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SMALL SCATTERED TOPOLOGICAL INVARIANTS

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We present a unified approach to define dimension functions like trind, trind_p, trt and $\mathfrak{p}$. We show how some similar facts on these functions can be proved similarly. Moreover, several new classes of infinite-dimensional spaces close to the classes of countable-dimensional and σ-hereditarily disconnected ones are introduced. We prove a compactification theorem for these classes.

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Мы предлагаем единый подход к определению таких размерностных функций, как trind, trind_p, trt и $\mathfrak{p}$. Мы показываем, как некоторые простые факты об этих функциях могут быть доказаны единообразно. Более того, вводится несколько новых классов бесконечномерных пространств близких к классам счётномерных пространств и σ-наследственно несвязных пространств. Мы также доказываем компактификационную теорему для этих классов.

1. Introduction. In [13] G. Steinke suggested and studied an integer valued inductive topological invariant, the separation dimension $\mathfrak{t}$. Recall that the separation dimension $\mathfrak{t}$ for a topological space X is defined inductively as follows: $\mathfrak{t}\,X = -1$ if and only if $X = \varnothing$; $\mathfrak{t}\,X = 0$ if $|X| = 1$; let $|X| > 1$ and n be an integer ≥ 0, if for each subset M of X with $|M| > 1$ there exist distinct points x, y of M and a partition L_M in the subspace M of X between x and y such that $\mathfrak{t}\,L_M \leq n-1$ then we write $\mathfrak{t}\,X \leq n$. One of the main property of $\mathfrak{t}$ is the following. If $\{X_i : i \in I\}$ is the family of all connected components of a non-empty space X then $\mathfrak{t}\,X = \sup\{\mathfrak{t}\,X_i : i \in I\}$. In particular, for any space X we have $\mathfrak{t}\,X = 0$ if and only if X is hereditarily disconnected.

Recall ([6]) that, the classes of strongly countable-dimensional metrizable compacta, countable-dimensional metrizable compacta and compact metrizable C-spaces are classical objects of infinite dimension theory. In [1] F. G. Arenas, V. A. Chatyrko and M. L. Puertas considered a natural transfinite extension of $\mathfrak{t}$, the topological invariant trt, and showed that each metrizable compact space X with $\mathrm{trt}\,X \neq \infty$ must be a C-space. Moreover, every strongly countable-dimensional metrizable compact space X has $\mathrm{trt}\,X \leq \omega_0$. However, there exist countable-dimensional metrizable compact spaces (namely, the infinite-dimensional Cantor manifolds) of dimension trt $> \omega_0$. Since the inequality $\mathrm{trt}\,X \leq \mathrm{trind}\,X$, where

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trind is the small transfinite inductive dimension ([6]), holds for each T_3-space X, every countable-dimensional metrizable compact space X satisfies $\operatorname{trt} X < \omega_1$. Set

$$\alpha_0 = \sup\{\operatorname{trt} K\colon K \text{ is a countable-dimensional metrizable compact space}\}.$$

It is clear that $\alpha_0 \leq \omega_1$ but the exact value of α_0 is still unknown.

In [10] T. M. Radul introduced an ordinal valued topological invariant, the dimension p, by modifying the definition of trt: the subsets M of the space X are supposed to be compact. It is easy to see that for any space X we have $\mathrm{p}\, X = \sup\{\operatorname{trt} K\colon K \text{ is a compact subset of } X\} \leq \operatorname{trt} X$. In [10] T. M. Radul proved that each σ-hereditarily disconnected hereditarily normal space X satisfies $\mathrm{p}\, X \neq \infty$. Recall (see [7] or [2]) that a space X is *σ-hereditarily disconnected* if X is a countable union of hereditarily disconnected subspaces. Since each zero-dimensional space in the sense of the small inductive dimension ind is hereditarily disconnected, each countable-dimensional in the sense of ind space is σ-hereditarily disconnected. Let us observe that for the subspace K^{ω_0} of the Hilbert cube I^{ω_0} consisting of points with finitely many non-zero coordinates (and so being strongly countable dimensional) we have $\mathrm{p}\, K^{\omega_0} = \omega_0$. Recall that $\operatorname{trt} K^{\omega_0} > \omega_0$ ([1]) but we do not know whether $\operatorname{trt} K^{\omega_0} \neq \infty$.

It is still unclear if each metrizable compact space X with $\operatorname{trt} X \neq \infty$ has to be σ-hereditarily disconnected. The well known R. Pol's example P ([9]) of a weakly infinite-dimensional uncountable-dimensional metrizable compact space is a σ-hereditarily disconnected C-space, and hence by Radul's result $\operatorname{trt} P \neq \infty$. (In fact, P can be constructed so that $\operatorname{trt} P = \omega_0$, see a remark in [1].) But it is unknown whether every compact metrizable C-space X is σ-hereditarily disconnected (resp. has $\operatorname{trt} X \neq \infty$).

In this paper we show that the dimension trind (as well as the transfinite inductive invariant trind_p from ([3])) can also be defined similarly to the definition of trt. One of the subjects of the paper is to unify proofs of some facts about the invariants trind, trind_p, trt, p and introduce new classes of infinite-dimensional spaces close to the classes of countable-dimensional spaces and σ-hereditarily disconnected ones. We prove a compactification theorem for these new classes. In particular, we show that, for any hereditarily disconnected separable completely metrizable space X there is a metrizable compactification Y of X such that $\operatorname{trt} Y \leq \omega_0 + 1$. Furthermore, for Renska's examples (see [11] (resp.[12])) of α-dimensional metrizable Cantor trind (resp. trInd)-manifolds, where α is any isolated countable ordinal, we have the values of trt are equal to $\omega_0 + 1$.

Our terminology follows [5] and [6].

2. Definitions and common properties. All considered topological spaces are assumed to be T_3-spaces. Let us fix for each space X a class $\mathcal{A}_X$ of subsets of X. The family of all classes $\mathcal{A}_X$ we denote by $\mathcal{A}$ and call it *a family of classes of subsets of spaces* (in short, *an SSC-family*).

Definition 1. Let X be a space and $\mathcal{A}$ be an SSC-family.

