УДК 515.12

I. V. Hetman

TOPOLOGICAL CLASSIFICATION OF THE HYPERSPACES OF POLYHEDRAL CONVEX SETS IN NORMED SPACES

I. V. Hetman. *Topological classification of the hyperspaces of polyhedral convex sets in normed spaces*, Mat. Stud. **39** (2013), 203–211.

We prove that, for a normed space X of dimension $\dim(X) \geq 2$ the space $\mathrm{PConv}_H(X)$ of non-empty polyhedral convex subsets of X endowed with the Hausdorff metric is homeomorphic to the topological sum $\{0\} \oplus |X^*| \times (\mathbb{R} \oplus (\mathbb{R} \times \bar{\mathbb{R}}_+) \oplus l_2^f)$, where the cardinal $|X^*|$ is endowed with the discrete topology.

И. В. Гетьман. *Топологическая классификация гиперпространств полиэдральных выпуклых множеств в нормированных пространствах* // Мат. Студії. – 2013. – Т.39, №2. – С.203–211.

Доказано, что пространство $\mathrm{PConv}_H(X)$ всех непустых полиэдральных выпуклых подмножеств нормированного пространства X размерности $\dim(X) \geq 2$ гомеоморфно топологической сумме $\{0\} \oplus |X^*| \times (\mathbb{R} \oplus (\mathbb{R} \times \bar{\mathbb{R}}_+) \oplus l_2^f)$, где кардинал $|X^*|$ рассматривается как дискретное топологическое пространство.

1. Introduction. The theory of Hyperspaces [10], [5] is an important area of Topology which has numerous applications in various branches of mathematics. One of classical results on the borderline of the Theory of Hyperspaces and Infinite-Dimensional Topology is Curtis-Shori-West Theorem ([6], [16]) saying that the hyperspace $\mathrm{Cld}(X)$ of non-empty closed subsets of a non-degenerate Peano continuum X is homeomorphic to the Hilbert cube $\mathbb{I}^\omega = [0,1]^\omega$. A similar result for hyperspaces $\mathrm{cc}(\mathbb{R}^n)$ of non-empty convex compact subsets of Euclidean spaces was obtained by S. Nadler, J. Quinn and N. M. Stavrakas ([11]). They proved that the space $\mathrm{cc}(\mathbb{R}^n)$ is homeomorphic to the product $\mathbb{I}^\omega \times \bar{\mathbb{R}}_+$ of the Hilbert cube $\mathbb{I}^\omega = [0,1]^\omega$ and the closed half-line $\bar{\mathbb{R}}_+ = [0,\infty)$. This result of S. Nadler, J. Quinn and N. M. Stavrakas has been developed in many different directions ([18], [13], [14], [12], [4], [2]).

In this paper, we shall be interested in the hyperspaces $\mathrm{Conv}(X)$ of all non-empty closed convex subsets in linear topological spaces X. There are many natural topologizations of the hyperspace $\mathrm{Conv}(X)$ (see [5]). One of the most important topologies on $\mathrm{Conv}(X)$ is the topology τ_H generated by the Hausdorff uniformity $\mathcal{U}_H$. This uniformity is generated by the base consisting of entourages $\{(A,B) \in \mathrm{Conv}(X) \times \mathrm{Conv}(X)\colon A \subset B + U,\quad B \subset A + U\}$, where U runs over open neighborhoods of zero in X. If (X,d) is a linear metric space then the Hausdorff uniformity on $\mathrm{Conv}(X)$ is generated by the Hausdorff distance

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \in [0,\infty] \quad \text{for} \quad A,B \in \mathrm{Conv}(X).$$

2010 *Mathematics Subject Classification:* 52A07, 54B20, 57N17.
Keywords: hyperspace; polyhedral convex set.

The space $\mathrm{Conv}(X)$ endowed with the topology τ_H (generated by the Hausdorff uniformity $\mathcal{U}_H$) is denoted by $\mathrm{Conv}_H(X)$.

The topological structure of the space $\mathrm{Conv}_H(X)$ was studied in [5], [9], [13], [12], [1] and [2]. In particular, in [2] it was shown that for any Banach space X the space $\mathrm{Conv}_H(X)$ is locally connected and every connected component $\mathcal{H}$ of $\mathrm{Conv}(X)$ is homeomorphic to one of the spaces: $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\bar{\mathbb{R}}_+$, $\mathbb{I}^\omega\times\bar{\mathbb{R}}_+$, l_2 or $l_2(\kappa)$ for some cardinal $\kappa\geq\mathfrak{c}$.

In this paper, given a normed space X, we shall study the topological structure of the subspace $\mathrm{PConv}_H(X)$ of $\mathrm{Conv}_H(X)$ consisting of all non-empty polyhedral convex subsets of X. We recall that a convex subset C of a normed space X is *polyhedral* if C can be written as the intersection $C=\cap\mathcal{F}$ of a finite family of closed half-spaces. A *half-space* in X is a convex set of the form $f^{-1}\big((-\infty,a]\big)$ for some real number a and some non-zero linear continuous functional $f\colon X\to\mathbb{R}$. The whole space X is a polyhedral set in X, being the intersection $X=\bigcap\mathcal{F}$ of the empty family $\mathcal{F}=\varnothing$ of closed half-spaces.

