УДК 513.88

Yu. I. Oliyar, O. G. Storozh

CRITERIA OF MUTUAL ADJOINTNESS OF PROPER EXTENSIONS OF LINEAR RELATIONS

Yu. I. Oliyar, O. G. Storozh. *Criteria of mutual adjointness of proper extensions of linear relations*, Mat. Stud. **40** (2013), 71–78.

In the paper the role of an initial object is played by a couple (L, L_0) of closed linear relations in a Hilbert space H, such that $L_0 \subset L$. Each closed linear relation $L_1(M_1)$ such that $L_0 \subset L_1 \subset L$ (respectively $L^* \subset M_1 \subset L_0^*$) is said to be a proper extension of $L_0(L^*)$. In the terms of abstract boundary operators i.e. bounded linear operator $U(V)$ acting from $L(M)$ to G (G is an auxiliary Hilbert space) such that the null space of $U(V)$ contains $L_0(L^*)$, criteria of mutual adjointness for mentioned above relations L_1 and M_1 are established.

Ю. И. Олияр, О. Г. Сторож. *Критерии взаимной сопряжености собственных расширений линейных отношений* // Мат. Студії. – 2013. – Т.40, №1. – С.71–78.

В работе роль исходного объекта играет пара (L, L_0) замкнутых линейных отношений в гильбертовом пространстве H, причем $L_0 \subset L$. Замкнутое линейное отношение $L_1(M_1)$ такое, что $L_0 \subset L_1 \subset L$ (соответственно $L^* \subset M_1 \subset L_0^*$) называется собственным расширениям отношения $L_0(L^*)$. В терминах абстрактных краевых операторов, т.е. линейных ограниченных операторов действующих из $L(M)$ в G (G — вспомогательное гильбертово пространство), многообразия нулей которых содержат $L_0(L^*)$, установлены критерии взаимной сопряженности упомянутых выше отношений L_1 и M_1.

1. Introduction and basic notations. The theory of linear relations in a Hilbert space was initiated by R. Arens ([1]) and has been developed by many mathematicians (see, for example [2–8] and the references therein). The present paper (as the majority of the mentioned above ones) is devoted to an application of the concept of a linear relation in the extension theory. The contents of the paper are as follows. Section 2 is devoted to the definition of a boundary pair. We also prove an abstract Lagrange formula in terms of boundary pairs there. Since Section 4 deals with proper extensions, we describe general form of a proper extensions in Section 3. In view of the results from Section 3, naturally arises the problem to establish a criterion of mutual adjointness of two proper extensions of linear relations. We consider this problem in Section 4.

In this paper we use the following notations:
$(\cdot \mid \cdot)_X$ is the inner product in a Hilbert space X;
$D(T)$, $R(T)$, $\ker T$ are, respectively, the domain, range, and kernel of a (linear) operator T;
$B(X, Y)$ is the set of linear bounded operators $A\colon X \to Y$ such that $D(A) = X$;
$A \downarrow E$ is the restriction of A to E;
1_X is the identity of X;

2010 *Mathematics Subject Classification:* 47A06, 47B99.
Keywords: extension; adjoint; Hilbert space.

$AE = \{Ax\colon x \in E\}$;
$\oplus$ and $\ominus$ are the symbols of the orthogonal sum and orthogonal complement, respectively;
if $A_i\colon X \to Y_i,\ i \in \{1,\dots,n\}$, are linear operators then the notation $A = A_1 \oplus \dots \oplus A_n$ means that $Ax = (A_1x, \dots, A_nx)$ for every $x \in X$;
$\bar{E}$ is the closure of E; $X^2 = X \oplus X$;
T^* is the operator (relation) adjoint to an operator (relation) T.

Let us recall that a linear manifold $T \subset X \oplus X$, where X is a Hilbert space, is called a linear relation on X. The adjoint T^* is defined as follows

$$T^* = \{\hat{z} = (z, z') \in X^2\colon \forall \hat{y} = (y, y') \in T \quad (y' \mid z)_X = (y \mid z')_X\}.$$

It is clear that $T^* = (\hat{J}T)^{\perp}$, where $\hat{J}(y, y') = (-iy', iy)$.

