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MOMENT CONDITIONS FOR FUNCTIONS WITH ZERO INTEGRALS OVER CONGRUENT BALLS

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We consider the question of precise conditions ensuring that a function having zero integrals over all balls of fixed radius is equal to zero. We completely investigate the case where together with zero integrals over congruent balls a function has zero first moments over these balls.

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Рассматривается вопрос о точных условиях, из которых следует, что функция, имеющая нулевые интегралы по всем шарам фиксированного радиуса, является нулевой. Полностью исследован случай, когда вместе с нулевыми интегралами по конгруэнтным шарам функция имеет равные нулю первые моменты по этим шарам.

1. Introduction. Let $\mathbb{R}^n$ be a real Euclidean space of dimension $n \geq 2$ with Euclidean norm $|\cdot|$, and let $B_r = \{x \in \mathbb{R}^n \colon |x| < r\}$. Assume that $f \in L_{loc}(\mathbb{R}^n)$, let r be a fixed positive number and let

$$\int_{B_r} f(x+u)du = 0 \tag{1}$$

for all $x \in \mathbb{R}^n$. Does this imply that $f = 0$? The answer is in the negative (see, for instance, [1, Part 2]); however, under some additional assumptions f is indeed zero function. One such an assumption is a sufficiently rapid decrease of f at infinity. For instance, it is known that if a function satisfying (1) belongs to the class $L^p(\mathbb{R}^n)$ for some $p \in [1, 2n/(n-1)]$, then $f = 0$, whereas for $p > 2n/(n-1)$ this does not hold any more (see [1], [2], where significantly more general and precise results in this direction were obtained). Another type of restrictions that ensure vanishing of f is related to an increase of the number of possible values of r in condition (1). In particular, the property $f = 0$ is recovered by using two balls of appropriately chosen radii (see [1–4] and the extensive bibliography therein).

In the present paper we consider condition (1) together with the vanishing first moments of f over balls for the case where f is defined in a ball B_R of radius $R > r$. In general we are asking about description and properties of the function space for f.

We present the precise statement of our main result in the following section. Its proof is presented in Section 4, while in Section 3 we develop the necessary apparatus.

2. Statement of the central result. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ with center at the origin and assume that $d\omega$ is the area element of $\mathbb{S}^{n-1}$. The polar coordinates of a point

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$x \in \mathbb{R}^n \backslash \{0\}$ are denoted by (ρ, σ) where $\rho = |x|$ and $\sigma = (\sigma_1, \dots, \sigma_n) = x/\rho \in \mathbb{S}^{n-1}$. Let $\mathcal{H}_k$ denote the space of the spherical harmonics of degree k (see [5, Chapter 4]). Suppose that a_k is the dimension of $\mathcal{H}_k$. It is known that $a_0 = 1$, $a_1 = n$, and

$$a_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \quad \text{for } k \geq 2,$$

(see [5, Chapter 4]). Let $\{Y_l^{(k)}\}$, $1 \leq l \leq a_k$, be an orthonormal basis in the space $\mathcal{H}_k$, which is regarded as a subspace of $L^2(\mathbb{S}^{n-1})$. To any function $f \in L_{loc}(B_R)$, there corresponds the Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(\rho) Y_l^{(k)}(\sigma), \ \rho \in (0, R), \tag{2}$$

where

$$f_{k,l}(\rho) = \int_{\mathbb{S}^{n-1}} f(\rho\sigma) \overline{Y_l^{(k)}(\sigma)} d\omega(\sigma).$$

If $R > r > 0$ then we denote by $V_r(B_R)$ the set of all functions $f \in L_{loc}(B_R)$ satisfying (1) for all $x \in B_{R-r}$. Let $M_r(B_R)$ be the set of all functions $f \in V_r(B_R)$ such that $x_j f \in V_r(B_R)$ for all $j \in \{1, \dots, n\}$.

Next, for $0 \leq a < b$ we set $B_{a,b} = \{x \in \mathbb{R}^n \colon a < |x| < b\}$.

We now present the main result of this paper.

