – **8**, No. 1. – P. 457–471.
71. *Green A. E., Naghdi P. M.* On thermal effects in the theory of shells // Proc. Roy. Soc. London. Ser. A. – 1979. – **365**, No. 1721. – P. 161–190.
72. *Grinfel'd M. A.* On heterogeneous equilibrium of nonlinear elastic phases and chemical potential tensors // Int. J. Eng. Sci. – 1981. – **19**, No. 7. – P. 1031–1039.
73. *Grinfeld M. A.* Thermodynamic methods in the theory of heterogeneous systems. – Harlow: Longman, 1991. – 399 p.
74. *Grot R. A.* Thermodynamics of a continuum with microstructure // Int. J. Eng. Sci. – 1969. – **7**, No. 8. – P. 801–814.
75. *Gurtin M. E., Pipkin A. C.* A general theory of heat conduction with finite wave speeds // Arch. Ration. Mech. Anal. – 1968. – **31**, No. 2. – P. 113–126.
76. *Herrmann R.* Fractional calculus: An introduction to physicists. – Singapore: World Scientific, 2014. – 261 p.
77. *Ieşan D.* Thermoelasticity of bodies with microstructure and microtemperatures // Int. J. Solids Struct. – 2007. – **44**, No. 25-26. – P. 8648–8662.
78. *Ieşan D., Quintanilla R.* On the theory of thermoelasticity with microtemperatures // J. Therm. Stresses. – 2000. – **23**, No. 3. – P. 199–215.
79. *Ieşan D., Quintanilla R.* On thermoelastic bodies with inner structure and micro-temperatures // J. Math. Anal. Appl. – 2009. – **354**, No. 1. – P. 12–23.
80. *Kilbas A. A., Srivastava H. M., Trujillo J. J.* Theory and applications of fractional differential equations. – Amsterdam: Elsevier, 2006. – 523 p.
81. *Kröner E.* Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen // Arch. Ration. Mech. Anal. – 1959. – **4**, No. 1. – P. 273–334.
82. *Kröner E.* Elasticity theory of materials with long-rang cohesive forces // Int. J. Solids Struct. – 1967. – **3**, No. 5. – 731–742.
83. *Kunin I. A.* Theory of elastic media with microstructure I. One-dimensional models. – Berlin: Springer, 1982. – viii+296 p.
84. *Kunin I. A.* Theory of elastic media with microstructure II. Three-dimensional models. – Berlin: Springer, 1983. – viii+274 p.