The role of the initial object is played by a couple (L, L_0) of closed linear relations such that $L_0 \subset L \subset H^2$ where H is a fixed complex Hilbert space equipped with the inner product $(\cdot \mid \cdot)$.

Put $M_0 = L^*$, $M = L_0^*$, $\hat{H}_L = L \ominus L_0$, $\hat{H}_M = M \ominus M_0$ and denote by $\hat{P}_L, \hat{P}_M$ the orthogonal projections $L \to \hat{H}_L$, $M \to \hat{H}_M$, respectively. Each closed linear relation $L_1(M_1)$ such that $L_0 \subset L_1 \subset L$ (respectively $M_0 \subset M_1 \subset M$) is said to be a proper extension of $L_0(M_0)$. It is easy to see that

$$\hat{H}_M = \hat{J}\hat{H}_L, \tag{1}$$

where

$$\hat{J} = \begin{pmatrix} 0 & -i1_H \\ i1_H & 0 \end{pmatrix}, \tag{2}$$

(see [9] for example). For the case of linear operators it was trivial because $\hat{J}$ is a unitary operator ([10]).

2. Boundary pair. Abstract Lagrange formula.

Definition 1. Let G be an (auxiliary) Hilbert space and $U \in B(L, G)$. The pair (G, U) is called a *boundary pair for* (L, L_0) if $R(U) = G$, $\ker U = L_0$. In this case G and U are said to be a boundary space and a total boundary operator, respectively.

Proposition 1. *A boundary pair for (L, L_0) exists and is unique provided the following implication holds: if $(G, U), (\hat{G}, \hat{U})$ are two boundary pairs for (L, L_0) then there exists a unique bijection $E_L \in B(G, \hat{G})$ such that $\hat{U} = E_L U$.*

Proof. Observe that $(\hat{H}_L, \hat{P}_L)$ is a boundary pair for (L, L_0). The uniqueness follows from the so called "Lemma on triple" [11] (see Remark 1). □

Remark 1. By the Lemma on triple we mean the following consequence of the Banach inverse operator theorem.

Let X, X_1, X_2 be Banach spaces, $A \in B(X, X_1)$, $B \in B(X, X_2)$, $R(A) = X_1$, $\ker A \subset \ker B$. Then there exists a unique $C \in B(X_1, X_2)$ such that $B = CA$.

Remark 2. Suppose that G_L and G_M are boundary spaces for L and M, respectively. Then $\dim G_L = \dim \hat{H}_L = \dim \hat{H}_M = \dim G_M$. This follows from (1)–(2) and Proposition 1.

Theorem 1. *Let (G_L, U) and (G_M, V) be boundary pairs for (L, L_0) and (M, M_0), respectively. Then there exists a unique mapping $E\colon G_L \to G_M$ satisfying the following requirements:*

$$E \in B(G_L, G_M), \quad E^{-1} \in B(G_M, G_L) \tag{3}$$

and

$$\forall \hat{y} = (y, y') \in L, \forall \hat{z} = (z, z') \in M$$

$$(y' \mid z) - (y \mid z') = (i\hat{J}\hat{y} \mid \hat{z})_{H^2} = (EU\hat{y} \mid V\hat{z})_{G_M} = (U\hat{y} \mid E^*V\hat{z})_{G_L}. \tag{4}$$

Conversely, if (G_L, U) is a boundary pair for (L, L_0), G_M is a boundary space for (M, M_0) (i.e. $\dim G_M = \dim \hat{H}_M$) and an operator E satisfies conditions (3) then there exists a unique $V \in B(M, G_M)$ such that (G_M, V) is a boundary pair for (M, M_0) and (4) holds.

Proof. In the case $(G_L, U) = (\hat{H}_L, \hat{P}_L)$, $(G_M, V) = (\hat{H}_M, \hat{P}_M)$ condition (4) is fulfilled with $E = i\hat{J} \downarrow \hat{H}_L$.

