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ON SOLUTIONS OF ONE CONVOLUTION EQUATION GENERATED BY A "DEEP ZERO"

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We consider a convolution type equation in the Smirnov spaces in a semi-strip. We obtain a description of solutions for the case when the characteristic function of the equation has a "deep zero" at infinity.

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Рассматривается уравнение типа свертки в пространствах Смирнова в полуполосе. Получено описание решений для случая, если характеристическая функция уравнения имеет "глубокий нуль" на бесконечности.

By definition, put $\mathbb{C}_+ = \{z\colon \operatorname{Re} z > 0\}$, $\mathbb{C}_- = \{z\colon \operatorname{Re} z < 0\}$, $\mathbb{C}^+ = \{z\colon \operatorname{Im} z > 0\}$, $\mathbb{C}^- = \{z\colon \operatorname{Im} z < 0\}$. By $H^p(\mathbb{C}_+), 0 < p < +\infty$, denote the Hardy space of analytic on $\mathbb{C}_+$ functions such that

$$\|f\|_{H^p(\mathbb{C}_+)} = \sup\Big\{\Big(\int_{-\infty}^{+\infty} |f(x+iy)|^p dy\Big)^{1/p}\colon\ x>0\Big\} < +\infty.$$

A function $f \in H^p(\mathbb{C}_+)$ (see [1]) has angular boundary values almost everywhere (a.e.) on $i\mathbb{R}$ which we denote by f and $f(iy) \in L^p(-\infty;+\infty)$. Here and below $\|\cdot\|$ denotes the norm for the case $p \geq 1$ and the quasi-norm for $0 < p < 1$. A. M. Sedletskii established ([2]) that the space $H^p(\mathbb{C}_+)$, $p > 0$, can be defined as a class of analytic on $\mathbb{C}_+$ functions such that

$$\|f\|^*_{H^p(\mathbb{C}_+)} = \sup\Big\{\Big(\int_0^{+\infty} |f(re^{i\varphi})|^p dr\Big)^{1/p}\colon\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\Big\} < +\infty$$

and

$$2^{-1/p}\|f\|_{H^p(\mathbb{C}_+)} \leq \|f\|^*_{H^p(\mathbb{C}_+)} \leq \|f\|_{H^p(\mathbb{C}_+)}. \tag{1}$$

Let M be the set of all segments lying in $D_\sigma = \{z\colon |\operatorname{Im} z| < \sigma, \operatorname{Re} z < 0\}$ and let M^* be the set of all segments lying in $D^*_\sigma = \mathbb{C}\backslash\overline{D}_\sigma$. Suppose $\widetilde{M}$ and $\widetilde{M^*}$ are the sets of all segments which are parallel to the coordinate axes and lying in D_σ and D^*_σ, respectively. We also denote by $E^p[D_\sigma]$ and $\widetilde{E}^p[D_\sigma]$, $0 < p < +\infty$, $\sigma > 0$, the spaces of analytic functions in D_σ such that

$$\sup\Big\{\int_\gamma |f(z)|^p |dz|\Big\}^{1/p} < +\infty, \tag{2}$$

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where the supremum is taken over all segments γ that belong to M and $\widetilde{M}$ respectively. We denote by $E^p_*[D_\sigma]$ and $\widetilde{E}^p_*[D_\sigma]$, $0 < p < +\infty$, $\sigma > 0$, the spaces of analytic functions in D^*_σ such that inequality (2) holds, where supremum is taken over all segments γ that belong to M^* and $\widetilde{M}^*$ respectively.

We claim that $f \in E^p[D_\sigma]$ if and only if $f \in \widetilde{E}^p[D_\sigma]$ for $p > 1$. Indeed, if $f \in \widetilde{E}^p[D_\sigma]$ then (see [3, Lemma 5]) $f = f_1 + f_2 + f_3$, where $f_1 \in H^p(\mathbb{C}^+ - i\sigma)$, $f_2 \in H^p(\mathbb{C}_-)$, $f_3 \in H^p(\mathbb{C}^- + i\sigma)$. Here for an arbitrary $\gamma \in M$ by Sedletskii's theorem, $\int_\gamma |f_j(z)|^p |dz| \le c_1 < +\infty$, $j \in \{1; 2; 3\}$. Therefore $f \in E^p[D_\sigma]$.

We claim that $f \in E^p_*[D_\sigma]$ if and only if $f \in \widetilde{E}^p_*[D_\sigma]$ for $p > 0$. Let $f \in \widetilde{E}^p_*[D_\sigma] = H^p(\mathbb{C}^+ + i\sigma) \cap H^p(\mathbb{C}_+) \cap H^p(\mathbb{C}^- - i\sigma)$. Then by Sedletskii's theorem $\int_\gamma |f(z)|^p |dz| \le c_2$, where c_2 does not depend on γ.