85. *Liu I. S.* On interface equilibrium and inclusion problem // Continuum Mech. Thermodyn. – 1992. – **4**, No. 3. – P. 177–186.
86. *Magin R. L.* Fractional calculus in bioengineering. – Connecticut: Begell House Publ., 2006. – 684 p.
87. *Mainardi F.* Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models. – London: Imperial College Press, 2010. – 347 p.
88. *Neumann F.* Vorlesungen über die Theorie des Elasticität des festen Körper und des Lichtäthers. – Leipzig: Teubner, 1895. – 374 S.
89. *Nigmatullin R. R.* To the theoretical explanation of the "universal response" // Phys. Stat. Sol. (b). – 1984. – **123**, No. 2. – P. 739–745.
90. *Nigmatullin R. R.* On the theory of relaxation for systems with "remnant" memory // Phys. Stat. Sol. (b). – 1984. – **124**, No. 1. – P. 389–393.
91. *Nowacki W.* Dynamical problems of thermodiffusion in solids I, II, III // Bull. Acad. Polon. Sci. Sér. Sci. Tech. – 1974. – **23**. – P. 55–64, 129–135, 257–266.
92. *Nowacki W.* Dynamic problems of thermodiffusion in elastic solids // Proc. Vibr. Probl. – 1974. – **15,** No. 2. – P. 105–128.
93. *Podio-Guidugli P.* Configurational balances via variational arguments // Interfaces Free Boundaries. – 2001. – **3**, No. 2. – P. 223–232.
94. *Podlubny I.* Fractional differential equations. – New York: Acad. Press, 1999. – 340 p.
95. *Povstenko Y. Z.* Axisymmetric ring loading in a nonlocal elastic space // Int. J. Eng. Sci. – 2001. – **39**, No. 3. – P. 285–302.
96. *Povstenko Y. Z.* Circular dislocation loops in non-local elasticity // J. Phys. D: Appl. Phys. – 1995. – **28**, No. 1. – P. 105–111.
97. *Povstenko Y. Z.* Fractional Cattaneo-type equations and generalized thermoelasticity // J. Therm. Stresses. – 2011. – **34**, No. 2. – P. 97–114.
98. *Povstenko Y. Z.* Fractional heat conduction equation and associated thermal stresses // J. Therm. Stresses. – 2005. – **28**, No. 1. – P. 83–102.
99. *Povstenko Y. Z.* Imperfections in nonlocal elasticity // J. Phys. (Paris). – 1998. – **8**, No. 8. – P. 309–316.
100. *Povstenko Y. Z.* Non-local equations in mathematics and physics. Theory of nonlocal elasticity // Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie. Matematyka. – 1997. – **5**. – P. 89–96.
101. *Povstenko Y. Z.* Straight disclinations in nonlocal elasticity // Int. J. Eng. Sci. – 1995. – **33**, No. 4. – P. 575–582.
102. *Povstenko Y. Z.* Stresses exerted by a source of diffusion in a case of a non-parabolic diffusion equation // Int. J. Eng. Sci. – 2005. – **43**, No. 11-12. – P. 977–991.
103. *Povstenko Y. Z.* The use of differential equations of nonlocal elasticity for description of crystal imperfections // Z. angew. Math. Mech. – 1996. – **75**, Suppl. 5. – P. 407–408.
104. *Povstenko Y. Z.* Thermoelasticity which uses fractional heat conduction equation // Мат. методи та фіз.-мех. поля. – 2008. – **51,** № 2. – С. 239–246.
J. Math. Sci. – 2009. – **162**, No. 2. – P. 296–305.
105. *Povstenko Y. Z.* Two-dimensional axisymmetric stresses exerted by instantaneous pulses and sources of diffusion in an infinite space in a case of time-fractional diffusion equation // Int. J. Solids Struct. – 2007. – **44**, No. 7-8. – P. 2324–2348.
106. *Povstenko Y. Z., Kubik I.* Concentrated ring loading in a nonlocal elastic medium // Int. J. Eng. Sci. – 2005. – **43**, No. 5-6. – P. 457–471.
107. *Povstenko Y. Z., Matkovskii O. A.* Circular disclination loops in nonlocal elasticity // Int. J. Solids Struct. – 2000. – **37**, No. 44. – P. 6419–6432.
108. *Povstenko Y.* Fractional thermoelasticity. – New York: Springer, 2015. – 253 p.
109. *Povstenko Y.* Generalized theory of diffusive stresses associated with the time-fractional diffusion equation and nonlocal constitutive equations for the stress tensor // Comput. Math. Appl. – Article in press. – doi.org/10.1016/j.camwa.2016.02.034.
110. *Povstenko Y.* Linear fractional diffusion-wave equation for scientists and engineers. – New York: Birkhäuser, 2015. – 460 p.
111. *Povstenko Y.* Non-axisymmetric solutions to time-fractional diffusion-wave equation in an infinite cylinder // Fract. Calcul. Appl. Anal. – 2011. – **14**, No. 3. – P. 418–435.
112. *Povstenko Y.* Theories of thermal stresses based on space-time-fractional telegraph equations // Comput. Math. Appl. – 2012. – **64**, No. 10. – P. 3321–3328.