(a) *The small inductive invariant* $\mathcal{A}(0)$-ind of the space X, denoted by $\mathcal{A}(0)\text{-ind}(X)$ is defined inductively as follows. $\mathcal{A}(0)\text{-ind}(X) = -1$ if and only if $X = \varnothing$; $\mathcal{A}(0)\text{-ind}(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and n be an integer ≥ 0, if for each element M of $\mathcal{A}_X$ with $|M| > 1$ and for every pair (A, x), where A is a closed subset of M and $x \in M - A$, there is a partition L_M in the space M between x and A such that $\mathcal{A}(0)\text{-ind}\, L_M \leq n - 1$ then we write $\mathcal{A}(0)\text{-ind}\, X \leq n$.

(b) *The small inductive invariant* $\mathcal{A}(1)$-ind of the space X, denoted by $\mathcal{A}(1)\text{-ind}(X)$ is defined inductively as follows. $\mathcal{A}(1)\text{-ind}(X) = -1$ if and only if $X = \varnothing$; $\mathcal{A}(1)\text{-ind}(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and n be an integer ≥ 0, if for each element M of $\mathcal{A}_X$ with $|M| > 1$ and for every pair (x, y) of distinct points of M there is a partition L_M in the space M between x and y such that $\mathcal{A}(1)\text{-ind}\, L_M \leq n-1$ then we write $\mathcal{A}(1)\text{-ind}\, X \leq n$.

(c) *The small inductive invariant* $\mathcal{A}(2)$-ind of the space X, denoted by $\mathcal{A}(2)\text{-ind}(X)$ is defined inductively as follows. $\mathcal{A}(2)\text{-ind}(X) = -1$ if and only if $X = \varnothing$; $\mathcal{A}(2)\text{-ind}(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and n be an integer ≥ 0, if for each element M of $\mathcal{A}_X$ with $|M| > 1$ there exists a point $x \in M$ possessing the following property: for every closed subset A of the space M with $x \notin A$ there is a partition L_M in the space M between x and A such that $\mathcal{A}(2)\text{-ind}\, L_M \leq n - 1$ then we write $\mathcal{A}(2)\text{-ind}\, X \leq n$.

(d) *The small inductive invariant* $\mathcal{A}(3)$-ind of the space X, denoted by $\mathcal{A}(3)\text{-ind}(X)$ is defined inductively as follows. $\mathcal{A}(3)\text{-ind}(X) = -1$ if and only if $X = \varnothing$; $\mathcal{A}(3)\text{-ind}(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and n be an integer ≥ 0, if for each element M of $\mathcal{A}_X$ with $|M| > 1$ there exists a proper closed subset A of the space M possessing the following property: for every point $x \in M - A$ there is a partition L_M in the space M between x and A such that $\mathcal{A}(3)\text{-ind}\, L_M \leq n - 1$ then we write $\mathcal{A}(3)\text{-ind}\, X \leq n$.

(e) *The small inductive invariant* $\mathcal{A}(4)$-ind of the space X, denoted by $\mathcal{A}(4)\text{-ind}(X)$ is defined inductively as follows. $\mathcal{A}(4)\text{-ind}(X) = -1$ if and only if $X = \varnothing$; $\mathcal{A}(4)\text{-ind}(X) = 0$ if $|X| = 1$. Let $|X| > 1$ and n be an integer ≥ 0, if for each element M of $\mathcal{A}_X$ with $|M| > 1$ there exist distinct points x, y of M and a partition L_M in the space M between x and y such that $\mathcal{A}(4)\text{-ind}\, L_M \leq n - 1$ then we write $\mathcal{A}(4)\text{-ind}\, X \leq n$.

The transfinite extension $\mathcal{A}(i)$-trind of the invariant $\mathcal{A}(i)$-trind is defined in the standard fashion, $i \in \{0, 1, 2, 3, 4\}$.

Let us introduce SSC-families $\mathcal{A}^j$, $j \in \{1, 2, 3, 4\}$, as follows: for every space X put $\mathcal{A}^1_X = \{X\}$, $\mathcal{A}^2_X = 2^X$, $\mathcal{A}^3_X = 2^X_{\text{comp}}$, $\mathcal{A}^4_X = 2^X_{\text{cl}}$, where 2^X (resp. 2^X_{comp} or 2^X_{cl}) is the family of all (resp. compact or closed) subsets of X. Note that one can suggest many other SSC-families $\mathcal{A}$ different from $\mathcal{A}^j$, $j \in \{1, 2, 3, 4\}$.

Remark 1. Note that

(a) $\mathcal{A}^1(0)\text{-trind}\, X = \operatorname{trind} X$ and $\mathcal{A}^1(1)\text{-trind}\, X = \operatorname{trind}_p X$ ([3]);

(b) $\mathcal{A}^2(4)\text{-trind}\, X = \mathcal{A}^4(4)\text{-trind}\, X = \operatorname{trt} X$ ([1] or Corollary 2);

(c) $\mathcal{A}^3(4)\text{-trind}\, X = \mathrm{p}\, X$ ([10]).

The following statement is evidently valid for every SSC-family $\mathcal{A}$ and every space X.

Proposition 1. *(a)* $\mathcal{A}(0)\text{-trind}\, X \geq \mathcal{A}(1)\text{-trind}\, X \geq \mathcal{A}(3)\text{-trind}\, X \geq \mathcal{A}(4)\text{-trind}\, X$ *and* $\mathcal{A}(0)\text{-trind}\, X \geq \mathcal{A}(2)\text{-trind}\, X \geq \mathcal{A}(3)\text{-trind}\, X$.

(b) $\mathcal{A}(i)\text{-trind}\, X \leq \sup\{\mathcal{A}(i)\text{-trind}\, A \colon A \in \mathcal{A}_X\}$ *for every* $i \in \{0, 1, 2, 3, 4\}$, *whenever* $A \in \mathcal{A}_A$ *for each* $A \in \mathcal{A}_X$ *with* $|A| > 1$.