The spaces $E^p[D_\sigma]$ and $E^p_*[D_\sigma]$ were studied in [3]. There it has been shown that a function f from either of these spaces has angular boundary values a.e. on ∂D_σ which will be denoted by $f(z)$ and $f \in L^p[\partial D_\sigma]$.

Let $H^p_\sigma(\mathbb{C}_+), \sigma \ge 0, 1 \le p < +\infty$, be the space of analytic on $\mathbb{C}_+$ functions, for which

$$\|f\| := \sup\Big\{\Big(\int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr\Big)^{1/p} :\ -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\Big\} < +\infty.$$

A function $f \in H^p_\sigma(\mathbb{C}_+)$ (see [3], Lemma 2 in [4]) has angular boundary values a. e. on $i\mathbb{R}$ which will be denoted by f and $f(iy)e^{-\sigma|y|} \in L^p(-\infty; +\infty)$. The space $H^p_\sigma(\mathbb{C}_+)$ for the case $\sigma = 0$ is the Hardy space $H^p(\mathbb{C}_+)$.

Let $T^2_\sigma(\mathbb{C}_-)$ be the set of all triples $F = (F_1, F_2, F_3)$, where $F_1(z)e^{-i\sigma z} \in H^2(\mathbb{C}_-)$, $F_3(z)e^{i\sigma z} \in H^2(\mathbb{C}_-)$, F_2 is an entire function of exponential type $\le \sigma$, $F_2 \in L^2(\mathbb{R})$, and $F_1(z) + F_2(z) + F_3(z) \equiv 0$ for $z \in \mathbb{C}_-$. The equalities

$$F_j(z) = \frac{1}{\sqrt{2\pi}} \int_{l_j} f(w)e^{-zw} dw, \quad f \in E^2[D_\sigma], \quad j \in \{1, 2, 3\}, \tag{3}$$

define (see [5, Theorem 1]) a bijection between the spaces $T^2_\sigma(\mathbb{C}_-)$ and $E^2[D_\sigma]$, where l_1, l_3 and l_2 are the legs of ∂D_σ respectively the rays laying under and above of the real axis, and the segment $[-i\sigma; i\sigma]$) and their orientation corresponds to the positive orientation of D_σ. The inverse formula

$$f(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} F_1(iy)e^{iyw} dy + \frac{1}{i\sqrt{2\pi}} \int_0^{+\infty} F_2(x)e^{xw} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 F_3(iy)e^{iyw} dy,\ w \in D_\sigma, \tag{4}$$

is also valid.

The equation

$$\int_{-\infty}^0 f(u+\tau)g(u)du = 0, \quad g \in L^2(-\infty; 0) \tag{5}$$

is studied in [6, 7, 8]. Its generalization

$$\int_{\partial D_\sigma} f(w+\tau)g(w)dw = 0, \quad \tau \le 0, \quad g \in E^2_*[D_\sigma], \tag{6}$$

is investigated in [5, 9, 10]. The equality

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_\sigma} g(w)e^{zw} dw \tag{7}$$

defines a bijection between the spaces $H^2_\sigma(\mathbb{C}_+)$ and $E^2_*[D_\sigma]$.

In [9, 10] it is shown that equation (6) has a nontrivial solution $f \in E^2[D_\sigma]$ if and only if one of the following conditions is valid:

1) G has at least one zero at $\lambda \in \mathbb{C}_+$;

2) the singular boundary function of G is not a constant;

3) $$\varlimsup_{x\to+\infty}\Big(\frac{\log|G(x)|}{x}+\frac{2\sigma}{\pi}\log x\Big)<+\infty. \tag{8}$$

The singular boundary function h of $\psi \in H^p_\sigma(\mathbb{C}_+)$ is defined up to an additive constant, and to the values at points of continuity by the equality

$$h(t_2)-h(t_1)=\lim_{x\to 0+}\int_{t_1}^{t_2}\log|\psi(x+iy)|dy-\int_{t_1}^{t_2}\log|\psi(iy)|dy.$$

The singular boundary function of analytic in $\overline{\mathbb{C}}_+$ function $\psi \in H^p_\sigma(\mathbb{C}_+)$ is a constant (see [1]). For the cases 1) and 2) solutions were studied in [3] and [11]. But the problem of constructing a representation of solutions in the third case left open. Following [12, p. 2, 1.2], we say that in this case the function G has a "deep zero".

Note that the case has no analog for $\sigma = 0$. Indeed, for $\sigma = 0$ equation (5) has a nontrivial solution $f \in L^2(-\infty; 0)$ if and only if ([6]) either $G(z) = 0$ for some $z \in \mathbb{C}_+$, or the singular boundary function of G is not a constant, or

$$\varlimsup_{x\to+\infty}\frac{\log|G(x)|}{x}<0.$$

In the present paper we describe the spectral analysis (see [7]) in $E^2[D_\sigma]$ for the case of "deep zero".