In Corollary 2 we shall show that the space $\mathrm{PConv}_H(X)$ is locally connected: any two polyhedral convex sets $A,B\in\mathrm{PConv}_H(X)$ with $d_H(A,B)<\infty$ can be linked by the arc $[A,B]=\big\{tA+(1-t)B\colon t\in[0,1]\big\}$ consisting of convex combinations $tA+(1-t)B=\big\{ta+(1-t)b\colon a\in A,\ b\in B\big\}$ of the sets A,B. The local connectedness of the space $\mathrm{PConv}_H(X)$ ensures that it decomposes into the topological sum of its connected components. This implies that the topological structure of $\mathrm{PConv}_H(X)$ is determined by the topological structure of its connected components. In this paper we shall prove that there are only four possible topological types of connected components of $\mathrm{PConv}_H(X)$ represented by: the singleton $\{0\}$, the real line $\mathbb{R}$, the closed half-plane $\mathbb{R}\times\bar{\mathbb{R}}_+$ and the linear hull $l_2^f=\{(x_n)_{n\in\omega}\in l_2\colon\exists n\in\omega\ \forall m\in\omega\ \ x_m=0\}$ of the standard orthonormal base in the separable Hilbert space l_2.

The following classification theorem is the main result of the paper.

Theorem 1. *Let X be a normed space. Each connected component $\mathcal{C}$ of the space $\mathrm{PConv}_H(X)$ is homeomorphic to $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\bar{\mathbb{R}}_+$ or l_2^f. More precisely, $\mathcal{C}$ is homeomorphic to:*

1) *$\{0\}$ iff $\mathcal{C}$ contains the whole space X;*
2) *$\mathbb{R}$, iff $\mathcal{C}$ contains a half-space;*
3) *$\mathbb{R}\times\bar{\mathbb{R}}_+$ iff $\mathcal{C}$ contains a hyperplane;*
4) *l_2^f in all other cases.*

This theorem will be proved in Section 3. Since each locally connected space X decomposes into the topological sum $X=\bigoplus_{\alpha\in A}X_\alpha$ of its connected components, Theorem 1 implies the following corollary, which will be proved in Section 4.

Corollary 1. *For any Banach space X the space $\mathrm{PConv}_H(X)$ is homeomorphic to:*

1) *$\{0\}$ iff $\dim(X)=0$;*
2) *$\{0\}\oplus\mathbb{R}\oplus(\mathbb{R}\times\bar{\mathbb{R}}_+)\oplus\mathbb{R}$ iff $\dim(X)=1$;*
3) *$\{0\}\oplus\big(|X^*|\times(\mathbb{R}\oplus(\mathbb{R}\times\bar{\mathbb{R}}_+)\oplus l_2^f)\big)$ iff $\dim(X)\geq 2$.*

Here X^* stands for the dual Banach space of X and the cardinal $|X^*|$ is endowed with the discrete topology.

2. Preliminary. In this section, we collect the information which will be used in the proofs of Theorem 1 and Corollary 1.

2.1. Components of the space $\mathrm{Conv}_H(X)$**.** As we know, for a Banach space X the topology τ_H of the space $\mathrm{Conv}_H(X)$ is generated by the Hausdorff distance d_H. The Hausdorff distance determines the equivalence relation $\sim_H$ on $\mathrm{Conv}_H(X)$ defined by $A \sim_H B$ iff $d_H(A,B) < \infty$. This equivalence relation decomposes the space $\mathrm{Conv}_H(X)$ into closed-and-open equivalence classes called *components* of $\mathrm{Conv}_H(X)$. The restriction of the Hausdorff distance d_H to each component $\mathcal{H}$ is a metric on $\mathcal{H}$. By Remark 4.8 of [2], the components of the space $\mathrm{Conv}_H(X)$ are connected and hence coincide with the connected components of $\mathrm{Conv}_H(X)$.

2.2. Characteristic cone and dual characteristic cone of a closed convex set. By a *convex cone* in a linear space X we understand a convex subset $C \subset X$ such that $t \cdot c \in C$ for all $c \in C$ and $t \in \bar{\mathbb{R}}_+$. Each subset $E \subset X$ generates the convex cone

$$\mathrm{cone}(E) = \left\{ \sum_{i=1}^n t_i x_i \colon n \in \mathbb{N},\ x_1, \dots, x_n \in E,\ \ t_1, \dots, t_n \in \bar{\mathbb{R}}_+ \right\}$$

in X which is the smallest convex cone containing E. The convex cone $\mathrm{cone}(E)$ contains the convex hull $\mathrm{conv}(E)$ of the set E.

For subsets A, B of a linear space X, a real number t, and a subset $T \subset \mathbb{R}$, we put $A + B = \{a + b \colon a \in A,\ b \in B\}$, $tA = \{ta \colon a \in A\}$ and $T \cdot A = \{ta \colon t \in T,\ \ a \in A\}$.

To each non-empty closed convex subset C of a normed space X we can assign

- the *characteristic space* $L_C = \{x \in X \colon \forall c \in C \quad c + \mathbb{R} \cdot x \subset C\}$;
- the *characteristic cone* $V_C = \{x \in X \colon \forall c \in C \quad c + \bar{\mathbb{R}}_+ \cdot x \subset C\}$;
- the *dual characteristic cone* $V^*_C = \{x^* \in X^* \colon \sup x^*(C) < \infty\}$.

The *dual characteristic cone* V^*_C lies in the dual Banach space X^* of the normed space X. It is clear that $L_C = V_C \cap (-V_C)$.

A closed convex subset $C \subset X$ is called *line-free* if it contains no affine line. This happen if and only if the characteristic space L_C of C is trivial.