Theorem 1. *Let $R > r > 0$ and assume that $f \in L_{loc}(B_R)$. Then the following assertions hold.*

(i) If $R \leq 2r$ then $f \in M_r(B_R)$ if and only if

$$\int_{B_{2r-R}} f(x)dx = \int_{B_{2r-R}} f(x) x_j dx = 0 \quad \text{for all} \quad j \in \{1, \dots, n\}, \tag{3}$$

and

$$f_{k,l}(|x|) = \sum_{m=0}^{k-3} c_{m,k,l} |x|^{2m-n-k+2} \tag{4}$$

for all non-negative integers k, $1 \leq l \leq a_k$, and almost all $x \in B_{2r-R,R}$, where $c_{m,k,l} \in \mathbb{C}$ and the sum is set to be equal to zero for $k \in \{0, 1, 2\}$.

(ii) If $R > 2r$ and $f \in M_r(B_R)$ then $f = 0$ in B_R.

We note that if $R = 2r$ then the condition $f \in L_{loc}(B_R)$ enables one to simplify the description of $M_r(B_R)$ in the first assertion of Theorem 1. In this case the constants $c_{m,k,l}$ vanish for $0 \leq m < (k-1)/2$ and (3) holds for each $f \in L_{loc}(B_R)$.

3. Notation and auxiliary statements. As usual we denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ the sets of positive integers, integers, and complex numbers, respectively.

For $k \in \mathbb{Z}$ we consider the differential operator d_k in the space $C^1(a, b)$, $0 < a < b$, defined as follows

$$(d_k f)(t) = f'(t) - \frac{k}{t} f(t), \text{ where } f \in C^1(a, b).$$

For $k \in \mathbb{N}$, let $D_k = d_k d_{k-1} \dots d_0$ and let $D_0 = d_0$. Next, let $\gamma \in \mathbb{R}^1$, let $\pi_m(\gamma) = \prod_{q=0}^{m} (\gamma - 2q)$, and assume that $t > 0$. Using induction on $k \in \{0, 1, \dots\}$ it is easy to see that the functions $u_\gamma(t) = t^\gamma$ and $v_\gamma(t) = t^\gamma \ln t$ satisfy the equalities

$$(D_k u_\gamma)(t) = \pi_k(\gamma) u_\gamma(t) t^{-k-1} \tag{5}$$

and

$$(D_k v_\gamma)(t) = v_\gamma(t)\frac{\pi_k(\gamma)}{t^{k+1}} + u_\gamma(t)\frac{\pi_k(\gamma)}{t^{k+1}}\sum_{m=0}^{k}\frac{1}{\gamma-2m}. \tag{6}$$

For $m \in \mathbb{N}$, let $w_{m,n}(t) = v_{2m-n}(t)$ if n is even and $2m \geq n$. Otherwise we set $w_{m,n}(t) = u_{2m-n}(t)$. A calculation shows that

$$\Delta^m\big(w_{m,n}(|x|)\big) = 0,\ x \in \mathbb{R}^n\backslash\{0\}, \tag{7}$$

where Δ is the Laplace operator in $\mathbb{R}^n$.

Let $\overline{G}$ be the closure of the set $G \subset \mathbb{R}^n$, $\widehat{f}$ the Fourier transform of the function $f \in L(\mathbb{R}^n)$, and $f * g$ the convolution of the functions f and g. We also write χ_r for the characteristic function (indicator) of the ball B_r.

We set $\mathfrak{H}_k^\infty(B_R) = \mathfrak{H}_k(B_R) \cap C^\infty(B_R)$, where $\mathfrak{H}_k(B_R)$ is the subspace of $L^2(B_R)$ spanned by the products of radial functions and spherical harmonics of degree k.

Also let

$$V_r^\infty(B_R) = V_r(B_R) \cap C^\infty(B_R), \quad M_r^\infty(B_R) = M_r(B_R) \cap C^\infty(B_R).$$

Let $T^n(\tau)$, $\tau \in O(n)$, be a quasi-regular representation of the orthogonal group $O(n)$ in $L^2(\mathbb{S}^{n-1})$. Then $T^n(\tau)$ is a direct sum of pairwise non-equivalent irreducible unitary representations $T^{n,k}(\tau)$ acting in $\mathcal{H}_k$ (see [6, Chapter 9]). We denote by $\big\{t_{i,j}^k(\tau)\big\}$ the matrix of $T^{n,k}(\tau)$, that is,

$$Y_j^{(k)}\big(\tau^{-1}\sigma\big) = \sum_{i=1}^{a_k} t_{i,j}^k(\tau) Y_i^{(k)}(\sigma), \quad \sigma \in \mathbb{S}^{n-1}.$$

This relation yields

$$t_{i,j}^k(\tau) = \int_{\mathbb{S}^{n-1}} Y_j^{(k)}\big(\tau^{-1}\sigma\big)\overline{Y_i^{(k)}(\sigma)}d\omega(\sigma), \quad \tau \in O(n),$$

whence $t_{i,j}^k$ is continuous on $O(n)$.