Theorem 1. *Let $f \in E^2[D_\sigma]$, $f \not\equiv 0$, be a solution of equation (6), a function G have no zero in $\mathbb{C}_+$ and the singular boundary function of G be a constant. Then the functions F_1 and F_3 defined by (3) are entire,*

$$F_1(z)=e^{i\sigma z}e^{a_1 z}\varkappa_1(z)\prod_{|\lambda_n|\le 1}\frac{z-\lambda_n}{z+\overline{\lambda}_n}\prod_{|\lambda_n|>1}\frac{1-z/\lambda_n}{1+z/\overline{\lambda}_n}\exp\Big(\frac{z}{\lambda_n}+\frac{z}{\overline{\lambda}_n}\Big),\ z\in\mathbb{C}_+,\ \lambda_n\in\mathbb{C}_+, \tag{9}$$

$$F_3(z)=e^{-i\sigma z}e^{a_3 z}\varkappa_3(z)\prod_{|\mu_n|\le 1}\frac{z-\mu_n}{z+\overline{\mu}_n}\prod_{|\mu_n|>1}\frac{1-z/\mu_n}{1+z/\overline{\mu}_n}\exp\Big(\frac{z}{\mu_n}+\frac{z}{\overline{\mu}_n}\Big),\ z\in\mathbb{C}_+,\ \mu_n\in\mathbb{C}_+, \tag{10}$$

where $a_1 \in \mathbb{R}$, $a_3 \in \mathbb{R}$, zeros λ_n and μ_n of the functions F_1 and F_3 satisfy the conditions

$$\sum_{|\lambda_n|\le 1}\operatorname{Re}\lambda_n<+\infty,\quad \varlimsup_{r\to+\infty}\left(\sum_{1<|\lambda_n|\le r}\Big(\frac{1}{|\lambda_n|}-\frac{|\lambda_n|}{r^2}\Big)\frac{\operatorname{Re}\lambda_n}{|\lambda_n|}-\frac{\sigma}{\pi}\log r\right)=\beta_1,\quad \beta_1\in\mathbb{R}, \tag{11}$$

$$\sum_{|\mu_n|\le 1}\operatorname{Re}\mu_n<+\infty,\quad \varlimsup_{r\to+\infty}\left(\sum_{1<|\mu_n|\le r}\Big(\frac{1}{|\mu_n|}-\frac{|\mu_n|}{r^2}\Big)\frac{\operatorname{Re}\mu_n}{|\mu_n|}-\frac{\sigma}{\pi}\log r\right)=\beta_3,\quad \beta_3\in\mathbb{R}, \tag{12}$$

$\varkappa_1 \in H^2(\mathbb{C}_+)$, $\varkappa_3 \in H^2(\mathbb{C}_+)$, the functions $\varkappa_1$ and $\varkappa_3$ have no zero in $\mathbb{C}_+$ and their singular boundary functions are constants. Moreover, $F_1(z)e^{-i\sigma z} \in H^2(\mathbb{C}_-)$, $F_3(z)e^{i\sigma z} \in H^2(\mathbb{C}_-)$.

The following statement, in some sense, is converse to the previous one.

Theorem 2. *Assume $G \in H^2_\sigma(\mathbb{C}_+)$, G has no zero in $\mathbb{C}_+$, the singular boundary function of G is a constant, and inequality (8) holds. If for the functions F_1, F_3, representations (9), (10) are valid, where $a_1 \in \mathbb{R}$, $a_3 \in \mathbb{R}$, $\varkappa_1 \in H^2(\mathbb{C}_+)$, $\varkappa_3 \in H^2(\mathbb{C}_+)$ conditions (11), (12) hold and $(F_1, -F_1 - F_3, F_3) \in T^2_\sigma(\mathbb{C}_-)$, then for some $c \geq 0$ the equality $F_2 = -F_1 - F_3$ and representation (4) give a solution of the equation*

$$\int_{\partial D_\sigma} f(w+\tau)\widehat{g}(w)dw = 0, \ \tau \leq 0, \tag{13}$$

where $\widehat{g}(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \widehat{G}(x)e^{-wx}dx,\ \widehat{G}(z) = G(z)e^{-cz}$.

Remark. Condition (11) cannot be replaced (see [3]) with the condition

$$\sum_{|\lambda_n|\leq 1} \operatorname{Re}\lambda_n < +\infty, \quad \varlimsup_{r\to+\infty}\left(\sum_{1<|\lambda_n|\leq r} \frac{\operatorname{Re}\lambda_n}{|\lambda_n|^2} - \frac{\sigma}{\pi}\log r\right) = \beta_1, \quad \beta_1 \in \mathbb{R}.$$

For the proof of Theorem 1 we provide some auxiliary results.