(Note that the SSC-families $\mathcal{A}^j, j \in \{1, 2, 3, 4\}$, *satisfy this condition.)*

Definition 2. We will say that an SSC-family $\mathcal{A}$ possesses *property* $(*)_1$ (resp. $(*)_2$ or $(*)_3$) if for every space X and every subspace Y of X the following assertions hold:

$(*)_1$: for each $A_Y \in \mathcal{A}_Y$ there exists an element $A_X \in \mathcal{A}_X$ such that $A_Y \subset A_X$;

($(*)_2$: for each $A_Y \in \mathcal{A}_Y$ we have $\mathrm{Cl}_X(A_Y) \in \mathcal{A}_X$, or $(*)_3$: $\mathcal{A}_Y \subset \mathcal{A}_X$).

Note that if an SSC-family $\mathcal{A}$ possesses property $(*)_2$ or property $(*)_3$ then it possesses also property $(*)_1$.

Remark 2. The families $\mathcal{A}^j, j \in \{1,2,3,4\}$, possess property $(*)_1$. The families $\mathcal{A}^j, j \in \{2,3,4\}$, possess property $(*)_2$. The families $\mathcal{A}^j, j \in \{2,3\}$, possess property $(*)_3$.

Proposition 2. *Let an SSC-family $\mathcal{A}$ possess the property $(*)_1$ (resp. $(*)_2$ or $(*)_3$). Then for every space X and every subspace Y of X we have*
$\mathcal{A}(i)\text{-trind}\, X \geq \mathcal{A}(i)\text{-trind}\, Y$ for $i \in \{0,1\}$ (resp. $i \in \{0,1,4\}$ or $i \in \{0,1,2,3,4\}$).

Proof. The case $(*)_1$. We will prove the statement only for $i = 0$.

Let $\mathcal{A}(0)\text{-trind}\, X = \alpha \geq -1$. Note that for $\alpha = -1$ or for $|Y| = 1$ the statement is valid. Apply induction on α. Consider the case: $\alpha \geq 0$ and $|Y| \geq 2$. Let $M_Y \in \mathcal{A}_Y$ with $|M_Y| \geq 2$. By the property $(*)_1$ there exists an element $M_X \in \mathcal{A}_X$ such that $M_Y \subset M_X$. Consider a point $x \in M_Y$ and a closed subset A_Y of the space M_Y such that $x \notin A_Y$. Choose any closed subset A_X of the space M_X such that $A_Y = A_X \cap M_Y$ and note that $x \notin A_X$. Since $\mathcal{A}(0)\text{-trind}\, X = \alpha$ there is a partition L_{M_X} in the space M_X between x and A_X such that $\mathcal{A}(0)\text{-ind}\, L_{M_X} < \alpha$. Note that the set $L_{M_Y} = L_{M_X} \cap M_Y$ is a partition in the space M_Y between x and A_Y. By the inductive assumption we have $\mathcal{A}(0)\text{-ind}\, L_{M_Y} \leq \mathcal{A}(0)\text{-ind}\, L_{M_X} < \alpha$ and so $\mathcal{A}(0)\text{-trind}\, Y \leq \alpha$.

The case $(*)_2$. We need to prove the statement only for $i = 4$ (see the sentence after Definition 2). Let $\mathcal{A}(4)\text{-trind}\, X = \alpha \geq -1$. Note that for $\alpha = -1$ or for $|Y| = 1$ the statement is valid. Apply induction on α. Consider the case: $\alpha \geq 0$ and $|Y| \geq 2$. Let $M_Y \in \mathcal{A}_Y$ with $|M_Y| \geq 2$. By the property $(*)_2$ the set $M_X = \mathrm{Cl}_X(M_Y)$ is an element of $\mathcal{A}_X$. Since $\mathcal{A}(4)\text{-trind}\, X = \alpha$ there exist distinct points x, y of M_X and a partition L_{M_X} in the space M_X between x and y such that $\mathcal{A}(4)\text{-ind}\, L_{M_X} < \alpha$. It also implies that there exist open disjoint subsets U_X and V_X of the space M_X such that $x \in U_X$, $y \in V_X$ and $L_{M_X} = M_X \setminus (U_X \cup V_X)$. Choose a point $a \in U_X \cap M_Y$ and a point $b \in V_X \cap M_Y$ and note that the set $L_{M_Y} = L_{M_X} \cap M_Y$ is a partition between the points a and b in the space M_Y. By the inductive assumption we have $\mathcal{A}(4)\text{-ind}\, L_{M_Y} \leq \mathcal{A}(4)\text{-ind}\, L_{M_X} < \alpha$ and so $\mathcal{A}(4)\text{-ind}\, Y \leq \alpha$.

The case $(*)_3$ is trivial. □

Applying Propositions 1 (b) and 2 we get such a statement.

Corollary 1. *Let an SSC-family $\mathcal{A}$ possess property $(*)_1$ (resp. $(*)_2$ or $(*)_3$), X be a space and $M(\mathcal{A}, X, i) = \sup\{\mathcal{A}(i)\text{-trind}\, A \colon A \in \mathcal{A}_X\}$. Then $\mathcal{A}(i)\text{-trind}\, X \geq M(\mathcal{A}, X, i)$ for $i \in \{0,1\}$ (resp. $i \in \{0,1,4\}$ or $i \in \{0,1,2,3,4\}$). Moreover, $\mathcal{A}(i)\text{-trind}\, X = M(\mathcal{A}, X, i)$ for $i \in \{0,1\}$ (resp. $i \in \{0,1,4\}$ or $i \in \{0,1,2,3,4\}$), whenever $A \in \mathcal{A}_A$ for each $A \in \mathcal{A}_X$ with $|A| > 1$.*

Definition 3. We will say that an SSC-family $\mathcal{A}$ is *an upper bound of degree* 1 (resp. *degree* 2 or *degree* 3) for an SSC-family $\mathcal{A}'$, in short, $\mathcal{A} \geq_1 \mathcal{A}'$ (resp. $\mathcal{A} \geq_2 \mathcal{A}'$ or $\mathcal{A} \geq_3 \mathcal{A}'$) if for every space X the following conditions hold:

(1) for each $A' \in \mathcal{A}'_X$ there exists an element $A \in \mathcal{A}_X$ such that $A' \subset A$;

(2) for each $A' \in \mathcal{A}'_X$ we have $\mathrm{Cl}_X(A') \in \mathcal{A}_X$, or (3) $\mathcal{A}'_X \subset \mathcal{A}_X$.