Let $d\tau$ be the Haar measure on $O(n)$ normalized so that the measure of $O(n)$ is 1. It is known that for each $f \in L_{loc}(B_R)$,

$$f_{k,l}(\rho)Y_p^{(k)}(\sigma) = a_k \int_{O(n)} f\big(\tau^{-1}x\big)\overline{t_{p,l}^k(\tau)}d\tau, \quad x \in B_R, \tag{8}$$

where $1 \leq l, p \leq a_k$ (see [2, formula (9.5)]). In what follows we assume that all functions that are defined and continuous in a punctured neighbourhood of zero in $\mathbb{R}^n$ and admit a continuous extension to 0 are defined at 0 by continuity.

The proof of Theorem 1 requires some preparation. The following lemmas are needed.

Lemma 1. *Let $f \in M_r^\infty(B_R)$. Then*

(i) $f(gx) \in M_r^\infty(B_R)$ for each $g \in O(n)$.

(ii) All partial derivatives of f are in the class $M_r^\infty(B_R)$.

(iii) $f_{k,l}(\rho)Y_p^{(k)}(\sigma) \in M_r^\infty(B_R)$ for the values of indices satisfying the inequalities $k \geq 0$ and $1 \leq l, p \leq a_k$.

Proof. To prove (i) and (ii) one only needs to use the definition of the class $M_r^\infty(B_R)$. Assertion (iii) follows from (i) and relation (8). □

Lemma 2. *Assume that $f(\rho)Y(\sigma) \in M_r^\infty(B_R)$ for some $Y \in \mathcal{H}_k$. Then*

(i) $\big(d_k f\big)(\rho)Y_l^{(k+1)}(\sigma) \in M_r^\infty(B_R)$ for all $1 \le l \le a_{k+1}$;

(ii) $\big(d_{2-k-n} f\big)(\rho)Y_l^{(k-1)}(\sigma) \in M_r^\infty(B_R)$ for $k \ge 1$ and all l, $1 \le l \le a_{k-1}$.

Proof. We have

$$\frac{\partial}{\partial x_1}(f(\rho)Y(\sigma)) = \big(d_k f\big)(\rho)\sigma_1 Y(\sigma) + \rho^{-k} f(\rho)U_\rho(\sigma), \tag{9}$$

where $U_\rho(\sigma) = \frac{\partial}{\partial x_1}(\rho^k Y(\sigma))$. Notice that $\rho^{1-k}U_\rho(\sigma) \in \mathcal{H}_{k-1}$. Using now the formula

$$\sigma_j Y(\sigma) = A(\sigma) + B(\sigma), \text{ where } A \in \mathcal{H}_{k-1}, B \in \mathcal{H}_{k+1} \tag{10}$$

(see [5, Chapter 6, Lemma 3.4]) we obtain (i) from Lemma 1.

Next, let $K(\sigma) = (\sigma_1 + i\sigma_2)^k$. If $n > 2$ then we also set $K_m(\sigma) = \sigma_m(\sigma_1 + i\sigma_2)^{k-1}$, $3 \le m \le n$. Since $K, K_m \in \mathcal{H}_k$, by the hypothesis and Lemma 1 we obtain $h, h_m \in M_r^\infty(B_R)$, where $h(x) = f(\rho)K(\sigma)$ and $h_m(x) = f(\rho)K_m(\sigma)$. We have

$$\big(d_{2-k-n} f\big)(\rho)(\sigma_1 + i\sigma_2)^{k-1} = \frac{\partial h}{\partial x_1} - i\frac{\partial h}{\partial x_2} + \sum_{m=3}^{n} \frac{\partial h_m}{\partial x_m}$$

(if $n = 2$ then we set the last sum equal to zero). This relation and Lemma 1 imply assertion (ii). □

Remark 1. Examining the above proofs we see that Lemmas 1 and 2 remain valid for the class $V_r^\infty(B_R)$.