Lemma 1. *If $f \in E^2[D_\sigma]$ is a solution of equation (6), then for each $c > 0$ the function f is also a solution of equation (13).*

Proof. Indeed, by Theorem 2 from [5]

$$\int_{\partial D_\sigma} f(w+\tau)\widehat{g}(w)dw = \int_0^{+i\infty} F_1(z)G(z)e^{-cz}e^{\tau z}dz + \int_{-i\infty}^0 F_3(z)G(z)e^{-cz}e^{\tau z}dz +$$
$$+\int_0^{+\infty} F_2(z)G(z)e^{-cz}e^{\tau z}dz = \int_0^{+i\infty} F_1(z)G(z)e^{(\tau-c)z}dz + \int_{-i\infty}^0 F_2(z)G(z)e^{(\tau-c)z}dz +$$
$$+\int_0^{+\infty} F_2(z)G(z)e^{(\tau-c)z}dz.$$

The right-hand side of the above equality is equal to zero for all $\tau \in (-\infty; 0)$. □

Lemma 2 ([13]). *If $f \in H^p_\sigma(\mathbb{C}_+), 1 \leq p < +\infty$, $\sigma > 0$ and $f \not\equiv 0$, then*

$$f(z) = e^{ia_0 + a_1 z} \cdot \Pi^*_f(z) \cdot S^*_f(z) \cdot \exp\Big\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\log|f(it)|dt\Big\}, \tag{14}$$

where a_0, a_1 are real constants,

$$\Pi^*_f(z) = \prod_{|\lambda_n|\leq 1} \frac{z-\lambda_n}{z+\overline{\lambda}_n} \prod_{|\lambda_n|>1} \frac{1 - z/\lambda_n}{1 + z/\overline{\lambda}_n} \exp\Big(\frac{z}{\lambda_n} + \frac{z}{\overline{\lambda}_n}\Big),\ S^*_f(z) = \exp\Big\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)dh(t)\Big\}, \tag{15}$$

(λ_n) is a zero sequence in $\mathbb{C}_+$ of f, $Q(t,z) = \frac{(tz+i)^2}{(1+t^2)^2(t+iz)}$, also the conditions

$$\sum_{|\lambda_n|\leq 1} \operatorname{Re}\lambda_n < \infty,\ \log|f(iy)| \in L^1(-1;1),\ f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R}), \tag{16}$$

$$\varlimsup_{r\to+\infty}\Big(S_f(r) + \Xi_f(r) - K_f(r)\Big) < +\infty, \tag{17}$$

are valid, where

$$S_f(r) = \sum_{1<|\lambda_n|\le r} \Big(\frac{1}{|\lambda_n|} - \frac{|\lambda_n|}{r^2}\Big)\frac{Re\lambda_n}{|\lambda_n|}, \quad \Xi_f(r) = \frac{1}{2\pi}\int_{1<|t|\le r}\Big(\frac{1}{t^2}-\frac{1}{r^2}\Big)|dh(t)|,$$

$$K_f(r) = \frac{1}{2\pi}\int_{1<|t|\le r}\Big(\frac{1}{t^2}-\frac{1}{r^2}\Big)\log|f(it)|dt,$$

and all products and integrals in (14) converge absolutely and uniformly on each compact subset of $\mathbb{C}_+$.

Lemma 3 ([14])**.** *Let* $g \in E^2_*[D_\sigma]$ *and* $G(x)\log(2+x) \in L^2(0;+\infty)$ *for* G*, defined by (7). Then* $f \in E^2[D_\sigma]$ *is a solution of (6) if and only if the following conditions are valid:*

1) *there exists a function* P_1, $P_1(z)e^{-i\sigma z} \in H^1_\sigma(\mathbb{C}_+)$*, such that the angular boundary values of* P_1/G *from* $\mathbb{C}_+$ *coincide with the angular boundary values of* F_1 *from* $\mathbb{C}_-$ *almost everywhere on* $i\mathbb{R}$*;*
2) *there exists a function* P_3, $P_3(z)e^{i\sigma z} \in H^1_\sigma(\mathbb{C}_+)$*, such that the angular boundary values of* P_3/G *from* $\mathbb{C}_+$ *coincide with the angular boundary values of* F_3 *from* $\mathbb{C}_-$ *almost everywhere on* $i\mathbb{R}$.

Lemma 4. *Suppose that the function* f *belongs to the Smirnov space* E^1 *in the domains* $\square_1 = \{z\colon 0 < \operatorname{Re} z < 1, a < \operatorname{Im} z < a+1\}$, $\square_2 = \{z\colon -1 < \operatorname{Re} z < 0, a < \operatorname{Im} z < a+1\}$ *and the angular boundary functions of* f *from* $\square_1$ *and* $\square_2$ *coincide* a. e. *on* $\{z = iy\colon y \in (a; a+1)\}$*. Then* f *belongs to the Smirnov space* E^1 *in the domain* $\square = \{z\colon -1 < \operatorname{Re} z < 1, a < \operatorname{Im} z < a+1\}$.