Note that if for SSC-families $\mathcal{A}$ and $\mathcal{A}'$ we have $\mathcal{A} \geq_2 \mathcal{A}'$ or $\mathcal{A} \geq_3 \mathcal{A}'$ then we have also $\mathcal{A} \geq_1 \mathcal{A}'$.

Remark 3. For every SSC-family $\mathcal{A}$ we have that $\mathcal{A}^1 \geq_1 \mathcal{A}$, $\mathcal{A}^4 \geq_2 \mathcal{A}$ and $\mathcal{A}^2 \geq_3 \mathcal{A}$.

The following statement is evidently valid.

Proposition 3. *Let $\mathcal{A}$ and $\mathcal{A}'$ be SSC-families such that $\mathcal{A} \geq_3 \mathcal{A}'$. Then $\mathcal{A}(i)\text{-trind}\, X \geq \mathcal{A}'(i)\text{-trind}\, X$ for every space X and every $i \in \{0,1,2,3,4\}$.*

Corollary 2. *For every space X and every $i \in \{0,1,2,3,4\}$ we have $\mathcal{A}^1(i)\text{-trind}\, X \leq \mathcal{A}^4(i)\text{-trind}\, X \leq \mathcal{A}^2(i)\text{-trind}\, X$ and $\mathcal{A}^3(i)\text{-trind}\, X \leq \mathcal{A}^4(i)\text{-trind}\, X$.*

Proposition 4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be SSC-families such that the family $\mathcal{A}$ possesses property $(*)_1$ (resp. $(*)_2$) and $\mathcal{A} \geq_1 \mathcal{A}'$ (resp. $\mathcal{A} \geq_2 \mathcal{A}'$). Then $\mathcal{A}(i)\text{-trind}\, X \geq \mathcal{A}'(i)\text{-trind}\, X$ for every space X, where $i \in \{0,1\}$ (resp. $i \in \{0,1,4\}$).*

Proof. The case $(*)_1$ and $\mathcal{A} \geq_1 \mathcal{A}'$. We will prove the statement only for $i = 0$. Let $\mathcal{A}(0)\text{-trind}\, X = \alpha \geq -1$. Note that for $\alpha = -1$ or for $|X| = 1$ the statement is valid. Apply induction on α. Consider the case: $\alpha \geq 0$ and $|X| \geq 2$. Let $M' \in \mathcal{A}'_X$ with $|M'| \geq 2$. Since $\mathcal{A} \geq_1 \mathcal{A}'$, there exists $M \in \mathcal{A}_X$ such that $M' \subset M$. Consider a point $x \in M'$ and a closed subset A' of the space M' such that $x \notin A'$. Choose any closed subset A of the space M such that $A' = A \cap M'$ and note that $x \notin A$. Since $\mathcal{A}(0)\text{-trind}\, X = \alpha$, there is a partition L_M in the space M between x and A such that $\mathcal{A}(0)\text{-ind}\, L_M < \alpha$. Note that the set $L_{M'} = L_M \cap M'$ is a partition in the space M' between x and A'. It follows from Proposition 2 that $\mathcal{A}(0)\text{-ind}\, L_{M'} \leq \mathcal{A}(0)\text{-ind}\, L_M$. Then by the inductive assumption we have $\mathcal{A}'(0)\text{-ind}\, L_{M'} \leq \mathcal{A}(0)\text{-ind}\, L_{M'} < \alpha$ and so $\mathcal{A}'(0)\text{-trind}\, X \leq \alpha$.

The case $(*)_2$ and $\mathcal{A} \geq_2 \mathcal{A}'$. We need to prove the statement only for $i = 4$ (see the sentences after Definitions 2 and 3). Let $\mathcal{A}(4)\text{-trind}\, X = \alpha \geq -1$. Note that for $\alpha = -1$ or for $|X| = 1$ the statement is valid. Apply induction on α. Consider the case: $\alpha \geq 0$ and $|X| \geq 2$. Let $M' \in \mathcal{A}'$ with $|M'| \geq 2$. Since $\mathcal{A} \geq_2 \mathcal{A}'$ the set $M = \mathrm{Cl}_X(M')$ is an element of $\mathcal{A}_X$. Recall that $\mathcal{A}(4)\text{-trind}\, X = \alpha$. So there exist distinct points x, y of M and a partition L_M in the space M between x and y such that $\mathcal{A}(4)\text{-ind}\, L_M < \alpha$. This also implies that there exist open disjoint subsets U and V of the space M such that $x \in U$, $y \in V$ and $L_M = M \setminus (U \cup V)$. Choose a point $a \in M' \cap U$ and a point $b \in M' \cap V$ and note that the set $L_{M'} = L_M \cap M'$ is a partition between the points a and b in the space M'. It follows from Proposition 2 that $\mathcal{A}(4)\text{-ind}\, L_{M'} \leq \mathcal{A}(4)\text{-ind}\, L_M$. By the inductive assumption we have $\mathcal{A}'(4)\text{-ind}\, L_{M'} \leq \mathcal{A}(4)\text{-ind}\, L_{M'} < \alpha$ and so $\mathcal{A}'(4)\text{-ind}\, X \leq \alpha$. □

Applying additionally Remarks 2 and 3 we get the following statement.

Corollary 3. *For every SSC-family $\mathcal{A}$ and for every space X we have the following inequalities:*

(a) $\mathcal{A}^1(i)\text{-trind}\, X \geq \mathcal{A}(i)\text{-trind}\, X$, *where* $i \in \{0,1\}$;

(b) $\mathcal{A}^4(i)\text{-trind}\, X \geq \mathcal{A}(i)\text{-trind}\, X$, *where* $i \in \{0,1,4\}$;

(c) $\mathcal{A}^2(i)\text{-trind}\, X \geq \mathcal{A}(i)\text{-trind}\, X$, *where* $i \in \{0,1,2,3,4\}$.