In fact, if $y \in L_0$ or $z \in M_0$, then $(i\hat{J}\hat{y} \mid z)_{H^2} = 0 = ((i\hat{J} \downarrow \hat{H}_L)\hat{P}_L\hat{y} \mid \hat{P}_M\hat{z})_{\hat{H}_M}$(let us recall that $\hat{J}L_0 = M^\perp, \hat{J}L = M_0^\perp$). Now assume that $\hat{y} \in \hat{H}_L, \hat{z} \in \hat{H}_M$. In view of (1) we obtain

$$(i\hat{J}\hat{y} \mid \hat{z})_{H^2} = (i\hat{J}\hat{P}_L\hat{y} \mid \hat{P}_M\hat{z})_{H^2} = ((i\hat{J} \downarrow \hat{H}_L)\hat{P}_L\hat{y} \mid \hat{P}_M\hat{z})_{\hat{H}_M}.$$

In the general case, taking into account Proposition 2, we see that there exist unique bijections $E_L \in B(G_L, \hat{H}_L)$, $E_M \in B(G_M, \hat{H}_M)$ satisfying the equalities $\hat{P}_L = E_L U$, $\hat{P}_M = E_M V$. Thus (4) holds with $E = E_M^*(i\hat{J} \downarrow \hat{H}_L)E_L$.

To obtain the converse statement, we reverse the previous steps. Indeed, suppose that operators E, U (and therefore $E_L = \hat{P}(U \downarrow \hat{H}_L)^{-1}$) are given and E_M is (the unique) solution of the latter equation. It is clear that the operator $V = E_M^{-1}\hat{P}_M$ (and only it) possesses all the required properties. □

Formula (4) is called an abstract Lagrange formula.

Theorem 2. *Let (G_L, U) and (G_M, V) be boundary pairs for (L, L_0) and (M, M_0), respectively, and let operator E satisfy the conditions (3). The following statements are equivalent:*

i) for each $\hat{y} = (y, y') \in L$ and $\hat{z} = (z, z') \in M$ equality (4) holds;

ii) $U(-i\hat{J} \downarrow \hat{H}_M)V^* = E^{-1}$;

iii) $V^*EU \downarrow \hat{H}_L = i\hat{J} \downarrow \hat{H}_L$;

iv) $V(i\hat{J} \downarrow \hat{H}_L)U^* = (E^{-1})^*$;

v) $U^*E^*V \downarrow \hat{H}_M = -i\hat{J} \downarrow \hat{H}_M$.

Proof. a) Preliminary remarks. Since U, V are normally solvable operators, U^* and V^* are normally solvable ones too (see [12]). Furthermore, $R(U) = G_L$, $\ker U = L_0$. This yields $\ker U^* = \{0\}$, $R(U^*) = \hat{H}_L$. Similar arguments show that $\ker V^* = \{0\}$, $R(V^*) = \hat{H}_M$. Taking into account the Banach inverse operator theorem, we conclude that

$$(U^*)^{-1} \in B(\hat{H}_L, G_L), \quad (V^*)^{-1} \in B(\hat{H}_M, G_M).$$

b) *ii)* ⇔ *iv)*. It follows from (1) that $\hat{J} \downarrow \hat{H}_L \in B(\hat{H}_L, \hat{H}_M)$ is a bijection. Let us find $(\hat{J} \downarrow \hat{H}_L)^* \in B(\hat{H}_M\hat{H}_L)$.

For each $\hat{u} \in \hat{H}_L, \hat{v} \in \hat{H}_M$ we have

$$((\hat{J} \downarrow \hat{H}_L)\hat{u} \mid \hat{v})_{H^2} = (\hat{J}\hat{u} \mid \hat{v})_{H^2} = (\hat{u} \mid \hat{J}\hat{v})_{H^2} = (\hat{u} \mid (\hat{J} \downarrow \hat{H}_L)\hat{v})_{H^2},$$

that is, $(\hat{J} \downarrow \hat{H}_L)^* = (\hat{J} \downarrow \hat{H}_M)$.

In addition, for each, $\hat{u} \in \hat{H}_L$ the equality $(\hat{J} \downarrow \hat{H}_M)(\hat{J} \downarrow \hat{H}_L)\hat{u} = \hat{J}^2\hat{u} = \hat{u}$ holds. In other words, $\hat{J} \downarrow \hat{H}_L,\ \hat{J} \downarrow \hat{H}_M$ are unitary operators and $(\hat{J} \downarrow \hat{H}_M)^{-1} = \hat{J} \downarrow \hat{H}_L$.