Proof. Indeed, the Smirnov space E^1 coincides (see [15, P. III, 7.1]) with the class of functions, representable by the Cauchy integral formula. Therefore

$$\frac{1}{2\pi i}\int_{\partial\square_1}\frac{f(t)}{t-z}dt = \begin{cases} f(z), & z\in\square_1;\\ 0, & z\in\square_2,\end{cases} \quad \frac{1}{2\pi i}\int_{\partial\square_2}\frac{f(t)}{t-z}dt = \begin{cases} f(z), & z\in\square_2;\\ 0, & z\in\square_1,\end{cases}$$

The function

$$\Xi(z) = \frac{1}{2\pi i}\int_{\partial\square}\frac{f(t)}{t-z}dt$$

is analytic on $\square$, coincides with f for $z\in\square_1$ and $z\in\square_2$, hence Ξ belongs to E^1 in $\square$. $\square$

Lemma 5 (Theorem 3 [10])**.** *Suppose* $\widetilde{F}_1(z)e^{-i\sigma z} \in H^2_\sigma(\mathbb{C}_+)$, $\widetilde{F}_3(z)e^{i\sigma z} \in H^2_\sigma(\mathbb{C}_+)$, $(\widetilde{F}_1(x)+\widetilde{F}_3(x))e^{\frac{2\sigma}{\pi}x\log x} \in L^2(0;+\infty)$*, and*

$$\lim_{x\to+\infty}\frac{\log|\widetilde{F}_j(x)|}{x} = -\infty, \quad j\in\{1;3\}. \tag{18}$$

Then there exists $c\in\mathbb{R}$*, such that*

$$\widetilde{F}_1(z)e^{-i\sigma z}e^{\frac{2\sigma}{\pi}z\log z}e^{-cz} \in H^2(\mathbb{C}_+), \quad \widetilde{F}_3(z)e^{i\sigma z}e^{\frac{2\sigma}{\pi}z\log z}e^{-cz} \in H^2(\mathbb{C}_+), \tag{19}$$

where $\log z$ *is the principal branch of the logarithm in* $\mathbb{C}_+$.

Lemma 6 (Lemma 9 [10])**.** *If* $(F_1, F_2, F_3) \in T^2_\sigma(\mathbb{C}_-)$*, the functions* F_1, F_3 *are entire and*

$$(\exists c_1\in\mathbb{R})\colon F_1(z)e^{-i\sigma z}e^{-c_1 z} \in H^2(\mathbb{C}_+), (\exists c_2\in\mathbb{R})\colon F_3(z)e^{i\sigma z}e^{-c_2 z}\in H^2(\mathbb{C}_+),$$

then $(F_1, F_2, F_3) \equiv (0,0,0)$.

Proof of Theorem 1. Let a function $f \in E^2[D_\sigma]$, $f \not\equiv 0$ be a solution of equation (6). Then by Lemma 1 f is a solution of equation (13) too. By Lemma 3 there exists a function P_1, $P_1(z)e^{-i\sigma z} \in H^1_\sigma(\mathbb{C}_+)$, such that the angular boundary values of $P_1/\widehat{G}$ from $\mathbb{C}_+$ coincide with the angular boundary values of F_1 from $\mathbb{C}_-$ almost everywhere on $i\mathbb{R}$. Then by Lemma 2

$$P_1(z) = e^{ia_0+a_1z+i\sigma z} \cdot \prod_{|\lambda_n|\le 1} \frac{z-\lambda_n}{z+\overline{\lambda}_n} \prod_{|\lambda_n|>1} \frac{1-z/\lambda_n}{1+z/\overline{\lambda}_n} \exp\Big(\frac{z}{\lambda_n}+\frac{z}{\overline{\lambda}_n}\Big) \cdot S^*_{P_1}(z)\times$$

$$\times \exp\Big\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\log|P_1(it)e^{\sigma t}|dt\Big\}, \quad z\in\mathbb{C}_+, \tag{20}$$

where (λ_n) is a sequence of zeros of P_1 in $\mathbb{C}_+$. By the assumptions of the theorem, $G(z)\neq 0$, $z\in\mathbb{C}_+$ and the singular boundary function of G is a constant, then by Lemma 2

$$G(z) = e^{i\widehat{a}_0+\widehat{a}_1 z}\cdot \exp\Big\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\log|G(it)|dt\Big\}, \quad z\in\mathbb{C}_+. \tag{21}$$

If we combine this statement with (20), we obtain for $z\in\mathbb{C}_+$

$$P_1(z)/G(z) = e^{i\sigma z}e^{i\widetilde{a}_0+\widetilde{a}_1 z}\Pi^*_{P_1}(z)S^*_{P_1}(z)\cdot \exp\Big\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\log\Big|P_1(it)e^{\sigma t}/G(it)\Big|dt\Big\}.$$

The function P_1/G coincides with the angular boundary values of $F_1(z)$ a.e. on $i\mathbb{R}$ from $\mathbb{C}_-$. But $F_1(z)e^{-i\sigma z}\in H^2(\mathbb{C}_-)$, hence ([1, VI. C.])