In particular, $\mathcal{A}^1(i)\text{-trind}\, X = \mathcal{A}^2(i)\text{-trind}\, X = \mathcal{A}^4(i)\text{-trind}\, X$, *where* $i \in \{0,1\}$, *and* $\mathcal{A}^2(4)\text{-trind}\, X = \mathcal{A}^4(4)\text{-trind}\, X$.

Now we have the following table of the functions $\mathcal{A}^j(i)$-trind, where $i \in \{0,1,2,3,4\}$, and $j \in \{1,2,3,4\}$; the dots in the cells of coordinates i, j replace the notations of invariants $\mathcal{A}^j(i)\text{-}t\,\text{trind}$'s.

Table 1.

j \ i	0	1	2	3	4
1	trind	trind_p	·	·	·
2	trind	trind_p	·	·	trt
3	·	·	·	·	p
4	trind	trind_p	·	·	trt

Let us mention some relationship between the table invariants. We start with equalities.

Proposition 5. *For each metrizable locally compact space X we have $\mathcal{A}^j(i)$-$\operatorname{ind} X = \operatorname{ind} X$ for every $j \in \{2, 4\}$ and every $i \in \{0, 1, 2, 3, 4\}$.*

Proof. Recall [13] that for each metrizable locally compact space X we have $\mathrm{t}\, X = \operatorname{ind} X$. Hence the statement follows from Proposition 1. □

Proposition 6. *For every metrizable space X with $\mathrm{p}(X) < \omega_0$ and every $i \in \{0, 1, 2, 3, 4\}$, we have $\mathcal{A}^3(i)$-$\operatorname{ind} X = \mathrm{p}(X)$.*

Proof. By Proposition 1 and Remark 1 it is sufficient to show that $\mathcal{A}^3(0)$-$\operatorname{ind} X \leq \mathrm{p}(X)$. Recall [10] that $\mathrm{p}(X) = \sup\{t(A)\colon \text{ A is compact subset of } X\}$. Since $\mathrm{p}(X) < \omega_0$, there exists a compact subset A of X with $|A| \geq 2$ such that $\mathrm{p}(X) = \mathrm{t}(A)$. Note ([13]) that $\operatorname{ind} A = \mathrm{t}(A)$. It follows from Remark 1 (a) and Corollary 3 (a) that $\operatorname{ind} A = \mathcal{A}^1(0)$-$\operatorname{ind} A \geq \mathcal{A}^3(0)$-$\operatorname{ind} A$. Hence $\mathrm{p}(X) = \mathrm{t}(A) = \operatorname{ind} A \geq \mathcal{A}^3(0)$-$\operatorname{ind} A$. □

The following statement is obvious.

Proposition 7. *For every compact space X we have $\mathcal{A}^3(i)$-$\operatorname{trind} X = \mathcal{A}^4(i)$-$\operatorname{trind} X$, where $i \in \{0, 1, 2, 3, 4\}$.*

We continue with inequalities.

Remark 4. Recall ([8]) that for each integer $n \geq 1$ there exists a totally disconnected separable metrizable space X_n such that $\operatorname{ind} X_n = n$. Note that $\operatorname{ind}_p X_n = \mathcal{A}^3(0)$-$\operatorname{ind} X_n = 0$.

Remark 5. Recall ([4]) that there exists a compact space Y with $\operatorname{ind} Y = \operatorname{ind}_p Y = 2$ such that each its component is homeomorphic to the closed interval $[0, 1]$. Hence, $\mathrm{t}\, Y = 1$. Moreover, $\mathcal{A}^1(2)$-$\operatorname{ind} Y = 1$ and $\mathcal{A}^1(3)$-$\operatorname{ind} Y = 0$.

Proposition 8. *For every strongly countable-dimensional compact metrizable space X we have $\mathcal{A}^4(2)$-$\operatorname{trind} X = \mathcal{A}^3(2)$-$\operatorname{trind} X \leq \omega_0$.*

Proof. Since every strongly countable-dimensional compact metrizable space has a non-empty open subset with $\operatorname{ind} < \infty$, we get the statement. □

Remark 6. Recall (cf. [6]) that for each $\alpha < \omega_1$ there exists a strongly countable-dimensional compact metrizable space X_α with $\operatorname{trind} X_\alpha = \alpha$.

Let X be a space and $X_{(k)}$ be the set of all points of X that have arbitrary small neighborhoods with boundaries of dimension $\operatorname{ind} \leq k-1$, where k is an integer ≥ 0. We call the space X *weakly n-dimensional* in the sense of ind if $\operatorname{ind} X = n$ and $\operatorname{ind}(X \setminus X_{(n-1)}) < n$.

Recall (cf. [8]) that for each integer $n \geq 1$ there exists a weakly n-dimensional in the sense of ind separable metrizable space Y_n. Note that the subset $Y_n \setminus (Y_n)_{n-1}$ of Y_n can not be closed. However there is a metrizable weakly 1-dimensional in the sense of ind space R such that $|R \setminus R_{(0)}| = 1$ (cf. [6, Problem 4.1.B]). This implies that $\mathcal{A}^2(1)$-$\operatorname{ind} R = \mathcal{A}^2(2)$-$\operatorname{ind} R = 0$.

Proposition 9. *Let X be a weakly n-dimensional in the sense of* ind *space, where $n \geq 1$. Then $\mathcal{A}^2(2)$-ind $X \leq n-1$.*

Proof. Consider a subset M of X with $|M| \geq 2$.

If $M \subset X \setminus X_{(n-1)}$ then $\operatorname{ind} M \leq n-1$. So for each point and any closed subset A of M such that $x \notin A$ there exists a partition L_M in M between x and A with $\operatorname{ind} L_M \leq n-2$. Note that $\mathcal{A}^2(2)$-ind $L_M \leq \operatorname{ind} L_M$ by Proposition 1.

If there is a point $x \in M \setminus (X \setminus X_{(n-1)})$, so x has arbitrary small neighborhoods with at most $(n-2)$-dimensional in the sense of ind boundaries. This implies that for every closed subset A of M such that $x \notin A$ there exists a partition L_M in M between x and A with $\operatorname{ind} L_M \leq n-2$. Recall again that $\mathcal{A}^2(2)$-ind $L_M \leq \operatorname{ind} L_M$.