The equivalence $ii) \Leftrightarrow iv)$ is proved.

c)$iii) \Leftrightarrow v)$. The proof of this equivalence is similar to the proof of the previous one.

d)$i) \Leftrightarrow iii)$ Equality (4) holds for each $\hat{y} \in L,\ \hat{z} \in M$ if it holds for $\hat{y} \in \hat{H}_L, \hat{z} \in \hat{H}_M$. But in this case the mentioned equality has the following form

$$((i\hat{J} \downarrow \hat{H}_L)\hat{u} \mid \hat{v})_{\hat{H}_M} = (V^*E(U \downarrow \hat{H}_L)\hat{u} \mid \hat{v})_{\hat{H}_M}.$$

Therefore, $i) \Leftrightarrow iii)$.

e) $ii) \Leftrightarrow iii)$. Since $\hat{J} \downarrow \hat{H}_M$ and $\hat{J} \downarrow \hat{H}_L$ are mutually adjoint and mutually inverse (unitary) operators and $(U \downarrow \hat{H}_L)^{-1} \in B(G, \hat{H}_L)$ (it follows from the equalities $R(U \downarrow \hat{H}_L) = G$, $\ker(U \downarrow \hat{H}_L) = \{0\}$ and the Banach inverse operator theorem), we obtain (taking into account the Preliminary remarks)

$$U(-i\hat{J} \downarrow \hat{H}_M)V^* = E^{-1} \Leftrightarrow (U \downarrow \hat{H}_L)(-i\hat{J} \downarrow \hat{H}_M)V^* = E^{-1} \Leftrightarrow$$

$$\Leftrightarrow (V^*)^{-1}(-i\hat{J} \downarrow \hat{H}_L)(\hat{U} \downarrow \hat{H}_L)^{-1} = E \Leftrightarrow V^*EU \downarrow \hat{H}_L = i\hat{J} \downarrow \hat{H}_L. \qquad \square$$

Corollary 1. *Let G_1, G_2 be Hilbert spaces, $U_i \in B(L, G_i), i \in \{1, 2\}, G = G_1 \oplus G_2, U = U_1 \oplus U_2$. Assume that (G, U) is a boundary pair for (L, L_0). Then*

a) There exist unique $\tilde{U}_1 \in B(M, G_2)$, $\tilde{U}_2 \in B(M, G_1)$ such that $(\tilde{G}, \tilde{U})$, where $\tilde{G} = G_2 \oplus G_1$, $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$, is a boundary pair for (M, M_0) and

$$\forall \hat{y} \in L, \forall \hat{z} \in M \quad (i\hat{J}\hat{y} \mid \hat{z})_{H^2} = (iJU\hat{y} \mid \tilde{U}\hat{z})_{\tilde{G}} = (U\hat{y} \mid -iJ^*\tilde{U}\hat{z})_G =$$

$$= (U_1\hat{y} \mid \tilde{U}_2\hat{z})_{G_1} - (U_2\hat{y} \mid \tilde{U}_1\hat{z})_{G_2}, \tag{5}$$

where $J \in B(G, \tilde{G})$ is defined as follows $\forall h_1 \in G_1, \forall h_2 \in G_2\ J(h_1, h_2) = (ih_2, -ih_1)$, i.e.

$$J = \begin{pmatrix} 0 & i1_{G_2} \\ -i_{G_1} & 0 \end{pmatrix}. \tag{6}$$

b) Let $(\tilde{G}, \tilde{U}) = (G_2 \oplus G_1, \tilde{U}_1 \oplus \tilde{U}_2)$ be a boundary pair for (M, M_0). The following statements are equivalent:

i) for each $\hat{y} \in L, \hat{z} \in M$ equality (5) holds;

ii) $U(\hat{J} \downarrow \hat{H}_M)\tilde{U}^* = J^*$;

iii) $\tilde{U}^*JU \downarrow \hat{H}_L = \hat{J} \downarrow \hat{H}_L$;

iv) $\tilde{U}(\hat{J} \downarrow \hat{H}_L)U^* = J$;

v) $U^*J^*\tilde{U} \downarrow \hat{H}_M = \hat{J} \downarrow \hat{H}_M$.