$$\int_{-\infty}^{+\infty} \frac{|\log|F_1(it)e^{\sigma t}||}{1+t^2}dt < +\infty. \tag{22}$$

Using

$$\frac{1}{i}Q(t,z) = \frac{-1}{it-z} - \frac{it(2+t^2)}{(1+t^2)^2} - \frac{zt^2}{(1+t^2)^2},$$

we get ([1, VI. C.])

$$(\exists c\in\mathbb{R}):\ \exp\Big\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q(t,z)\log\Big|P_1(it)e^{\sigma t}/G(it)\Big|dt - cz\Big\}\in H^2(\mathbb{C}_+). \tag{23}$$

It is clear that condition (17) of Lemma 2 is valid for functions f, such that $f(z)e^{i\sigma z}\in H^2_\sigma(\mathbb{C}_+)$, because $S_f\equiv S_{f(z)e^{i\sigma z}}$, $\Xi_f\equiv \Xi_{f(z)e^{i\sigma z}}$. Since $\int_{1<|t|\le r}(1/t^2-1/r^2)\sigma t dt = 0$, we get $K_f(r) = K_{f(z)e^{i\sigma z}}(r)$ for all $r>1$. Hence we will write S_{P_1} instead of $S_{P_1(z)e^{-i\sigma z}}$, $\Xi_{P_1}(r)$ instead of $\Xi_{P_1(z)e^{-i\sigma z}}$, and $K_{P_1}(r)$ instead of $K_{P_1(z)e^{-i\sigma z}}$. Condition (17) is valid also for functions f, such that $f(z)e^{-i\sigma z}\in H^2_\sigma(\mathbb{C}_+)$. Therefore we obtain

$$\varlimsup_{r\to+\infty}\Big(S_{P_1}(r)+\Xi_{P_1}(r)-K_{P_1}(r)\Big) < +\infty. \tag{24}$$

We obviously have $K_{P_1}(r) = K_{P_1/G}(r)+K_G(r)$ and using the notation $\log^+ t = \max\{\log t; 0\}$ we obtain by (22)

$$K_{P_1/G}(r) = K_{P_1(z)/G(z)e^{-i\sigma z}} \le \frac{1}{2\pi}\int_{1<|t|\le r}\Big(\frac{1}{t^2}-\frac{1}{r^2}\Big)\log^+|P_1(it)/G(it)e^{\sigma t}|dt \le$$

$$\le \frac{1}{2\pi}\int_{1<|t|\le r}\frac{1}{t^2}\log^+|P_1(it)/G(it)e^{\sigma t}|dt \le \frac{1}{\pi}\int_{1<|t|\le r}\frac{|\log|P_1(it)/G(it)e^{\sigma t}||}{t^2+1}dt < +\infty.$$

And as

$$K_G(r) \leq \frac{1}{2\pi}\int_{1<|t|\leq r}\Big(\frac{1}{t^2}-\frac{1}{r^2}\Big)\log^+|G(it)|dt \leq$$
$$\leq \frac{1}{2\pi}\int_{1<|t|\leq r}\Big(\frac{1}{t^2}-\frac{1}{r^2}\Big)\log^+\Big|G(it)e^{-\sigma|t|}\Big|dt + \frac{1}{2\pi}\int_{1<|t|\leq r}\frac{1}{t^2}\sigma|t|dt \leq$$
$$\leq \frac{1}{4\pi}\int_{1<|t|\leq r}\frac{1}{t^2}\Big|G(it)e^{-\sigma|t|}\Big|^2dt + \frac{\sigma}{2\pi}\int_{1<|t|\leq r}\frac{1}{|t|}dt \leq c_1 + \frac{\sigma}{\pi}\log r,$$

that

$$\overline{\lim_{r\to+\infty}}\Big(S_{P_1}(r)+\Xi_{P_1}(r)-\frac{\sigma}{\pi}\log r\Big)\leq \overline{\lim_{r\to+\infty}}\Big(S_{P_1}(r)+\Xi_{P_1}(r)-K_{P_1}(r)\Big)+$$
$$+\overline{\lim_{r\to+\infty}}K_{P_1/G}(r)+\overline{\lim_{r\to+\infty}}\Big(K_G(r)-\frac{\sigma}{\pi}\log r\Big)< +\infty.$$

Obviously, the functions $S_{P_1/G}$ and $\Xi_{P_1/G}$ are nonnegative, therefore

$$\overline{\lim_{r\to+\infty}}\Big(S_{P_1}(r)-\frac{\sigma}{\pi}\log r\Big)\leq \overline{\lim_{r\to+\infty}}\Big(S_{P_1}(r)+\Xi_{P_1}(r)-\frac{\sigma}{\pi}\log r\Big)< +\infty,$$
$$\overline{\lim_{r\to+\infty}}\Big(\Xi_{P_1}(r)-\frac{\sigma}{\pi}\log r\Big)\leq \overline{\lim_{r\to+\infty}}\Big(S_{P_1}(r)+\Xi_{P_1}(r)-\frac{\sigma}{\pi}\log r\Big)< +\infty.$$