Lemma 3. *Let $f \in L(B_R)$ and suppose that this function has the form $f(x) = u(\rho)Y(\sigma)$ for some $Y \in \mathcal{H}_k$. Then there exists a function U on $[0, R-r]$ such that $(f * \chi_r)(x) = U(\rho)Y(\sigma)$.*

Proof. We set $f = 0$ outside B_R. Then $\widehat{f * \chi_r} = \widehat{f}\,\widehat{\chi_r}$. It follows from the assumption of the lemma that (see [5, Chapter 4, Theorem 3.10])

$$\widehat{f * \chi_r}(x) = W(\rho)Y(\sigma)$$

for some function W on $[0, +\infty)$. Hence (see [5, Chapter 4, Theorem 3.10]) we arrive at the desired assertion. □

Lemma 4. *Let $1 < R < 2$, let $f \in L(B_R)$ and assume that $f(x) = x_j\rho^{-n}$ in $B_{2-R,R}$ for some $j \in \{1, \dots, n\}$. Then $f \notin V_1(B_R)$.*

Proof. For $x \in \mathbb{R}^n \backslash \{0\}$ we set

$$\gamma_n(x) = \begin{cases} \frac{1}{2\pi}\ln|x|, & \text{if } n = 2; \\ ((2-n)\omega_{n-1})^{-1}|x|^{2-n}, & \text{if } n \ge 3, \end{cases}$$

where ω_{n-1} is the surface area of $\mathbb{S}^{n-1}$. Then one has

$$\omega_{n-1}\frac{\partial \gamma_n}{\partial x_j} = f \text{ in } B_{2-R,R}, \tag{11}$$

and

$$\int_{B_{2-R}} \frac{\partial \gamma_n}{\partial x_j}\, dx = 0. \tag{12}$$

The function γ_n is a fundamental solution for the operator Δ, that is, $\Delta\gamma_n = \delta_0$, where δ_0 is the Dirac measure supported at the origin (see [7, Chapter 5.2]). Hence $\Delta(\gamma_n * \chi) = \Delta\gamma_n * \chi = \chi$. In addition, the convolution $\gamma_n * \chi$ is a radial continuous function. This yields $(\gamma_n * \chi)(x) = c + |x|^2/(2n)$ in B_1 for some $c \in \mathbb{C}$, and

$$\left(\frac{\partial \gamma_n}{\partial x_j} * \chi\right)(x) = \frac{x_j}{n} \quad \text{in} \quad B_1. \tag{13}$$

Next, for each unit ball $B \subset B_R$ it follows that $B_{2-R} \subset B$ and

$$\int_B f(x)dx = \int_{B_{2-R}} f(x)dx + \int_{B\setminus B_{2-R}} f(x)dx. \tag{14}$$

However, equalities (11) and (12) imply that

$$\int_{B\setminus B_{2-R}} f(x)dx = \omega_{n-1} \int_B \frac{\partial \gamma_n}{\partial x_j} dx.$$

This together with (14) and (13) shows that $f \notin V_1(B_R)$, as contended. □

Lemma 5. *Let $Y \in \mathcal{H}_k$, $k \geq 2$, and let $1 < R < 2$. Suppose that a function $f \in L(B_R)$ possesses the following properties:*

(i) $f(x) = u(\rho)Y(\sigma)$ for some function $u\colon [0, R) \to \mathbb{C}$;

(ii) $u(\rho) = \rho^{k-n-2}$ for $\rho \in (2-R, R)$.

Then $x_1 f \notin V_1(B_R)$.

Proof. Let $\varepsilon \in (0, R-1)$. Consider a function $\varphi \in \mathfrak{H}_0^\infty(B_R)$ such that $\varphi = 0$ in $B_{2-R+\varepsilon/2}$ and $\varphi = 1$ in $B_{2-R+\varepsilon,R}$. Assume that $x_1 f \in V_1(B_R)$ and define $\psi = f\varphi$. Then by the definition of φ it follows that $x_1 f\varphi \in V_1^\infty(B_{R-\varepsilon})$. Using now (10) and Remark 1 we see from Lemma 1(iii) that $\rho u(\rho)\varphi(x)A(\sigma) \in V_1^\infty(B_{R-\varepsilon})$ for some $A \in \mathcal{H}_{k-1}$. If $k = 2$ this contradicts Lemma 4 (see Lemma 1(iii) and Remark 1). Suppose now that $k \geq 3$. Applying assertion (ii) of Lemma 2 repeatedly for $k-1, \ldots, 2$ we obtain a contradiction in the same way. □

Lemma 6. *Let $k, m \in \mathbb{N}$, $m \leq k$, let $Y \in \mathcal{H}_{k+1}$, and assume that $1 < R < 2$. Assume also that a function $f \in L_{loc}(B_R)$ satisfies the following conditions:*

(i) $\int_{B_{2-r}} f(x)dx = 0$;

(ii) $f(x) = \rho^{2m-n-k-1}Y(\sigma)$ for $x \in B_{2-R,R}$.