The proof of Corollary 1 can be obtained from Theorems 1 and 2 by substituting $G_L = G, G_M = \tilde{G}, E = iJ$ in the corresponding formulas.

Remark 3. Since $R(\tilde{U}^*) = \hat{H}_M$, the expression $\hat{J} \downarrow \hat{H}_M$ in equality $ii)$ can be replaced by $\hat{J}$, therefore this equality is equivalent to the following system

$$U_1\hat{J}\tilde{U}_1^* = 0,\ U_1\hat{J}\tilde{U}_2^* = i1_{G_1},\ U_2\hat{J}\tilde{U}_1^* = -i1_{G_2},\ U_2\hat{J}\tilde{U}_2^* = 0. \tag{7}$$

3. The general form of the proper extension of L_0.

Proposition 2. *Let $G_i, U_i, \tilde{U}_i$ $(i \in \{1,2\})$ be as in Corollary 1 and $L_1 \overset{def}{=} \ker U_1$. Then $L_1^* = \ker \tilde{U}_1$.*

Proof. The inclusion $L_0 \subset L_1$ implies $L_1^* \subset M$. Further, $L_1^* = \{\hat{z} \in H^2 \colon \forall \hat{y} \in L_1 = \ker U_1$ $(i\hat{J}\hat{y} \mid \hat{z})_{H^2} = 0\}$, consequently, (5) yields $\ker \tilde{U}_1 \subset L_1^*$. Furthermore, it follows from the first of equalities (7) that $\tilde{U}_1\hat{J}U_1^* = 0$, therefore

$$L_1^* = \hat{J}L_1^{\perp} = \hat{J}(\ker U_1)^{\perp} = \hat{J}\overline{R(U_1^*)} = \overline{\hat{J}R(U_1^*)} = \overline{\ker \tilde{U}_1} = \ker \tilde{U}_1. \qquad \square$$

Theorem 3. *Assume that $L_0 \subset L_1 = \bar{L}_1 \subset L$ and G is a boundary space for (L, L_0). Then*

a) there exists an orthogonal decomposition $G = G_1 \oplus G_2$ and operators

$$U_1 \in B(L, G_1),\ V_1 \in B(M, G_2) \tag{8}$$

such that

$$L_1 = \ker U_1,\ L_1^* = \ker V_1, \tag{9}$$

and, as a result,

$$\ker U_1 \supset L_0,\ \ker V_1 \supset M_0. \tag{10}$$

b) without loss of generality we may assume that

$$R(U_1) = G_1,\ R(V_1) = G_2. \tag{11}$$

Proof. Let (G, U) be a boundary pair for (L, L_0). Put

$$G_2 \overset{def}{=} \{U\hat{y} \colon \hat{y} \in L_1\} = \{(U \downarrow \hat{H}_L)\hat{y} \colon \hat{y} \in L_1 \ominus L_0\}.$$

Since $U \downarrow \hat{H}_L$ is a homeomorphism $\hat{H}_L \to G_2 \subset G$, G_2 is a closed linear subspace of G. Put $G_1 = G \ominus G_2$, $U_i = P_iU$, where P_i is the orthogonal projection $G \to G_i, (i \in \{1,2\})$, and denote by $\tilde{U}_1, \tilde{U}_2$ the operators uniquely determined by U_1, U_2 from (5).

To complete the proof, it is sufficient to substitute $V_1 = \tilde{U}_1$ and apply Proposition 2. $\square$

4. Criteria of mutual adjointness. Taking into account Theorem 3, we see that the following problem arises in a natural way.

Let

i) $G = G_1 \oplus G_2$ be a boundary space for (L, L_0) and operators U_1, V_1 satisfy conditions (8), (10);

ii)
$$L_1 = \ker U_1, M_1 = \ker V_1 \tag{12}$$

(cf.(9)). The problem is to establish a criterion of mutual adjointness of L_1, and M_1.