But by the theorem assumptions, G has no zero in $\mathbb{C}_+$, and the singular boundary function of G is a constant, hence $S_G\equiv 0$, $\Xi_G\equiv 0$ and $S_{P_1/G}\equiv S_{P_1}$, $\Xi_{P_1/G}\equiv \Xi_{P_1}$. Therefore

$$\overline{\lim_{r\to+\infty}}\Big(S_{P_1/G}(r)-\frac{\sigma}{\pi}\log r\Big)< +\infty,\quad \overline{\lim_{r\to+\infty}}\Big(\Xi_{P_1/G}(r)-\frac{\sigma}{\pi}\log r\Big)< +\infty.$$

From this statement, considering first condition (16), imply (see [3, 16]) the estimates

$$\Big|\Pi^*_{P_1/G}(z)e^{-i\sigma z}\Big|\leq \exp\Big(\frac{2\sigma}{\pi}x\log r + c_1x\Big),\quad \Big|S^*_{P_1/G}(z)e^{-i\sigma z}\Big|\leq \exp\Big(\frac{2\sigma}{\pi}x\log r + c_2x\Big) \tag{25}$$

for $z = x+iy = re^{i\varphi}\in\mathbb{C}_+$. From the above formulas and (23) it follows that the function P_1/G belongs to the Smirnov class $E^2\subset E^1$ in $\triangle_c(0;1)$ for each $c\in\mathbb{R}$, where $\triangle_c(a;b)=\{z\colon a<\mathrm{Re}z<b, c<\mathrm{Im}z<c+1\}$. Since the angular boundary values of P_1/G and F_1 coincide almost everywhere on $i\mathbb{R}$, we (see [15, P. IV, 2.5]) can consider P_1/G and F_1 as an analytic function on the complex plane and we will write F_1 instead of P_1/G. Since $F_1(z)e^{-i\sigma z}\in H^2(\mathbb{C}_-)$, this function belongs (see [17]) also to the class $E^2\subset E^1$ in $\triangle_c(-1;0)$ for each $c\in\mathbb{R}$. Then, by Lemma 4, F_1 is analytic on $\triangle_c(-1;1)$ for each $c\in\mathbb{R}$. Hence we have that F_1 is an entire function. Therefore the singular boundary function of the function F_1 is a constant. But the singular boundary function of F_1 coincides with the singular boundary function of P_1, hence $S^*_{P_1}(z)\equiv 1$, $\Xi_{P_1}(r)\equiv 0$. Therefore representation (14) implies formula (9) and conditions

$$\sum_{|\lambda_n|\leq 1}\mathrm{Re}\,\lambda_n<+\infty,\quad \overline{\lim_{r\to+\infty}}\Bigg(\sum_{1<|\lambda_n|\leq r}\Big(\frac{1}{|\lambda_n|}-\frac{|\lambda_n|}{r^2}\Big)\frac{\mathrm{Re}\,\lambda_n}{|\lambda_n|}-\frac{\sigma}{\pi}\log r\Bigg)<+\infty,$$

are valid for P_1/G. Also these formulas hold for F_1, because P_1/G is the analytic continuation of F_1 to $\mathbb{C}_+$.

Assume on the contrary to (11) that

$$\overline{\lim_{r\to+\infty}}\Bigg(\sum_{1<|\lambda_n|\le r}\Big(\frac{1}{|\lambda_n|}-\frac{|\lambda_n|}{r^2}\Big)\frac{\operatorname{Re}\lambda_n}{|\lambda_n|}-\frac{\sigma}{\pi}\log r\Bigg)=-\infty.$$

Hence (see [18])

$$\lim_{x\to+\infty}\frac{\log\Big|\prod\limits_{|\lambda_n|\le 1}\frac{x-\lambda_n}{x+\bar\lambda_n}\prod\limits_{|\lambda_n|>1}\frac{1-x/\lambda_n}{1+x/\bar\lambda_n}\exp\Big(\frac{x}{\lambda_n}+\frac{x}{\bar\lambda_n}\Big)\Big|}{x}-\frac{2\sigma}{\pi}\log x=-\infty.$$

Then after the designation $\widetilde F_j(z)=F_j(z)e^{-\frac{2\sigma}{\pi}z\log z}$, $j\in\{1;3\}$ we obtain estimation (18) for $j=1$, because

$$\overline{\lim_{x\to+\infty}}\frac{\log|e^{i\sigma z}e^{a_1 z}\varkappa_1(z)|}{x}<+\infty.$$