By Lemma 3.1 of [2], the characteristic cone V_C coincides with the convex cone $\{x \in X \colon \forall x^* \in V^*_C \ \ x^*(x) \le 0\}$. This implies that for any component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$ and any two convex sets $A, B \in \mathcal{H}$ we get $V^*_A = V^*_B$, $V_A = V_B$, and $L_A = L_B$, which allows us to define the *characteristic space* $L_\mathcal{H}$, the *characteristic cone* $V_\mathcal{H}$ and the *dual characteristic cone* $V^*_\mathcal{H}$ of $\mathcal{H}$ letting $L_\mathcal{H} = L_C$, $V_\mathcal{H} = V_C$ and $V^*_\mathcal{H} = V^*_C$ for any convex set $C \in \mathcal{H}$. A component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$ will be called *line-free* if its characteristic linear space $L_\mathcal{H}$ is trivial in the sense that $L_\mathcal{H} = \{0\}$ i. e., every $C \in \mathcal{H}$ is line-free.

The equality $V_\mathcal{H} = \{x \in X \colon \forall x^* \in V^*_\mathcal{H} \ \ x^*(x) \le 0\}$ implies that the characteristic cone $V_\mathcal{H}$ and the characteristic space $L_\mathcal{H} = V_\mathcal{H} \cap (-V_\mathcal{H})$ are closed in X. Consequently, we can consider the quotient normed space $X/L_\mathcal{H}$ and the quotient operator $q_\mathcal{H} \colon X \to X/L_\mathcal{H}$. The operator q induces a map $\bar{q} \colon \mathcal{H} \to \mathrm{Conv}_H(X/L_\mathcal{H})$, $\bar{q} \colon C \mapsto q(C)$. The equality $C = C + L_C = C + L_\mathcal{H} = q^{-1}(q(C))$ implies that the convex set $\bar{q}(C) = q(C)$ is closed in the quotient normed space $X/L_\mathcal{H}$, so the map $\bar{q}$ is well-defined. Moreover, the map $\bar{q}$ is an isometry as is shown by the following lemma proved in [2, 5.2]:

Lemma 1. *The image* $\mathcal{H}/L_\mathcal{H} = \bar{q}(\mathcal{H})$ *of* $\mathcal{H}$ *coincides with a line-free component of the space* $\mathrm{Conv}_H(X/L_\mathcal{H})$ *and the map* $\bar{q} \colon \mathcal{H} \to \mathcal{H}/L_\mathcal{H}$ *is an isometry.*

This lemma allows to reduce the study of components of the space $\mathrm{Conv}_H(X)$ to studying its line-free components.

For polyhedral convex sets, we can additionally assume that the normed space X is finite-dimensional. Indeed, by the definition, a non-empty polyhedral convex set C in a normed space X can be written as the intersection of half-spaces $C = \bigcap_{i=1}^n f_i^{-1}\big((-\infty, a_i]\big)$ for some linear continuous functional $f_1, \dots, f_n \colon X \to \mathbb{R}$ and some real numbers $a_1, \dots, a_n$. Then the characteristics cone V_C of C coincides with $\bigcap_{i=1}^n f_i^{-1}\big((-\infty, 0]\big)$ and the characteristic linear subspace L_C of C coincides with $\bigcap_{i=1}^n f_i^{-1}(0)$ and hence has finite codimension in X. Consequently, the component $\mathcal{H}$ containing C is isometric to the component $\mathcal{H}/L_{\mathcal{H}}$ of the space $\mathrm{Conv}_H(X/L_{\mathcal{H}})$ of convex sets of the finite-dimensional normed space $X/L_{\mathcal{H}}$.

We shall often use the following classical characterization of polyhedral convex sets in finite-dimensional normed spaces (see [19, 1.2] or [8, §4.3]).

Lemma 2. *A convex subset C of a finite-dimensional normed space X is polyhedral if and only if $C = \mathrm{conv}(F) + \mathrm{cone}(E)$ for some finite sets $F, E \subset X$.*

2.3. Embedding components of $\mathrm{Conv}_H(X)$ into a Banach space. In [2] it was shown that for a normed space X each component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$ is isometric to a convex set of the Banach space $l_\infty(S^* \cap V^*_{\mathcal{H}})$ of all bounded functions defined on the subset $S^* \cap V^*_{\mathcal{H}} = \{x^* \in V^*_{\mathcal{H}} \colon \|x^*\| = 1\}$ of the unit sphere S^* of the dual Banach space X^*. The Banach space $l_\infty(S^* \cap V^*_{\mathcal{H}})$ is endowed with the standard sup-norm $\|f\| = \sup\{|f(x^*)| \colon x^* \in S^* \cap V^*_{\mathcal{H}}\}$.

For a component $\mathcal{H}$ containing a polyhedral convex set, the isometric embedding $\delta \colon \mathcal{H} \to l_\infty(S^* \cap V^*_{\mathcal{H}})$ can be defined by assigning to each convex set $C \in \mathcal{H}$ the function $\delta_C \colon S^* \cap V^*_{\mathcal{H}} \to \mathbb{R}$, $\delta_C(x^*) = \sup x^*(C)$. Theorem 1.1 of [1] and Proposition 2.1 of [2] imply that δ is a well-defined isometric embedding of $\mathcal{H}$ into the Banach space $l_\infty(S^* \cap V^*_{\mathcal{H}})$. Moreover, the embedding δ is affine in the sense that $\delta(\overline{(1-t)A + tB}) = (1-t)\delta(A) + t\delta(B)$ for any convex sets $A, B \in \mathcal{H}$ and any real number $t \in [0,1]$, see Section 4 of [2]. This implies that $\delta(\mathcal{H})$ is a convex subset of $l_\infty(S^* \cap V^*_{\mathcal{H}})$.