Before solving the problem let us introduce the following notation

$$\begin{cases} X_1 = L_1 \ominus L_0, & X_2 = L \ominus L_1; \\ Y_1 = M_1 \ominus M_0, & Y_2 = M \ominus M_1. \end{cases} \tag{13}$$

It is clear that

$$\hat{H}_L = X_1 \oplus X_2,\ \hat{H}_M = Y_1 \oplus Y_2, \tag{14}$$

$$L_0 \oplus X_1 = L_1 = \ker U_1,\ M_0 \oplus Y_1 = M_1 = \ker V_1. \tag{15}$$

Taking into account (1) and (14), we see that

$$\hat{H}_L = \hat{J}\hat{H}_M = \hat{J}[Y_1 \oplus Y_2] = \hat{J}Y_1 \oplus \hat{J}Y_2. \tag{16}$$

Lemma 1.

$$M_1^* = L_0 \oplus \hat{J}Y_2 = L_0 \oplus \hat{J}\overline{R(V_1^*)}. \tag{17}$$

Proof. Applying (1) to the pair (M, M_1) (instead of (L, L_0)), we obtain $M_1^* = L_0 \oplus \hat{J}[M \ominus M_1]$.

Taking into account (12), (13), we have $Y_2 = M \ominus M_1 = M \ominus \ker V_1 = \overline{R(V_1^*)}$. □

Lemma 2. *The following statements are equivalent:*

i) $L_1 \supset M_1^*$;

ii) $U_1\hat{J}V_1^* = 0$;

iii) $\ker U_1 \supset L_0 \oplus \hat{J}\overline{R(V_1^*)}$;

iv) $X_1 \supset \hat{J}Y_2$.

In each of the cases

$$L_1 \ominus M_1^* = \ker U_1 \ominus [L_0 \oplus \hat{J}\overline{R(V_1^*)}] = X \ominus \hat{J}Y_2. \tag{18}$$

Proof. Taking into account (14)–(17) and the inclusion $\ker U_1 \supset L_0$, we obtain

$$U_1\hat{J}V_1^* = 0 \Leftrightarrow \ker U_1 \supset \hat{J}R(V_1^*) \Leftrightarrow \ker U_1 \supset \hat{J}\overline{R(V_1^*)} \Leftrightarrow \ker U_1 \supset L_0 \oplus \hat{J}\overline{R(V_1^*)} \Leftrightarrow$$
$$\Leftrightarrow L_1 \supset M_1^* \Leftrightarrow L_1 \ominus L_0 \supset M_1^* \ominus L_0 \Leftrightarrow X_1 \supset \hat{J}Y_2.$$

Therefore, conditions *i)-iv)* are equivalent. Suppose that these conditions hold. From (15) and (17) equalities (18) are derived.

Corollary 2. *The following equivalences hold*

$$L_1 = M_1^* \Leftrightarrow \ker U_1 = L_0 \oplus \hat{J}\overline{R(V_1^*)} \Leftrightarrow X_1 = \hat{J}Y_2. \tag{19}$$

Proof. The corollary immediately follows from (18). □

Lemma 3. *Assume that equalities (11) hold and* $U_1\hat{J}V_1^* = 0$. *Put* $\hat{U} = U_1 \oplus V_1\hat{J}\hat{P}_L$. *Then* $R(\hat{U}) = G, \ker\hat{U} = L_0 \oplus [X_1 \ominus \hat{J}Y_2]$.

Proof. Let us show that $R(V_1\hat{J}\hat{P}_L \downarrow R(V_1^*)) = G_2$. For this purpose note that $R(V_1 \downarrow R(V_1^*)) = R(V_1) = G_2$, i.e. $(\forall h \in G_2)(\exists f \in G_2)\colon V_1V_1^*f = h$.

Put $\hat{H}_L \ni y = \hat{J}V_1^*f$. Taking into account the inclusion $R(V_1^*) \subset \hat{H}_M$ and applying (1), we obtain $U_1y = U_1\hat{J}V_1^*f = 0,\ V_1\hat{J}\hat{P}_L\hat{J}V_1^*f = V_1\hat{J}\hat{J}V_1^*f = V_1V_1^*f = h$.