The same is valid for $j=3$. Moreover, by the first condition of (25) we have $|\widetilde F_1(z)e^{-i\sigma z}|\le$ $\le|F_1(z)|\exp(\frac{2\sigma}{\pi}x\log r+d_1x)\exp(-\frac{2\sigma}{\pi}x\log r+\frac{2\sigma}{\pi}y\arg z)=|F_1(z)|\exp(\frac{2\sigma}{\pi}y\arg z+d_1x)$, $d_1\in\mathbb{R}$. Therefore $\widetilde F_1(z)e^{-i\sigma z}e^{-d_1z}\in H^2_\sigma(\mathbb{C}_+)$, analogously $\widetilde F_3(z)e^{i\sigma z}e^{-d_3z}\in H^2_\sigma(\mathbb{C}_+)$ for some $d_3\in\mathbb{R}$. If $d_1<0$ and $d_3<0$, then by Lemma 6 we have $(F_1,F_2,F_3)\equiv(0,0,0)$. If $d_1>0$ or $d_3>0$, then we consider functions $\widetilde F_1(z)e^{-dz}$, $\widetilde F_3(z)e^{-dz}$ instead of $\widetilde F_1(z)$, $\widetilde F_3(z)$, where $d=\max\{d_1,d_3\}$. It is clear, that $(\widetilde F_1(x)+\widetilde F_3(x))e^{\frac{2\sigma}{\pi}x\log x}\in L^2(0;+\infty)$. Then by Lemma 5 formulas (19) are valid. Hence by Lemma 6 we have $(F_1,F_2,F_3)\equiv(0,0,0)$ that is impossible. □

Proof of Theorem 2. If $(F_1,-F_1-F_3,F_3)\in T^2_\sigma(\mathbb{C}_-)$, $F_2=-F_1-F_3$, then the function f defined by (4) belongs to $E^2[D_\sigma]$. Condition (11) implies inequalities

$$\sum_{|\lambda_n|\le 1}\operatorname{Re}\lambda_n<+\infty,\quad \overline{\lim_{r\to+\infty}}\Bigg(\sum_{1<|\lambda_n|\le r}\Big(\frac{1}{|\lambda_n|}-\frac{|\lambda_n|}{r^2}\Big)\frac{\operatorname{Re}\lambda_n}{|\lambda_n|}-\frac{\sigma}{\pi}\log r\Bigg)<+\infty.$$

Then (see [3]) we obtain the first inequality of (25). Condition (8) is equivalent (see [19]) to

$$(\exists c_0\in\mathbb{R})\colon G(z)\exp\Big\{\frac{2\sigma}{\pi}z\log z-c_0z\Big\}\in H^2(\mathbb{C}_+).$$

Hence

$$|F_1(z)\widehat G(z)e^{-i\sigma z}|\le e^{c_1x}\Big|\varkappa_1(z)G(z)\exp\Big\{\frac{2\sigma}{\pi}z\log z-cz\Big\}\Big|\cdot\exp\Big\{\frac{2\sigma}{\pi}x\log r\Big\}\times$$
$$\times\exp\Big\{-\frac{2\sigma}{\pi}x\log r+\frac{2\sigma}{\pi}y\varphi\Big\}\le e^{c_1x}\Big|\varkappa_1(z)G(z)\exp\Big\{\frac{2\sigma}{\pi}z\log z-cz\Big\}\Big|\cdot\exp\Big\{\frac{2\sigma}{\pi}y\varphi\Big\}$$

for $z=x+iy=re^{i\varphi}\in\mathbb{C}_+$. But $\varkappa_1(z)G(z)\exp\{\frac{2\sigma}{\pi}z\log z-cz\}\in H^1(\mathbb{C}_+)$, therefore for some c_2 we have $F_1(z)\widehat G(z)e^{-i\sigma z}\in H^1_\sigma(\mathbb{C}_+)$, analogously $F_3(x)\widehat G(z)e^{i\sigma z}\in H^1_\sigma(\mathbb{C}_+)$. Hence by Lemma 3, f is a solution of (13). □

Example. The function $\psi_1(z)=\int_{l_1}\exp(-e^{-\frac{\pi}{2\sigma}w})e^we^{-wz}dw$ is entire, it has representation (9) with condition (11).

Indeed, the function $G_0(z)=\exp\{-\frac{2\sigma}{\pi}z\log z\}(1+z)^{-2}$ has no zero in $\mathbb{C}_+$ and

$$\overline{\lim_{x\to+\infty}}\Big(\frac{\log|G_0(x)|}{x}+\frac{2\sigma}{\pi}\log x\Big)=\overline{\lim_{x\to+\infty}}\Big(-\frac{\log(1+x)^2}{x}\Big)<+\infty.$$

Also G_0 is analytic in $\overline{\mathbb{C}}_+$, hence the singular boundary function of G is a constant. Therefore G_0 satisfies the conditions of Theorem 1. But the function (see [5]) $f(w) = \exp(-e^{-\frac{\pi}{2\sigma}w})e^w$ is a solution of (6) for

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} G_0(x) e^{-xw} dx.$$

Hence by Theorem 1 ψ_1 is entire, it has representation (9) with condition (11).

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