Lemma 3. *For any normed space X and a component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$ containing a polyhedral convex set, the images $\delta(\mathcal{H})$ and $\delta(\mathcal{H} \cap \mathrm{PConv}_H(X))$ are convex subsets of the Banach space $l_\infty(S^* \cap V^*_{\mathcal{H}})$.*

Proof. We already know that $\delta(\mathcal{H})$ is a convex set. It remains to show that so is $\delta(\mathcal{H} \cap \mathrm{PConv}_H(X))$.

Since $\mathcal{H}$ contains a polyhedral convex set C, the characteristic space $L_{\mathcal{H}} = L_C$ of $\mathcal{H}$ has finite codimension in X and hence the quotient normed space $Y = X/L_{\mathcal{H}}$ is finite-dimensional. Consider the quotient linear operator $q \colon X \to Y$ and the induced isometry $\bar{q} \colon \mathcal{H} \to \mathcal{H}'$ of $\mathcal{H}$ onto the component $\mathcal{H}' = \mathcal{H}/L_{\mathcal{H}}$ of the space $\mathrm{Conv}_H(Y)$. Lemma 3.3 of [1] implies that $\mathcal{H} \cap \mathrm{PConv}_H(X) = \bar{q}^{-1}(\mathcal{H}' \cap \mathrm{PConv}_H(Y))$.

Since $q \colon X \to Y$ is a quotient operator, its dual $q^* \colon Y^* \to X^*$ is an isometric embedding of Y^* into X^*. This embedding induces the restriction operator $q^*_\infty \colon l_\infty(S^*_X \cap V^*_{\mathcal{H}}) \to l_\infty(S^*_Y \cap V^*_{\mathcal{H}'})$, $q^*_\infty \colon f \mapsto f \circ q^*$. Here S^*_X and S^*_Y denote the unit sphere in the dual spaces X^* and Y^*.

Consider the isometric embeddings $\delta \colon \mathcal{H} \to l_\infty(S^*_X \cap V^*_{\mathcal{H}})$ and $\delta' \colon \mathcal{H}' \to l_\infty(S^*_X \cap V^*_{\mathcal{H}})$ which fit into the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{H} & \xrightarrow{\ \delta\ } & \delta(\mathcal{H}) & \hookrightarrow & l_\infty(S^*_X \cap V^*_{\mathcal{H}}) \\
\big\downarrow {\scriptstyle \bar{q}|\mathcal{H}} & & \big\downarrow {\scriptstyle q^*_\infty|\delta(\mathcal{H})} & & \big\downarrow {\scriptstyle q^*_\infty} \\
\mathcal{H}' & \xrightarrow[\ \delta'\ ]{} & \delta'(\mathcal{H}') & \hookrightarrow & l_\infty(S^*_Y \cap V^*_{\mathcal{H}'})
\end{array}$$

Taking into account that $\bar{q}$, δ and δ' are isometric embeddings and $\bar{q}(\mathcal{H}\cap \mathrm{PConv}_H(X)) = \mathcal{H}'\cap \mathrm{PConv}_H(Y)$, we conclude that the map $q^*_\infty|\delta(\mathcal{H})\colon \delta(\mathcal{H})\to\delta'(\mathcal{H}')$ is a bijective isometry mapping the set $\delta(\mathcal{H}\cap \mathrm{PConv}_H(X))$ onto the set $\delta'(\mathcal{H}'\cap \mathrm{PConv}_H(Y))$. Since q^*_∞ is a linear operator, the convexity of the set $\delta(\mathcal{H}\cap \mathrm{PConv}_H(X))$ will be established as soon as we check that the set $\delta'(\mathcal{H}'\cap \mathrm{PConv}_H(Y))$ is convex in $l_\infty(S^*_Y\cap V^*_{\mathcal{H}'})$.

As we already know, the isometric embedding $\delta'\colon \mathcal{H}'\to l_\infty(S^*_Y\cap V^*_{\mathcal{H}'})$ is affine in the sense that $\delta'(\overline{(1-t)A+tB}) = (1-t)\delta'(A)+t\delta'(B)$ for any convex sets $A,B\in\mathcal{H}'$ and $t\in[0,1]$. So, it suffices to check that the set $\mathcal{H}'\cap \mathrm{PConv}_H(Y)$ is convex in the sense that $(1-t)A+tB\in\mathcal{H}'\cap \mathrm{PConv}_H(Y)$ for any polyhedral convex sets $A,B\in\mathcal{H}'\cap \mathrm{PConv}_H(Y)$ and any real number $t\in[0,1]$. By Lemma 2, there are finite sets $F_A,E_A,F_B,E_B\subset Y$ such that $A=\mathrm{conv}(F_A)+\mathrm{cone}(E_A)$ and $B=\mathrm{conv}(F_B)+\mathrm{cone}(E_B)$. Since $\mathrm{cone}(E_A)=V_A=V_{\mathcal{H}'}=V_B=\mathrm{cone}(E_B)$, we can assume that $E_A=E_B=E$ for some finite set E. Consider the finite subset $F=(1-t)F_A+tF_B=\{(1-t)a+tb\colon a\in F_A,\ b\in F_B\}\subset Y$ and observe that the convex set

$$\begin{aligned}(1-t)A+tB &= (1-t)\mathrm{conv}(F_A)+t\mathrm{conv}(F_B)+\mathrm{cone}(E)=\\ &= \mathrm{conv}((1-t)F_A+tF_B)+\mathrm{cone}(E)=\mathrm{conv}(F)+\mathrm{cone}(E)\end{aligned}$$

is polyhedral according to Lemma 2. So, the sets $\mathcal{H}'\cap \mathrm{PConv}_H(Y)$, $\delta'(\mathcal{H}'\cap \mathrm{PConv}_H(Y))$ and $\delta(\mathcal{H}\cap \mathrm{PConv}_H(X))$ are convex. □