113. *Povstenko Y.* Theory of thermoelasticity based on the space-time-fractional heat conduction equation // Phys. Scr. T. – 2009. – **136**. – 014017–1–6.
114. *Povstenko Y.* Thermoelasticity based on fractional heat conduction equation // Proc. 6th Int. Congr. Thermal Stresses (26–29 May 2005, Vienna, Austria) / Eds. F. Ziegler, R. Heuer, C. Adam. – Vol. 2. – Vienna: Vienna University of Technology, 2005. – P. 501–504.
115. *Rubin M. B.* A uniqueness theorem for thermoelastic shells with generalized boundary conditions // Quart. Appl. Math. – 1986. – **44**, No. 3. – P. 431–440.
116. *Rusanov A. I.* Advances in thermodynamics of solid surfaces // Pure Appl. Chem. – 1989. – **61**, No. 11. – P. 1945–1948.
117. *Rusanov A. I.* Problems on surface thermodynamics // Pure Appl. Chem. – 1992. – **64**, No. 1. – P. 111–124.
118. *Rusanov A. I.* Thermodynamics of solid surfaces // Surface Sci. Rep. – 1996. – **23**, No. 6-8. – P. 173–247.
119. *Rusanov A. I.* Surface thermodynamics revisited // Surface Sci. Rep. – 2005. – **58**, No. 5-8. – P. 111–239.
120. *Rusanov A. I., Shchekin A. K., Tatyanenko D. V.* Grand potential in thermodynamics of solid bodies and surfaces // J. Chem. Phys. – 2009. – **131**, No. 16. – 161104.
121. *Rushchitsky J.* Theory of waves in materials. – Copenhagen: Ventus Publishing ApS, 2011. – 270 p.
122. *Samohýl I.* Thermodynamics of mixtures of reacting and non-reacting fluids with heat conduction, diffusion and viscosity // Int. J. Non-Linear Mech. – 1997. – **32**, No. 2. – P. 241–257.
123. *Samohýl I.* Thermodynamics of non-reacting mixtures of any symmetry with heat conduction, diffusion and viscosity // Int. J. Non-Linear Mech. – 1997. – **32**, No. 2. – P. 235–240.
124. *Samohýl I., Pabst W.* Phase equilibrium in non-ꝑuids and non-ꝑuid mixtures // Int. J. Non-Linear Mech. – 2004. – **39**, No. 2. – P. 247–263.
125. *Samohýl I., Pabst W.* The Eshelby relation in mixtures // Int. J. Non-Linear Mech. – 1997. – **32**, No. 2. – P. 227–233.
126. *Scalia A., Svanadze M.* On the representations of solutions of the theory of thermoelasticity with microtemperatures // J. Therm. Stresses. – 2006. – **29**, No. 9. – P. 849–863.
127. *Scalia A., Svanadze M.* Potential method in the linear theory of thermoelasticity with microtemperatures // J. Therm. Stresses. – 2009. – **32**, No. 10. – P. 1024–1042.
128. *Scalia A., Svanadze M., Tracina R.* Basic theorems on the equilibrium theory of thermoelasticity with microtemperatures // J. Therm. Stresses. – 2010. – **33**, No. 8. – P. 721–753.
129. *Shimizu I.* Nonhydrostatic and nonequilibrium thermodynamics of deformable materials // J. Geophys. Res. – 1992. – **97**, No. B4. – P. 4587–4597.
130. *Stuke B.* Allgemeine Rahmengleichungen der Kontinuumsdynamik // Phys. Lett. – 1966. – **21**, No. 6. – P. 649–650.
131. *Stuke B.* Tensorielle chemische Potential: eine notwendige Erweiterung der Gibbs'schen Thermodynamik // Z. Naturforsch. A. – 1975. – **30**, No. 11. – P. 1433–1440.
132. *Svanadze M.* Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures // J. Therm. Stresses. – 2004. – **27**, No. 2. – P. 151–170.
133. *Trusdell C.* Rational thermodynamics. – New York: McGraw-Hill, 1969. – 208 p.
134. *Uchaikin V. V.* Fractional derivatives for physicists and engineers. – Heidelberg: Springer, 2013. – 385 p.
135. *Wei C.* A theoretical framework for modeling the chemomechanical behavior of unsaturated soils // Vadose Zone J. – 2014. – **13**, No. 9. – P. 1–21.
136. *West B. J., Bologna M., Grigolini P.* Physics of fractal operators. – New York: Springer, 2003. – 354 p.
137. *Woźniak C.* Thermoelasticity of bodies with micro-structrure // Arch. Mech. Stos. – 1967. – **19**, No. 3. – P. 335–365.
138. *Woźniak C.* Thermoelasticity of non-simple oriented materials // Int. J. Eng. Sci. – 1967. – **5**, No. 8. – P. 605–612.
139. *Wyrwał J., Marynowicz A., Świrska J.* On tensorial forms of thermodynamic potentials in mixtures theory // Int. J. Solids Struct. – 2009. – **46**, No. 11-12. – P. 2293–2297.

ВІД ТЕНЗОРА ХІМІЧНОГО ПОТЕНЦІАЛУ І ТЕНЗОРА КОНЦЕНТРАЦІЇ ДО НЕЛОКАЛЬНИХ ТЕОРІЙ СУЦІЛЬНОГО СЕРЕДОВИЩА

Нетрадиційна термодинамічна пара (тензор хімічного потенціалу і тензор концентрації) була введена у піонерських дослідженнях Підстригача. Виключивши тензор хімічного потенціалу і тензор концентрації з конститутивних рівнянь для тензора напружень, Підстригач отримав для тензора напружень нелокальні рівняння за часом і простором. У статті розглядається розвиток наукових ідей Підстригача.

Ключові слова: *тензор хімічного потенціалу, тензор концентрації, нелокальна теорія пружності, інтеграли і похідні дробового порядку.*

ОТ ТЕНЗОРА ХИМИЧЕСКОГО ПОТЕНЦИАЛА И ТЕНЗОРА КОНЦЕНТРАЦИИ К НЕЛОКАЛЬНЫМ ТЕОРИЯМ СПЛОШНОЙ СРЕДЫ

Нетрадиционная термодинамическая пара (тензор химического потенциала и тензор концентрации) была введена в пионерских исследованиях Подстригача. Исключив тензор химического потенциала и тензор концентрации из конститутивних уравнений для тензора напряжений, Подстригач получил для тензора напряжений нелокальные уравнения по времени и пространству. В обзоре рассматривается развитие научных идей Подстригача.

Ключевые слова: *тензор химического потенциала, тензор концентрации, нелокальная теория упругости, интегралы и производные дробного порядка.*

Inst. of Math. and Comput. Sci.
Jan Długosz Univ. in Częstochowa, Poland

Received
18.03.18