Then $f \in V_1(B_R)$.

Proof. Let $\varphi \in \mathfrak{H}_0^\infty(B_R)$ be a function such that $\varphi = 0$ in $B_{1-R/2}$ and $\varphi = 1$ in $B_{2-R,R}$, and let $\psi(x) = \varphi(x)w_{m,n}(|x|)$. For $x \in B_{R-1}$ we set

$$\Phi(x) = \int_{B_1} \psi(x-y)dy. \tag{15}$$

Then $\Phi \in \mathfrak{H}_0^\infty(B_{R-1})$. Since $\Delta^m \psi = 0$ in $B_{2-R,R}$ (see (7)), we deduce from (15) that $\Delta^m \Phi$ is identically constant. Hence $\Delta^{m+1}\Phi = 0$ and $\Phi(x) = \sum_{q=0}^m c_q |x|^{2q}$. Thus, the function $(\partial/\partial x_1)^{m+1}\Phi$ is a polynomial of degree at most $m-1$. On the other hand, using properties of φ and arguments of the proof of Lemma 2(i) one can show that

$$\left(\frac{\partial}{\partial x_1}\right)^{m+1} \psi = \sum_{q=0}^{m+1} h_q,$$

where $h_q \in \mathfrak{H}_q^\infty(B_R)$ and $h_q = 0$ in $B_{1-R/2}$. Moreover, h_{m+1} can be represented in the following form $h_{m+1}(x) = (D_m w_{m,n})(\rho) Y^{(m+1)}(\sigma)$, where $\rho \in [2-R, R)$ and $Y^{(m+1)} \in \mathcal{H}_{m+1}$ (see (9)). By Lemma 3 the convolution $h_{m+1} * \chi_1$ vanishes in B_{R-1}. This means that for each $X \in \mathcal{H}_{m+1}$ the function $h(x) = h_{m+1}(x) X(\sigma)/Y^{m+1}(\sigma)$ belongs to $V_1(B_R)$ (see Lemma 1 and Remark 1). Bearing in mind that $\big(D_m w_{m,n}\big)(\rho) = c\rho^{m-n-1}$, where $c \in \mathbb{C}\backslash\{0\}$ (see (5) and (6)), and

$$\int_{B_{2-R}} h(x)dx = 0$$

(see [5, Chapter 4, Corollary 2.4]), we see that for each unit ball $B \subset B_R$ the integral of the function $\rho^{m-n-1} Y(\sigma)$ over $B \backslash B_{2-R}$ vanishes. This proves Lemma 6 for $k = m$. Applying assertion (i) of Lemma 2 to h (see Remark 1) we obtain in a similar way the assertion of Lemma 6 for all $m \le k$. □

Corollary 1. *Let $k, m \in \mathbb{Z}_+$, $k \ge 3$, $m \le k-3$, let $Y \in \mathcal{H}_k$, and assume that $1 < R < 2$. Suppose that a function $f \in L(B_R)$ satisfies (3) and $f(x) = \rho^{2m-n-k+2} Y(\sigma)$ for $x \in B_{2-R,R}$. Then $f \in M_r(B_R)$.*

The proof follows from Lemma 6 and equality (10).

We shall now study some properties of expansions in the Gegenbauer polynomials $C_k^{n/2}$ (see [6, Chapter 9]). We shall use the well known result: the Fourier–Jacobi series of functions in the class $C^\infty[-1,1]$ are uniformly convergent on $[-1,1]$ (see, for instance, [8, Chapter 7]).

Lemma 7. *Assume that $n \ge 3$, $0 < \varepsilon < 1$, let $f(|x|) \in C^\infty(\overline{B}_{1-\varepsilon, 1+\varepsilon})$, and let*

$$f(\sqrt{1+s^2+2st}) = \sum_{k=0}^{\infty} f_k(s) C_k^{(n/2)-1}(t) \tag{16}$$

for all $t \in [-1,1]$, $s \in [0, \varepsilon]$. If $f_0(s) = 0$ on $[0,\varepsilon]$ then

$$s f_k(s) = (n+2k-4)^{-1}\big(s f'_{k-1}(s) - (k-1) f_{k-1}(s)\big) - \\ -(n+2k)^{-1}\big(s f'_{k+1}(s) + (n+k-1) f_{k+1}(s)\big)$$

for $k \ge 1$.