Both cases imply that $\mathcal{A}^2(2)$-ind $X \leq n-1$. □

3. Zero-dimensionality with respect to $\mathcal{A}(i)$-trind. In this section, let $\mathcal{A}$ be any SSC-family, and put $\mathcal{C}_\alpha(\mathcal{A}(i)) = \{X \colon \mathcal{A}(i)\text{-trind}\, X \leq \alpha\}$ for $i \in \{0,1,2,3,4\}$.

Question 1. *Determine the class $\mathcal{C}_0(\mathcal{A}(i))$, where $i \in \{0,1,2,3,4\}$.*

Proposition 1 (a) and Corollaries 2 and 3 easily imply the next statement.

Proposition 10. *The following assertions hold.*

(a) $\mathcal{C}_0(\mathcal{A}(0)) \subset \mathcal{C}_0(\mathcal{A}(1)) \subset \mathcal{C}_0(\mathcal{A}(3)) \subset \mathcal{C}_0(\mathcal{A}(4))$ *and* $\mathcal{C}_0(\mathcal{A}(0)) \subset \mathcal{C}_0(\mathcal{A}(2)) \subset \mathcal{C}_0(\mathcal{A}(4))$.

(b) For every $i \in \{0,1,2,3,4\}$ we have $\mathcal{C}_0(\mathcal{A}^2(i)) \subset \mathcal{C}_0(\mathcal{A}^4(i)) \subset \mathcal{C}_0(\mathcal{A}^1(i))$ and $\mathcal{C}_0(\mathcal{A}^4(i)) \subset \mathcal{C}_0(\mathcal{A}^3(i))$.

(c) $\mathcal{C}_0(\mathcal{A}^1(i)) \subset \mathcal{C}_0(\mathcal{A}(i))$, *where $i \in \{0,1\}$;*
$\mathcal{C}_0(\mathcal{A}^4(i)) \subset \mathcal{C}_0(\mathcal{A}(i))$, *where $i \in \{0,1,4\}$;*
$\mathcal{C}_0(\mathcal{A}^2(i)) \subset \mathcal{C}_0(\mathcal{A}(i))$, *where $i \in \{0,1,2,3,4\}$.*
In particular, $\mathcal{C}_0(\mathcal{A}^1(i)) = \mathcal{C}_0(\mathcal{A}^2(i)) = \mathcal{C}_0(\mathcal{A}^4(i))$, where $i \in \{0,1\}$, and $\mathcal{C}_0(\mathcal{A}^2(4)) = \mathcal{C}_0(\mathcal{A}^4(4))$.

Additionally, we have the following proposition.

Proposition 11. *(a)* $\mathcal{C}_0(\mathcal{A}(3)) = \mathcal{C}_0(\mathcal{A}(4))$.

(b) $\mathcal{C}_0(\mathcal{A}^3(0)) = \mathcal{C}_0(\mathcal{A}^3(1)) = \mathcal{C}_0(\mathcal{A}^3(2)) = \mathcal{C}_0(\mathcal{A}^3(3)) = \mathcal{C}_0(\mathcal{A}^3(4))$.

Proof. (a) By Proposition 3.1 (a), it is sufficient to show that $\mathcal{C}_0(\mathcal{A}^3(4)) \subset \mathcal{C}_0(\mathcal{A}^3(3))$. Let $X \in \mathcal{C}_0(\mathcal{A}^3(4))$ and $M \in \mathcal{A}_X$. Note that the subspace M of the space X is disconnected. So there are clopen disjoint non-empty subsets M_1 and M_2 of M such that $M = M_1 \cup M_2$. Put $A = M_1$ and observe that every $x \in M_2$ can be separated from A in M by the empty set. This implies that $\mathcal{A}(3)$-ind $X = 0$.

(b) By Proposition 10 (a), it is sufficient to show that $\mathcal{C}_0(\mathcal{A}^3(4)) \subset \mathcal{C}_0(\mathcal{A}^3(0))$. Let $X \in \mathcal{C}_0(\mathcal{A}^3(4))$ and $M \in \mathcal{A}^3_X$ with $|M| > 1$. Note that X must be punctiform ([10]). Consider $M \in \mathcal{A}^3_X$, i.e. M is a compact subspace of X. Note that M is punctiform too. By [6, Theorem 1.4.5] we have $\operatorname{ind} M = 0$. This implies that $\mathcal{A}^3(0)$-ind $X = 0$. □

We summarize the classes $\mathcal{C}_0(\mathcal{A}^j(i)) \setminus \{X\colon |X| \le 1\}$, where $i \in \{0,1,2,3,4\}$, and $j \in \{1,2,3,4\}$ in the following table.

Table 2.

j \ i	0	1	2	3	4
1	$\mathcal{Z}$	$\mathcal{D}_t$	$\mathcal{Z}_p$	$\mathcal{D}$	$\mathcal{D}$
2	$\mathcal{Z}$	$\mathcal{D}_t$	$\mathcal{X}$	$\mathcal{D}_h$	$\mathcal{D}_h$
3	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$
4	$\mathcal{Z}$	$\mathcal{D}_t$	$\mathcal{Y}$	$\mathcal{D}_h$	$\mathcal{D}_h$

where

$\mathcal{Z}$ is the class of zero-dimensional spaces in the sense of ind with $|X| > 1$ (see Remark 1);

$\mathcal{D}_t$ is the class of totally disconnected spaces with $|X| > 1$ (see [6, Definition 1.4.1] and Remark 1);

$\mathcal{D}_h$ is the class of hereditarily disconnected spaces with $|X| > 1$ (see [6, Definition 1.4.2], Remark 1 and [13]);

$\mathcal{P}$ is the class of punctiform spaces with $|X| > 1$ (see [6, Definition 1.4.3], Remark 1 and [10]);

$\mathcal{D}$ is the class of disconnected spaces;

$\mathcal{Z}_p$ is the class of non-trivial spaces having at least one point at which the dimension ind is zero.