Thus $R(V_1\hat{J}\hat{P}_L \downarrow \ker U_1) = G_2$. Then using Lemma 4.5.2 from [13] we see that

$$R(\hat{U}) = R(U_1) \oplus R(V_1\hat{J}\hat{P}_L) = G_1 \oplus G_2 = G.$$

Furthermore, $\ker U \supset L_0$ and (see Lemma 2) $X_1 \supset \hat{J}Y_2$. Moreover, $X_1 \ominus \hat{J}Y_2 = X_1 \cap \hat{J}Y_1$. Therefore, to complete the proof, it is sufficient to verify the equality

$$\ker U_1 \cap \ker V_1\hat{J}\hat{P}_L \cap \hat{H}_L = X_1 \cap \hat{J}Y_1. \tag{20}$$

In order to prove (20) assume first that $y \in X_1 \cap \hat{J}Y_1$. Evidently, $y \in \ker U_1 \cap \hat{H}_L$ and there exists $z \in Y_1$ such that $y = \hat{J}z$. Thus we have $V_1\hat{J}\hat{P}_Ly = V_1\hat{J}\hat{P}_L\hat{J}z = V_1z = 0$, therefore $y \in \ker V_1\hat{J}\hat{P}_L$.

Conversely, if $y \in \ker U_1 \cap \ker V_1\hat{J}\hat{P}_L \cap \hat{H}_L$ then $y \in \ker U_1 \cap \hat{H}_L = X_1$, therefore $0 = V_1\hat{J}\hat{P}_Ly = V_1\hat{J}y$. In other words, $\hat{J}y \in \ker V_1 \cap \hat{H}_M = Y_1$, sequently $y = \hat{J}\hat{J}y \in \hat{J}Y_1$. □

Corollary 3. *Assume that* $\dim \hat{H}_L < \infty$ *and equalities (11) hold. Then* $L_1 = M_1^*$ *if and only if* $U_1\hat{J}V_1^* = 0$.

Proof. Let $\hat{U}$ be the operator from Lemma 3. This operator maps the finite-dimensional space $\hat{H}_L$ onto a finite-dimensional space G. Moreover $\dim \hat{H}_L = \dim G$ (see Remark 2) therefore $X_1 \ominus \hat{J}Y_2 = \ker \hat{U} \downarrow \hat{H}_L = \{0\}$. Now the proof follows from Lemma 2. □

Remark 4. In a general case the condition $U_1\hat{J}V_1^* = 0$ is necessary but not sufficient for the mutual adjointness of L_1 and M_1.

Theorem 4. *Let* $(G, \Lambda), (\tilde{G}, \Pi)$, *where* $G = G_1 \oplus G_2$, $\tilde{G} = G_2 \oplus G_1$, *be boundary pairs for* (L, L_0) *and* (M, M_0), *respectively,* $E \in B(G, \tilde{G})$, $E^{-1} \in B(\tilde{G}, G)$ *and*

$$\forall \hat{y} \in L,\ \forall \hat{z} \in M\ (i\hat{J}\hat{y} \mid \hat{z})_{H^2} = (E\Lambda\hat{y} \mid \Pi\hat{z})_{\tilde{G}} = (\Lambda\hat{y} \mid E^*\Pi\hat{z})_G.$$

Assume that $A_1 \in B(G, G_1)$, $B_1 \in B(\tilde{G}, G_2)$ *and put*

$$L_1 = \ker A_1\Lambda = \{\hat{y} \in \hat{L}\ :\ A_1\Lambda\hat{y} = 0\}, \tag{21}$$

$$M_1 = \ker B_1\Pi = \{\hat{z} \in \hat{M}\ :\ B_1\Pi\hat{z} = 0\}. \tag{22}$$

Then

i) $L_1 \supset M_1^*$ *if and only if*

$$A_1E^{-1}B_1^* = 0; \tag{23}$$

ii) $L_1 = M_1^*$ *if and only if* $\ker A_1 = \overline{R(E^{-1}B_1^*)}$.