Proof. Since $f \in C^\infty$, series (16) is uniformly convergent in t on $[-1,1]$ for each $s \in [0,\varepsilon]$. Let $u(s,t) = f\big(\sqrt{1+s^2+2st}\big)$; then

$$f_k(s) = c_{k,n} \int_{-1}^{1} u(s,t) C_k^{(n/2)-1}(t)(1-t^2)^{(n-3)/2} dt, \tag{17}$$

where

$$c_{k,n}=\frac{k!(n+k-2)\Gamma^2\left(\frac{n}{2}-1\right)2^{n-4}}{\pi\Gamma(k+n-2)}$$

(see [6, Chapter 9, § 3, i. 4]). By the definition of u we have $s\frac{\partial u}{\partial s}=(s+t)\frac{\partial u}{\partial t}$. In view of the equality

$$dC_k^{(n/2)-1}(t)/dt=(n-2)C_{k-1}^{n/2}(t)$$

(see [6, Chapter 9, § 3, i. 2]), it follows from the assumptions of the lemma and (17) that

$$(t+s)\frac{\partial u}{\partial t}=\sum_{k=1}^{\infty}(n-2)(t+s)f_k(s)C_{k-1}^{n/2}(t) \tag{18}$$

and

$$s\frac{\partial u}{\partial s}=\sum_{k=1}^{\infty}sf_k'(s)C_k^{(n/2)-1}(t), \tag{19}$$

where series (18) and (19) are uniformly convergent in t on $[-1,1]$ for each $s\in[0,\varepsilon]$. Using formulae

$$C_{k+2}^{\lambda}(t)=\frac{\lambda}{\lambda+k+2}\left(C_{k+2}^{\lambda+1}(t)-C_k^{\lambda+1}(t)\right),$$
$$tC_{k+1}^{\lambda}(t)=\frac{k+2}{2(\lambda+k+1)}C_{k+2}^{\lambda}(t)+\frac{2\lambda+k}{2(\lambda+k+1)}C_k^{\lambda}(t)$$

for $\lambda=(n/2)-1$ (see [6, Chapter 9, § 3, i. 2]) we can represent the difference between the series in (18) and (19) as a Fourier-Jacobi series in the polynomials $C_k^{n/2}$. The coefficients of this series vanish, which gives us the assertion of Lemma 7. □

The following result is an analogue of Lemma 7 for $n=2$.

Lemma 8. *Let $n=2$, assume that $0<\varepsilon<1$, let $f(|x|)\in C^{\infty}(\overline{B}_{1-\varepsilon,1+\varepsilon})$, and let*

$$f\left(\sqrt{1+2s\cos\theta+s^2}\right)=\sum_{k=0}^{\infty}f_k(s)\cos k\theta \tag{20}$$

for $s\in[0,\varepsilon]$ and $\theta\in[0,\pi]$. If $f_0(s)=0$ on $[0,\varepsilon]$ then

$$2ksf_k(s)=sf_{k-1}'(s)-sf_{k+1}'(s)-(k-1)f_{k-1}(s)-(k+1)f_{k+1}(s)$$

for $k\geq 1$.

Proof. We set $v(s,\theta)=f\left(\sqrt{1+2s\cos\theta+s^2}\right)$; then

$$(s+\cos\theta)\frac{\partial v}{\partial\theta}+\frac{\partial v}{\partial s}s\,\sin\theta=0.$$

Using (20) we can expand the function on the left hand side of this equality in a Fourier series in the system $\{\sin k\theta\}$ on $[0,\pi]$. The coefficients of the series vanish, which proves Lemma 8. □