Remark 7. (a) Recall that $\mathcal{Z} \subset \mathcal{D}_t \subset \mathcal{D}_h \subset \mathcal{P}$ and there are subspaces of the real plane which exhibit the difference between the classes (see [6, Examples 1.4.6-8]).

(b) Note that $\mathcal{Z} \subset \mathcal{Z}_p \subset \mathcal{D}$ and $\mathcal{D}_h \subset \mathcal{D}$.

(c) Let $X \oplus Y$ be the free union of topological spaces X and Y, I the closed interval $[0,1]$ and P a one-point space. Then observe that $P \oplus I \in \mathcal{Z}_p \setminus \mathcal{P}$, $I \oplus I \in \mathcal{D} \setminus \mathcal{Z}_p$, the space Z from [6, Example 1.4.8] is in $\mathcal{P} \setminus D$ and the Erdös' space H_0 from [6, Example 1.2.15] is in $\mathcal{D}_t \setminus \mathcal{Z}_p$.

(d) It follows from Proposition 10 (a) and (b) that $\mathcal{Z} \subset \mathcal{X} \subset \mathcal{Y} \subset \mathcal{D}_h$ and $\mathcal{Y} \subset \mathcal{Z}_p$. Note also that every weakly 1-dimensional in the sense of ind space is in $\mathcal{X} \setminus \mathcal{Z}$ (see Proposition 9), the Erdös' space H_0 is in $\mathcal{D}_t \setminus \mathcal{Y}$ and the space $P \oplus I$ from (c) is in $\mathcal{Z}_p \setminus \mathcal{Y}$.

We have the following additional facts about the classes $\mathcal{X}$ and $\mathcal{Y}$:

(i) if $X \in \mathcal{X}$, $X' \subset X$ and $|X'| > 1$ then $X' \in \mathcal{X}$;

(ii) if $Y \in \mathcal{Y}$, Y' is a closed subset of Y and $|Y'| > 1$ then $Y' \in \mathcal{Y}$;

(iii) $\mathcal{X} = \mathcal{Y} = \mathcal{Z}$ in the realm of locally compact spaces.

Problem 1. *Describe the classes $\mathcal{X}$ and $\mathcal{Y}$ in the realm of separable metrizable spaces (resp. metrizable spaces or topological T_3-spaces).*

4. Countable unions of spaces of $\mathcal{A}(i)$-trind ≤ 0, $i \in \{0,1,2,3,4\}$. Let $\mathcal{A}$ be any SSC-family.

Definition 4. A space X is said to be σ-$\mathcal{C}_0(\mathcal{A}(i))$, where $i \in \{0,1,2,3,4\}$, if $X = \bigcup_{j=1}^{\infty} X_j$, where $X_j \in \mathcal{C}_0(\mathcal{A}(i))$ for each j.

Problem 2. *Describe the class σ-$\mathcal{C}_0(\mathcal{A}(i))$, where $i \in \{0,1,2,3,4\}$.*

We will restrict now our discussion to the realm of separable metrizable spaces.

Proposition 12. *Let X be a separable completely metrizable σ-$\mathcal{C}_0(\mathcal{A}^j(i))$ space, where $i \in \{0,1,2,3,4\}$ and $j \in \{1,2,3,4\}$. Then there is a metrizable compactification Y of X such that Y is also σ-$\mathcal{C}_0(\mathcal{A}^j(i))$.*

Proof. Recall ([6, Lemma 5.3.1]) that there is metrizable compactification Y of X such that the remainder $Y \setminus X$ is strongly countable-dimensional. Note that the space $Y \setminus X$ is σ-$\mathcal{C}_0(\mathcal{A}^j(i))$. Hence, Y is also σ-$\mathcal{C}_0(\mathcal{A}^j(i))$. □

Remark 8. Let us recall [10] that the R. Pol's metrizable compactum P is a compactification of some complete A-strongly infinite-dimensional totally disconnected space P_0 with the reminder $P \setminus P_0 = \bigcup_{k=1}^{\infty} P_k$, where P_k is a finite-dimensional compactum for each k. Note that P is σ-$\mathcal{C}_0(\mathcal{A}^j(i))$ for every $i \in \{1,3,4\}$ and every $j \in \{1,2,3,4\}$, and for the pair: $i = 0$ and $j = 3$. We note also that P is not σ-$\mathcal{C}_0(\mathcal{A}^j(i))$ for $i = 0$ and every $j \in \{1,2,4\}$.

Question 2. *Is P σ-$\mathcal{C}_0(\mathcal{A}^2(2))$ (resp. σ-$\mathcal{C}_0(\mathcal{A}^4(2))$)?*

The following statement is evident.

Lemma 1. *Let Y be a metrizable compact space, $X \subset Y$ and $Y \setminus X = \bigcup_{i=1}^{\infty} X_i$, where for each i the set X_i is compact and $\operatorname{ind} X_i < \infty$. Assume that M is a closed subset of Y. Then either $|M \cap X| > 1$ or M is strongly countable dimensional.*

Lemma 2. *Let X be a separable metrizable space, $M \subset X$, $x, y \in M$ and L_M a partition of M between the points x, y. Then there is a partition L_X of X between x, y such that $L \cap M = L_M$.*

Proof. Let O_x and O_y be disjoint open subsets of M such that $x \in O_x$, $y \in O_y$ and $M \setminus (O_x \cup O_y) = L_M$. Put $L = \operatorname{Cl}_X(L_M \cup O_x) \cap \operatorname{Cl}_X(L_M \cup O_y)$. Note that L is a partition of the subspace $\operatorname{Cl}_X M$ of X between the points x, y such that $L \cap M = L_M$. By [6, Lemma 1.2.9] there is a partition L_X of X between x, y such that $L_X \cap \operatorname{Cl}_X M = L$. Note that $L_X \cap M = L_M$. □

Proposition 13. *Let Y be a metrizable compact space, $X \subset Y$ and $Y \setminus X = \bigcup_{i=1}^{\infty} X_i$, where for each i the set X_i is compact and $\operatorname{ind} X_i < \infty$.*

Assume that $\operatorname{trt} X = \alpha \neq \infty$. Then

$$\operatorname{trt} Y \leq \begin{cases} \omega_0 + \alpha + 1, & \text{if } \alpha < \omega_0^2; \\ \alpha + 1, & \text{if } \alpha \geq \omega_0^2. \end{cases}$$

(One can omit 1 in the formula if α is an infinite limit ordinal.)