Proof. Put $U_1 = A_1\Lambda,\ V_1 = B_1\Pi$. Theorem 4 implies

$$U_1(-i\hat{J} \downarrow \hat{H}_M)V_1^* = A_1\Lambda(-i\hat{J} \downarrow \hat{H}_M)\Pi^*B_1^* = A_1E^{-1}B_1^*,$$

in other words,

$$-iU_1\hat{J}V_1^* = A_1E^{-1}B_1^*. \tag{24}$$

i) This item follows from (24) and Lemma 2.

ii) Applying Theorem 2 and Corollary 2 and taking into account (24) we obtain

$$L_1 = M_1^* \Leftrightarrow \ker(A_1(\Lambda \downarrow \hat{H}_L)) = \hat{J}\overline{R(\Pi^* B_1^*)} \Leftrightarrow \ker A_1 = (\Lambda \downarrow \hat{H}_L)\hat{J}\overline{R(\Pi^* B_1^*)} \Leftrightarrow$$
$$\Leftrightarrow \ker A_1 = \overline{R((\Lambda_{\downarrow \hat{H}_L})\hat{J}\Pi^* B_1^*)} \Leftrightarrow \ker A_1 = \overline{R(\Lambda \hat{J}\Pi^* B_1^*)} \Leftrightarrow$$
$$\Leftrightarrow \ker A_1 = \overline{R(E^{-1}B_1^*)} = E^{-1}\overline{R(B_1^*)}.$$

(recall that $\Lambda \downarrow \hat{H}_L$,$\Pi \downarrow \hat{H}_M$ and E are homeomorphisms) □

Corollary 4. *Let, in addition to the conditions of Theorem 4,* $\dim \hat{H}_L < \infty$ *and equalities (11) hold.*

Then relations (21) and (22) are mutually adjoint if and only if $A_1 E^{-1} B_1^* = 0$.

Proof. Use Corollary 3 and (24). □

REFERENCES

1. R. Arens, *Operational calculus of linear relations*, Pacific J. Math., **11** (1961), 9–23.
2. E.A. Coddington, *Self-adjoint subspace extensions of nondensely defined symmetric operators*, Advances in Math., **14** (1974), 309–332.
3. A. Dijksma, H.V. de Snoo, *Self-adjoint extensions of symmetric subspaces*, Pacific J. Math., **54** (1974), №1, 71–100.
4. A.N. Kochubei, *On the extensions of nondensely defined symmetric operator*, Siber. Math. J., **18** (1977), №2, 314–320. (in Russian)
5. A.V. Kuzhel, S.A. Kuzhel, Regular extensions of hermitian operators. – VSP, Utrech, 1988.
6. V.A. Derkach, S. Hassi, M.M. Malamud, H.S.V. de Snoo, *Generalized resolvents and admissibility*, Methods of Functional Anal. and Topology, **6** (2000), №3, 24–55.
7. Yu.M. Arlinski, S. Hassi, Z. Sebestyen, H.S.V. de Snoo, *On the clas of extremal extensions of a nonegative operators*, Oper. Theory Adv. Appl., **127** (2001), 41–81.
8. V.M. Bruk, *On linear relations generated by differential expression and a Nevanlinna operator function*, J. of Math Physics, Analysis, Geometry, **7** (2011), №2, 114–140.
9. O.G. Storozh, *The correlation between two pairs of linear relations and dissipative extensions of some nondensely defined symmetric operators*, Carpathian Math. Publications, **1** (2009), №2, 207–213. (in Ukrainian)
10. V.E. Lyantse, *On some relations among closed operators*, Rep. Academy of Science USSR, **204** (1972), №3, 431–495. (in Russian)
11. A.N. Kolmogorov, C.V. Fomin, Elements of the theory of functions and functional analysis. – Nauka, Moscow, 1976. (in Russian)
12. T. Kato, Perturbation theory for linear operators. – Mir, Moscow, 1972. (in Russian)
13. V.E. Lyantse, O.G. Storozh, Methods of the theory of unbounded operators. – Naukova Dumka, Kyiv, 1983. (in Russian)

Department of Mathematical and Functional Analysis
Lviv Ivan Franko National University
aruy14@ukr.net
storog@ukr.net

Received 19.04.2012
Revised 14.06.2013