4. Proof of the central result. We now proceed to the proof of the first assertion of Theorem 1.

Proof. Necessity. Assume that $R \leq 2r$ and let $f \in M_r(B_R)$. First, we prove (4). Without loss of generality we can assume that $r = 1$, $R < 2$, and $f \in C^\infty(B_R)$ (see [5, Chapter 1, Theorem 1.18]). Thus we have $x_j f \in V_1^\infty(B_R)$ for each $j \in \{1, \dots, n\}$. Hence $\frac{\partial}{\partial x_j}(x_j f) \in V_1^\infty(B_R)$ and $\frac{\partial f}{\partial x_j} \in V_1^\infty(B_R)$. Therefore, for all $y \in B_{R-1}$ we obtain

$$\int_{B_1} \frac{\partial}{\partial x_j}\big(x_j f(x+y)\big)dx = \int_{B_1} \frac{\partial}{\partial x_j}\big((x_j + y_j) f(x+y)\big)dx - y_j \int_{B_1} \frac{\partial}{\partial x_j}\big(f(x+y)\big)dx = 0.$$

By the Gauss divergence theorem this implies that

$$\int_{\mathbb{S}^{n-1}} f(\sigma + y)\sigma_j^2 d\omega(\sigma) = 0.$$

Summation over the set of all $j \in \{1, \dots, n\}$ yields

$$\int_{\mathbb{S}^{n-1}} f(\sigma + y) d\omega(\sigma) = 0 \quad \text{for all} \quad y \in B_{R-1}.$$

The same equality holds if f is replaced with $f_{0,1}(|x|)$ because of Lemma 1. Consequently,

$$\int_{-1}^{1} f_{0,1}\big(\sqrt{1+s^2+2st}\big)(1-t^2)^{(n-3)/2} dt = 0, \quad 0 < s < R-1. \tag{21}$$

Next, since $\frac{\partial}{\partial x_1}(f_{0,1}(|x|)) \in V_1(B_R)$, one has

$$\int_{-1}^{1} f_{0,1}\big(\sqrt{1+s^2+2st}\big)t(1-t^2)^{(n-3)/2} dt = 0, \quad 0 < s < R-1. \tag{22}$$

Taking (21), (22) and (17) into account we conclude from Lemmas 7 and 8 that $f_{0,1}(|x|) = 0$ for all $x \in B_{2-R,R}$.

Next, $f_{1,l}(\rho)Y_l^{(1)}(\sigma) \in M_1^\infty(B_R)$ for all $l \in \{1, \dots, a_1\}$ in view of Lemma 1. Use of Lemma 2(ii) and the result for $f_{0,1}$ which obtained above then leads to the conclusion that $d_{1-n} f_{1,l}(\rho) = 0$ for $\rho \in (2-R, R)$. Together with Lemma 4 this shows that $f_{1,l}(|x|) = 0$ for all $x \in B_{2-R,R}$, $l \in \{1, \dots, a_1\}$. Similarly, $f_{2,l}(|x|) = 0$ for all $x \in B_{2-R,R}$, $l \in \{1, \dots, a_2\}$ because of Lemmas 2(ii) and 5. Finally, in the case $k \geq 3$ equality (4) is obtained by induction on k (see Lemmas 2(ii), 5, and Corollary 1).

We now prove (3). Using the fact that $\mathcal{H}_{k_1}$ is orthogonal to $\mathcal{H}_{k_2}$ for $k_1 \neq k_2$ we see from equality (4) with $k = 0$ that

$$\int_{B_{2-R}} f(x)dx = \int_{B_{2-R}} f_{0,1}(|x|)dx = \int_{B_1} f(x)dx = 0.$$

Similarly, it follows from (4) with $k = 1$ that

$$\int_{B_{2-R}} f(x)x_j dx = \sum_{l=1}^{n} \int_{B_{2-R}} f_{1,l}(|x|)Y_l^{(1)}(\sigma)x_j dx =$$

$$= \sum_{l=1}^{n} \int_{B_1} f_{1,l}(|x|)Y_l^{(1)}(\sigma)x_j dx = \int_{B_1} f(x)x_j dx = 0,$$

which proves (3).

Sufficiency. Clearly, it suffices to consider the case where $R < 2r$. By hypothesis and Corollary 1 we conclude that $f_{k,l}(\rho)Y_l^{(k)}(\sigma) \in M_r(B_R)$ for all $k \in \mathbb{Z}_+$, $l \in \{1, \ldots, a_k\}$. However, the series on the right-hand side of (2) converges to f in the space $\mathcal{D}'(B_R)$ of distributions on B_R (see [9, § 1]). This ensures us that $f \in M_r(B_R)$. The first assertion of Theorem 1 is thereby established.

Finally, we note that the second assertion of Theorem 1 can easily be derived from its first assertion and Lemma 1(iii) by means of the standard smoothing procedure (see [5, Chapter 1, Theorem 1.18]). $\square$

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