Corollary 2. *The space* $\mathrm{PConv}_H(X)$ *is locally connected and each connected component of the space* $\mathrm{PConv}_H(X)$ *coincides with the intersection* $\mathcal{H}\cap \mathrm{PConv}_H(X)$ *for a unique component* $\mathcal{H}$ *of the space* $\mathrm{Conv}_H(X)$.

Proof. For any polyhedral convex set $C\in \mathrm{PConv}_H(X)$, the component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$ containing C is a closed-and-open subset of $\mathrm{Conv}_H(X)$. By Lemma 3, the intersection $\mathcal{H}\cap \mathrm{PConv}_H(X)$ is homeomorphic to a convex subset of a Banach space and hence is connected and locally connected. Then C has a locally connected neighborhood $\mathcal{H}\cap \mathrm{PConv}_H(X)$ in $\mathrm{PConv}_H(X)$, which implies that the space $\mathrm{PConv}_H(X)$ is locally connected.

The intersection $\mathcal{H}\cap \mathrm{PConv}_H(X)$, being a connected clopen subset of $\mathrm{PConv}_H(X)$, coincides with the connected component of $\mathrm{PConv}_H(X)$ containing the set C. □

2.4. Extremal points of closed convex sets. A point x of a convex set C is called *extremal* if the set $C\setminus\{x\}$ is convex. By $\mathrm{ext}(C)$ we denote the set of all extremal points of C.

Let us recall that a convex set C in a normed space L is called *line-free* if it contains no affine line. This holds if and only if the characteristic linear space $L_C=V_C\cap(-V_C)=\{0\}$ is trivial. The following result of Krein-Milman type can be found in [15, 1.4.4].

Lemma 4. *Each line-free closed convex subset* C *of a finite-dimensional Banach space* X *can be written as the sum* $C=\mathrm{conv}(\mathrm{ext}(C))+V_C$.

Lemma 5. *If* C *is a line-free polyhedral convex set in a finite-dimensional Banach space* X*, then its set of extreme points* $\mathrm{ext}(C)$ *coincides with the smallest finite subset* $F\subset X$ *such that* $C=\mathrm{conv}(F)+V_C$.

Proof. By Lemma 4, $C=\mathrm{conv}(\mathrm{ext}(C))+V_C$ and by Lemma 2, $C=\mathrm{conv}(F)+\mathrm{cone}(E)=\mathrm{conv}(F)+V_C$ for some finite sets $F,E\subset X$.

The lemma will follow as soon as we check that $\mathrm{ext}(C)\subset F$ for each finite subset $F\subset X$ with $C=\mathrm{conv}(F)+V_C$. Given any point $x\in\mathrm{ext}(C)\subset C=\mathrm{conv}(F)+V_C$, find two points $y\in$

$\mathrm{conv}(F)$ and $v \in V_C$ such that $x = y + v$. We claim that $v = 0$. In the opposite case, x is not extremal being the midpoint of the segment $[y, y+2v] \subset C$. So, $v = 0$ and $x = y \in \mathrm{conv}(F)$. Assuming that $x \notin F$, we would conclude that $x \in \mathrm{conv}(F) \subset \mathrm{conv}(C \setminus \{x\}) = C \setminus \{x\}$, which is a contradiction. So, $x \in F$ and $\mathrm{ext}(C) \subset F$. □

For a metric space (X, d) let $[X]^{<\omega}$ be the family of non-empty finite subsets of X, endowed with the Hausdorff metric that induces Vietoris topology. It is clear that $[X]^{<\omega} = \bigcup_{n=1}^{\infty} [X]^n$, where $[X]^n$ is the space of all n-subsets of X.

Lemma 6. *For every $n \in \mathbb{N}$ and a line-free polyhedral convex cone V in a finite-dimensional Banach space X, the set $\mathcal{O}_n = \big\{F \in [X]^n \colon F = \mathrm{ext}(\mathrm{conv}(F) + V)\big\}$ is open in $[X]^n$.*

Proof. For $n = 1$ the set $\mathcal{O}_1$ coincides with $[X]^1$ and so is open. Now assume that $n > 1$. It suffices to prove that the set $\mathcal{N} = \big\{F \in [X]^n \colon F \neq \mathrm{ext}(\mathrm{conv}(F) + V)\big\}$ is closed in $[X]^n$. Given any sequence $\{F_k\}_{k\in\omega} \subset \mathcal{N}$ convergent to some set $F_\infty \in [X]^n$, we need to show that $F_\infty \in \mathcal{N}$.