Proof. Apply induction on $\alpha \geq 0$. Assume that $\alpha = 0$. Consider a closed subset M of Y with $|M| > 1$. By Lemma 1, we have two possibilities:

$(a)_0$ $|M \cap X| > 1$ or $(b)_0$ M is strongly countable dimensional.

The case $(a)_0$. Since $\operatorname{trt} X = 0$, we have $\operatorname{trt}(M \cap X) = 0$ and the empty set is a partition of $M \cap X$ between some points x, y of $M \cap X$. By Lemma 2, there is a partition L of M

between the points x, y such that $L \subset M \setminus X$. Note that the space L is strongly countable dimensional and hence $\operatorname{trt} L \leq \omega_0$ (see [1]).

The case $(b)_0$. Since M is strongly countable dimensional, we have $\operatorname{trt} M \leq \omega_0$. Note that the both cases imply $\operatorname{trt} X \leq \omega_0 + 1$.

Assume that the statement is valid for all $\alpha < \gamma \geq 1$. Let now $\alpha = \gamma$. Consider a closed subset M of Y with $|M| > 1$. Again by Lemma 2, we have two possibilities:

$(a)_\gamma$ $|M \cap X| > 1$ or $(b)_\gamma$ M is strongly countable dimensional.

The case $(a)_\gamma$. Since $\operatorname{trt} X = \gamma$, we have $\operatorname{trt}(M \cap X) \leq \gamma$. Hence, there is a partition $L_{M\cap X}$ of $M \cap X \subset M$ between some points x, y of $M \cap X$ such that $\operatorname{trt} L_{M\cap X} < \gamma$. By Lemma 2 there is a partition L_M of M between the points x, y such that $L_M \cap (M \cap X) = L_{M\cap X}$. Note that $L_{M\cap X} = L_M \cap X$ and the space $L_M \setminus L_{M\cap X} = \bigcup_{i=1}^{\infty}(L_M \cap X_i)$ is strongly countable dimensional. Hence, by induction, we have $\operatorname{trt} L_M \leq \omega_0 + \operatorname{trt} L_{M\cap X} + 1 < \omega_0 + \gamma + 1$. (Let us observe that if γ is an infinite limit ordinal then $\omega_0 + \operatorname{trt} L_{M\cap X} + 1 < \omega_0 + \gamma$.)

The case $(b)_\gamma$. Since M is strongly countable dimensional, $\operatorname{trt} M \leq \omega_0$. Note that both cases imply $\operatorname{trt} X \leq \omega_0 + \gamma + 1$.

Let us recall that if $\alpha \geq \omega_0^2$ then $\omega_0 + \alpha = \alpha$. □

Corollary 4. *Let X be a separable completely metrizable space and $\operatorname{trt} X = \alpha \neq \infty$. Then there is a compactification Y of X such that*

$$\operatorname{trt} Y \leq \begin{cases} \omega_0 + \alpha + 1, & \text{if } \alpha < \omega_0^2; \\ \alpha + 1, & \text{if } \alpha \geq \omega_0^2. \end{cases}$$

(One can omit 1 in the formula if α is an infinite limit ordinal.)

Proof. Recall ([6, Lemma 5.3.1]) that there is a metrizable compactification Y of X such that the remainder $Y \setminus X$ is strongly countable dimensional. Now, apply Proposition 13 to the space Y. □

Corollary 5. *For any hereditarily disconnected separable completely metrizable space X there is a metrizable compactification Y of X such that $\operatorname{trt} Y \leq \omega_0 + 1$.*

Recall (see [11] (resp. [12])) that for each isolated countable infinite ordinal α there exists an α-dimensional metrizable Cantor trind-manifold Y^α (resp. trInd-manifold Z^α) which is a disjoint union of countably many Euclidean cubes and the irrationals. It follows now from Proposition 13 that for each isolated countable infinite ordinal α, $\operatorname{trt} Y^\alpha = \operatorname{trt} Z^\alpha = \omega_0 + 1$.

Problem 3. *Is there a countable-dimensional separable metrizable space X such that $\operatorname{trt} X > \omega_0 + 1$ (and $\operatorname{trt} X \neq \infty$)?*

REFERENCES

1. F.G. Arenas, V.A. Chatyrko, M.L. Puertas, *Transfinite extension of Steinke's dimension*, Acta Math. Hungar., **88** (2000), №1-2, 105–112.
2. T. Banakh, R. Cauty, *On universality of countable and weak products of sigma hereditarily disconnected spaces*, Fund. Math., **167** (2001), 97–109.
3. V.A. Chatyrko, *Ordinal products of topological spaces*, Fund. Math., **144** (1994), 85–117.

4. V.A. Chatyrko, *On properties of subsets of* $[0, \omega_c] \times I$, Q & A in General Topology, **26** (2008), 97–104.
5. R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
6. R. Engelking, Theory of dimensions, finite and infinite, Heldermann Verlag, Lemgo, 1995.
7. J. Krasinkewicz, Essential mappings onto products of manifolds, in: Geometric and Algebraic Topology, Banach Center Publ. 18, PWN, 1986, 377–406.
8. J. van Mill, The Infinite-Dimensional Topology of Function Spaces, Elsevier, Amsterdam, 2001.
9. R. Pol, *A weakly infinite-dimensional compactum which is not countable-dimensional*, Proc. AMS, **82** (1981), 634–636.
10. T.M. Radul, *On classification of sigma hereditary disconnected spaces*, Mat. Stud. **26** (2006), №1, 97—100.
11. M. Renska, *On Cantor manifolds for the large transfinite dimension*, Topology Appl., **112** (2001), 1–11.
12. M. Renska, *On Cantor manifolds for the small transfinite dimension*, Commentationes Mathematicae, **XLVI** (2006), №2, 163–168.
13. G. Steinke, *A new dimension by means of separating sets*, Arch. Math., **40** (1983), 273–282.

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