Consider the map $\pi \colon [X]^{<\omega} \to \mathrm{Conv}(X)$ assigning to each finite set $F \in [X]^{<\omega}$ the closed convex subset $\pi(F) = \mathrm{conv}(F) + V$ of X. It follows that the map π is non-expanding (with respect to the Hausdorff metric) and hence continuous. Then the sequence $(C_k)_{k\in\omega}$ of the convex sets $C_k = \pi(F_k)$ converges to the convex set $C_\infty = \pi(F_\infty)$ in the space $\mathrm{Conv}_H(X)$.

For every $k \in \omega$, fix a point $x_k \in F_k \setminus \mathrm{ext}(C_k)$ (such a point exists as $\mathrm{ext}(C_k) \subsetneq F_k \in \mathcal{N}$). Replacing the sequence (F_k) with a suitable subsequence, we can assume that the sequence x_k converges to some point $x_\infty \in F_\infty$. Taking into account that $F_\infty = \lim_{k\to\infty} F_k \in [X]^n$, we conclude that the sequence $(F_k \setminus \{x_k\})_{k\in\omega}$ converges to $F_\infty \setminus \{x_\infty\}$ in the space $[X]^{n-1}$.

It follows from

$$C_k = \mathrm{conv}(F_k) + V = \mathrm{conv}(\mathrm{ext}(C_k)) + V \subset \mathrm{conv}(F_k \setminus \{x_k\}) + V \subset \mathrm{conv}(F_k) + V = C_k$$

that $C_k = \mathrm{conv}(F_k \setminus \{x_k\}) + V$ and hence

$$C_\infty = \lim_{k\to\infty} C_k = \lim_{k\to\infty} (\mathrm{conv}(F_k \setminus \{x_k\}) + V) = \mathrm{conv}(F_\infty \setminus \{x_\infty\}) + V.$$

By Lemma 5, $\mathrm{ext}(C_\infty) \subset F_\infty \setminus \{x_\infty\}$, which means that $F_\infty \neq \mathrm{ext}(C_\infty) = \mathrm{ext}(\mathrm{conv}(F_\infty) + V)$ and $F_\infty \in \mathcal{N}$. □

A topological space X is called *strongly countable-dimensional* if it can be written as a countable union of closed finite-dimensional subspaces. Observe that a topological space X is σ-compact and strongly countable-dimensional if and only if it can be written as a countable union of finite-dimensional compact subsets.

Lemma 7. *Let X be a normed space. For any component $\mathcal{H}$ of $\mathrm{Conv}_H(X)$, the intersection $\mathcal{H} \cap \mathrm{PConv}_H(X)$ is σ-compact and strongly countable-dimensional.*

Proof. If the intersection $\mathcal{H} \cap \mathrm{PConv}_H(X)$ is empty then there is nothing to prove. So, we assume that $\mathcal{H}$ contains some polyhedral convex set.

Since the component $\mathcal{H}$ is isometric to the line-free component $\mathcal{H}/L_{\mathcal{H}}$ of the space $\mathrm{Conv}_H(X/L_{\mathcal{H}})$ and $X/L_{\mathcal{H}}$ is finite-dimensional as observed after Lemma 1, we can assume that X is a finite-dimensional space and the component $\mathcal{H}$ is line-free, and hence its characteristic space $L_{\mathcal{H}}$ is trivial. Since the component $\mathcal{H}$ contains some polyhedral convex set, the characteristic cone $V_{\mathcal{H}}$ of $\mathcal{H}$ is polyhedral.

In this case, the space X can be written as the union $X = \bigcup_{k\in\mathbb{N}} B_k$ of an increasing sequence of compact sets. For every $k, n \in \mathbb{N}$, the space $[B_k]^{\le n} = \{F \in [X]^{<\omega} : F \subset B_k, |F| \le n\}$ is compact and finite-dimensional (more precisely, has dimension $\dim([B_k]^{\le n}) \le n\cdot\dim(B_k)$ according to [17, 4.1.1]). Moreover, the space $[B_k]^n = [B_k]^{\le n} \setminus [B_k]^{\le n-1}$, being an open subset of $[B_k]^{\le n}$ is σ-compact and strongly countable-dimensional and so is the space $[X]^n = \bigcup_{k\in\mathbb{N}}[B_k]^n$.

Lemma 2 implies that the space $\mathcal{H}\cap\mathrm{PConv}_H(X)$ is the image of the space $[X]^{<\omega}$ under the (non-expanding) map $\pi\colon [X]^{<\omega} \to \mathcal{H}\cap\mathrm{PConv}_H(X)$, $\pi\colon F \mapsto \mathrm{conv}(F) + V_{\mathcal{H}}$. By Lemma 4, the map π has a section $\mathrm{ext}\colon \mathcal{H}\cap\mathrm{PConv}_H(X) \to [X]^{<\omega}$ assigning to each polyhedral convex set $C \in \mathcal{H}$ its set of extremal points $\mathrm{ext}(C)$. By Lemma 6, for every $n \in \mathbb{N}$ the set $\mathcal{O}_n = \{F \in [X]^n : F = \mathrm{ext}\circ\pi(F)\} = \mathrm{ext}(\mathcal{H}\cap\mathrm{PConv}_H(X)) \cap [X]^n$ is open in the space $[X]^n$. Since $[X]^n$ is σ-compact and finite-dimensional (more precisely $\le n\cdot\dim X$) by the Subset Theorem of dimension. Hence, its open subspace $\mathcal{O}_n$ is strongly countable-dimensional (more precisely $\le n\cdot\dim X$) by the Countable Sum Theorem. Write $\mathcal{O}_n$ as the countable union $\mathcal{O}_n = \bigcup_{i\in\mathbb{N}} K_{n,i}$ of finite-dimensional compact sets $K_{n,i}$. For every $i \in \mathbb{N}$ the restriction $\pi|K_{n,i}$, being injective and continuous, is a topological embedding. Consequently, the set $\pi(K_{n,i})$ is compact and finite-dimensional, and hence the space

$$\mathcal{H}\cap\mathrm{PConv}_H(X) = \pi\circ\mathrm{ext}(\mathcal{H}\cap\mathrm{PConv}_H(X)) = \bigcup_{n\in\mathbb{N}}\pi(\mathcal{O}_n) = \bigcup_{n,i\in\mathbb{N}}\pi(K_{n,i})$$

is σ-compact and strongly countable-dimensional. □

3. Proof of Theorem 1. Given a normed space X and a connected component $\mathcal{C}$ of the space $\mathrm{PConv}_H(X)$, we need to prove that it is homeomorphic to $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\bar{\mathbb{R}}_+$ or l_2^f. Fix a polyhedral convex set $C \in \mathcal{C}$. By Corollary 2, the component $\mathcal{C}$ coincides with the intersection $\mathcal{H}\cap\mathrm{PConv}_H(X)$ where $\mathcal{H}$ is the component of the space $\mathrm{Conv}_H(X)$ containing C. Since the set C is polyhedral, its characteristic cone V_C is polyhedral and its characteristic linear subspace L_C has finite codimension in X. Let $Y = X/L_{\mathcal{H}}$ be the quotient space and $q\colon X \to Y$ be the quotient operator. By Lemma 1, it induces an isometry $\bar{q}\colon \mathcal{H} \to \mathcal{H}'$ from $\mathcal{H}$ onto the component $\mathcal{H}' = \mathcal{H}/L_{\mathcal{H}}$ of the space $\mathrm{Conv}_H(Y)$ containing the polyhedral convex set $q(C) \subset Y$. By Lemma 3.3 of [1], this isometry maps the component $\mathcal{C} = \mathcal{H}\cap\mathrm{PConv}_H(X)$ onto the component $\mathcal{C}' = \mathcal{H}'\cap\mathrm{PConv}_H(Y)$.

Now we consider four cases appearing in Theorem 1.

1. If $\mathcal{C}$ contains the whole space then we can assume that $C = X$. In this case, since $L_X = X$, the quotient space Y is trivial, and hence the components $\mathcal{C}'$ and $\mathcal{C}$ are singletons.

2. If $\mathcal{C}$ contains a half-space then we can assume that the set C is a half-space. In this case, the characteristic space L_C has codimension 1 and the quotient space Y is 1-dimensional. Moreover, the convex set $C' = q(C)$ is a half-line, and hence its component $\mathcal{H}' = \mathcal{C}'$ is isometric to the real line. Then the isometric copy $\mathcal{C}$ of $\mathcal{C}'$ is also isometric to the real line.

3. If $\mathcal{C}$ contain a hyperplane then we can assume that the set C is a linear subspace of codimension 1. In this case, the quotient space Y is 1-dimensional, the convex set $C' = q(C) = \{0\}$ is a singleton, and its component $\mathcal{C}' = \mathcal{H}'\cap\mathrm{PConv}_H(X) = \mathcal{H}'$ is homeomorphic to the closed half-plane $\mathbb{R}\times\bar{\mathbb{R}}_+$ (see Case 1a in the proof of Theorem 1 of [2]).

4. Finally assume that the component $\mathcal{C}$ contains neither the whole space nor a half-space nor a hyperplane. In this case, the quotient space Y has dimension ≥ 2. By Lemma 3.1 of [1], the set $\mathcal{C} = \mathcal{H}\cap\mathrm{PConv}_H(Y)$ is dense in $\mathcal{H}$ and by Theorem 1 of [2], the component $\mathcal{H}$ is infinite-dimensional. Now consider the isometric embedding $\delta\colon \mathcal{H} \to l_\infty(S^*\cap V_{\mathcal{H}}^*)$ discussed

in Subsection 2.2. By Lemma 3, the sets $\delta(\mathcal{H})$ and $\delta(\mathcal{C})$ are convex subsets of the Banach space $l_\infty(S^* \cap V^*_{\mathcal{H}})$. The space $\delta(\mathcal{C})$, being a dense convex subset of the infinite-dimensional convex set $\delta(\mathcal{H})$, is infinite-dimensional. By Lemma 7, the spaces $\mathcal{C}$ and $\delta(\mathcal{C})$ are σ-compact and strongly countable dimensional. Applying a theorem of T. Dobrowoslki ([7] or [3, 5.3.12] which says that, each convex infinite-dimensional σ-compact strongly countable dimensional subset of a normed space is homeomorphic to the pre-Hilbert space l_2^f), we conclude that the convex set $\delta(\mathcal{C})$ is homeomorphic to l_2^f. Then its isometric copy $\mathcal{C}$ is homeomorphic to l_2^f too.

4. Proof of Corollary 1. The cases (1) and (2) follow immediately from Corollary 2 of [2] (describing the topology of the space $\mathrm{Conv}_H(X)$) because $\mathrm{Conv}_H(X)$ coincides with $\mathrm{PConv}_H(X)$ if $\dim(X) < 2$.

So, assume that $\dim(X) \geq 2$. The space $\mathrm{PConv}_H(X)$ is locally connected and hence can be written as the topological sum of its connected components. So, we can write $\mathrm{PConv}_H(X)$ as the topological sum $\{X\} \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$ where the set $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) consists of all closed convex sets whose component contains a half-space (resp. a hyperplane), and $\mathcal{A}_3 = \mathrm{PConv}_H(X) \setminus \big(\{X\} \cup \mathcal{A}_1 \cup \mathcal{A}_2\big)$. By Theorem 1, the spaces $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are homeomorphic to the products $\kappa_1 \times \mathbb{R}$, $\kappa_2 \times (\mathbb{R} \times \bar{\mathbb{R}}_+)$, $\kappa_3 \times l_2^f$ for some cardinals κ_1, κ_2, κ_3 endowed with the discrete topology. It remains to show that $\kappa_1 = \kappa_2 = \kappa_3 = |X^*|$. Taking into account that each polyhedral convex set is the intersection of finitely many half-spaces, we conclude that the cardinality of the set $\mathrm{PConv}(X)$ does not exceed the cardinality of the union $\bigcup_{n\in\omega}(X^* \times \mathbb{R})^n$, which is equal to $|X^*|$. So, $\max\{\kappa_1, \kappa_2, \kappa_3\} \leq |X^*|$.

To see that $\min\{\kappa_1, \kappa_2, \kappa_3\} \geq |X^*|$, consider the following families of finite sets: $\mathcal{F}_1 = \big\{\{f\} \colon f \in S^*_X\big\}$, $\mathcal{F}_2 = \big\{\{f, -f\} \colon f \in S^*_X\big\}$ and $\mathcal{F}_3 = \big\{\{f, g\} \colon f, g \in S^*_X, f \notin \{g, -g\}\big\}$. Observe that for every $i \in \{1, 2, 3\}$ and $F \in \mathcal{F}_i$ the convex cone $C_F = \bigcap_{f\in F} f^{-1}\big((-\infty, 0]\big)$ belongs to the set $\mathcal{A}_i$. Moreover, for two distinct sets $F, E \in \mathcal{F}_i$ the cones C_F and C_E belong to distinct components of the space $\mathrm{PConv}_H(X)$, which implies that $\kappa_i \geq |\mathcal{F}_i| = |X^*|$.

5. Acknowledgement. The author would like to express his thanks to Taras Banakh for help in writing this paper.

REFERENCES

1. T. Banakh, I. Hetman, *A "hidden" characterization of approximatevely polyhedral convex sets in Banach spaces*, Studia Math., **210** (2012), 137–157.
2. T. Banakh, I. Hetman, K. Sakai, Topology of the hyperspace of closed convex subsets of a Banach space, available at: http://arxiv.org/abs/1112.6374.
3. T. Banakh, T. Radul, M. Zarichnyi, Adsorbing sets in infinite-dimensional manifolds. – VNTL Publ., Lviv, 1996.
4. L. Bazylevych, D. Repovš, M. Zarichnyi, *Hyperspace of convex compacta of nonmetrizable compact convex subspaces of locally convex spaces*, Topology Appl., **155** (2008), №8, 764–772.
5. G. Beer, Topologies on closed and closed convex sets. – Kluwer Academic Publ. Group, Dordrecht, 1993.
6. D.W. Curtis, R.M. Schori, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math., **101** (1978), №1, 19–38.
7. T. Dobrowolski, *The compact Z-set property in convex sets*, Topology Appl., **23** (1986), №2, 163–172.
8. J. Gallier, Notes on Convex Sets, Polytopes, Polyhedra, Combinatorial Topology, Voronoi Diagrams and Delaunay Triangulations, book in progress available at: http://arxiv.org/abs/0805.0292.

9. A. Granero, M. Jimenez, J. Moreno, *Convex sets in Banach spaces and a problem of Rolewicz*, Studia Math., **129** (1998), №1, 19–29.
10. A. Illanes, S. Nadler, Hyperspaces, Marcel Dekker, Inc., New York, 1999.
11. S. Nadler, J. Quinn, N.M. Stavrakas, *Hyperspaces of compact convex sets,* Pacific J. Math., **83** (1979), 441–462.
12. K. Sakai, *The spaces of compact convex sets and bounded closed convex sets in a Banach space*, Houston J. Math., **34** (2008), №1, 289–300.
13. K. Sakai, M. Yaguchi, *The AR-property of the spaces of closed convex sets*, Colloq. Math., **106** (2006), №1, 15–24.
14. K. Sakai, Z. Yang, *The spaces of closed convex sets in Euclidean spaces with the Fell topology*, Bull. Pol. Acad. Sci. Math., **55** (2007), №2, 139–143.
15. R. Schneider, Convex Bodies: The Brunn-Minkowski Theory. – Cambridge Univ. Press, 1993.
16. R. Schori, J. West, *The hyperspace of the closed unit interval is a Hilbert cube*, Trans. Amer. Math. Soc., **213** (1975), 217–235.
17. A. Teleiko, M. Zarichnyi, Categorical topology of compact Hausdorff spaces. – VNTL Publ., Lviv, 1999.
18. M.M. Zarichnyi, S. Ivanov, *Hyperspaces of convex compact subsets of the Tychonoff cube*, Ukrainian Math. J., **53** (2001), №5, 809–813.
19. G. Ziegler, Lectures on polytopes, Springer-Verlag, New York, 1995.

Ivan Franko National University of Lviv
ihromant@gmail.com

Received 5.11.2012