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DECOMPOSITION OF FINITELY GENERATED PROJECTIVE MODULES OVER BEZOUT RING

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It is shown that a commutative Bezout ring R of stable range 2 is an elementary divisor ring if and only if for each ideal I every finitely generated projective R/I-module is a direct sum of principal ideals generated by idempotents.

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Показано, что коммутативное кольцо Безу R стабильного ранга 2 есть кольцом элементарных делителей тогда и только тогда, когда для произвольного идеала I каждый конечнопорожденный проективный R/I-модуль является прямой сумой главных идеалов, порожденных идемпотентами.

1. Preliminaries. In this paper we consider the following question: is every commutative Bezout domain an elementary divisor ring? This question was posed by M. Henriksen in 1955 ([1]). In [2, 3] the following is showed: an elementary divisor ring is complete characterizing those rings whose finitely presented modules are a direct sum of cyclic modules.

All rings in this paper are commutative rings with identity; all modules are unitary. A ring R is a Bezout ring if every its finitely generated ideal is principal. Let A be an $m \times n$ matrix over R, denote by $\operatorname{Ker}(A)$ the set of columns $X \in R^n$ for each $AX = 0$ and let $\operatorname{Im}(A)$ be a submodule of R^n generated by the columns of A. We say that an R-module R is named by a matrix A if $M \cong \operatorname{Coker}(A) \cong R^m/\operatorname{Im}(A)$. A module M is said to be finitely presented if, for some finitely generated free module F and finitely generated submodule K of F, we have $M \cong F/K$. An important simple observation is: if there are finitely presented and $M \cong F/K$, where F is a finitely generated free module, then K is also finitely generated.

A matrix representation of $\phi \in \operatorname{Hom}(R^m, R^n)$ with respect to bases $e_1, e_2, \dots, e_m$ and $f_1, f_2, \dots, f_n$ of R^m and R^n respectively, is a matrix $A = (a_{ij})$ where $\phi(e_i) = a_{1i}f_1 + a_{2i}f_2 + \dots + a_{ni}f_n$ for $i \in \{1, 2, \dots m\}$. We say that A names (with respect to bases E, F, f, R^m, R^n respectively) if A is a matrix representation of a homomorphism ϕ with respect to the bases E, F and ϕ names ϕ.

Initially, we can afford to be careless about bases; if A names M with respect to bases E, F and names M' with respect to bases to bases E', F', then M is isomorphic to M'. Let E, F be bases of R^m, R^n respectively and let Q^{-1}, P are change of bases matrices which conver E

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to E' and F, F' respectively. If A names M with respect to F, F, then PAQ names M with respect to E', E'.

A module M over R is in canonical form if $M \cong R/A_1 \oplus \ldots \ominus R/A_n$, where $A_1 \ldots A_n$ are ideals of R and $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \neq R$. In [3] it is shown that if M has another canonical form $M \cong R/B_1 \oplus R/B_2 \oplus \ldots \oplus R/B_m$ then $n = m$ and each $A_i \cong B_i$.

Recall that a cyclic R-module of the form R/rR, where $r \in R$ is called a cyclically presented R-module. Using [5] one can show that a finitely presented cyclic module over a Bezout ring is cyclically presented.

Proposition 1 ([4]). *A commutative ring R is a Bezout ring if every finitely presented R-module can be named by a square matrix.*

Corollary 1. *Let R be a Bezout ring, then every finitely presented R-module which is a direct sum of cyclic modules, has canonical form.*

A ring R is Hermite if every matrix over R is equivalent to an upper triangular matrix. If every matrix is equivalent to a diagonal matrix (a_{ii}) in which a_{ii} divides $a_{i+1,i+1}$, then R is an elementary divisor ring. Notice that every elementary divisor ring is an Hermite and Bezout ring ([2]).

Theorem 1 ([2, 3]). *A ring R is an elementary divisor ring if and only if every finitely presented R-module is a direct sum of cyclic modules which are cyclically presented.*

A ring R has stable range $2(st.r(R) = 2)$ if for each $a, b, c \in R$ such that $aR+bR+cR = R$ there exists $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$. ([6])

Theorem 2 ([6]). *A commutative Bezout ring R is an Hermite ring if and only if $st.r(R) = 2$.*

2. Main result. Let R be an elementary divisor ring. Then by Theorem 1 we obtain that all finitely presented R-modules are direct sums of cyclically presented modules (i.e. modules of the form R/rR). If these cyclically presented modules are projective, rR must split in R_R, and thus rR is generated by an idempotent. By [4], for an Hermite ring R, if for every ideal I every finitely generated projective R/I-module is a direct sum of a cyclic then R is an elementary divisor ring.

Theorem 3. *A commutative Bezout ring R of stable range 2 is an elementary divisor ring if and only if for each ideal I every finitely generated projective R/I-module is a direct sum of a principal ideals generated by idempotents.*

Definition 1. A ring R is said to be an *ID-ring* if every idempotent matrix over R is diagonalized by a similarity transformation ([7]).

Theorem 4 ([7]). *If R is an elementary divisor ring then R is an ID-ring.*

Using this theorem we obtain the following result.

Theorem 5. *A commutative Bezout domain R is an elementary divisor ring if and only if for each ideal I, the ring R/I is an ID-ring.*

Proof. By [10] each nonzero finitely generated projective R/I module is a direct sum of a principal ideals generated by idempotents. By Theorem 3 and [4] we obtain that R is an elementary divisor ring. If R is an elementary divisor ring then the ring R/I is an elementary divisor ring for each ideal I. By Theorem 4, we obtain that R/I is an ID-ring. □

Definition 2. An element a of a ring R is called an *adequate element* if for any element $b \in R$ it can be represented as a product $a = r \cdot s$, where $rR + bR = R$ and for any element $s' \in R$ such that $sR \subset s'R \neq R$, we have $s'R + bR \neq R$ [3]. A commutative Bezout ring in which any nonzero element is adequate is called an adequate ring ([8]).

Theorem 6 ([8])**.** *Let a be an adequate element of a commutative Bezout ring. Then $\overline{0}$ is an adequate element of the factor-ring R/aR.*

Theorem 7 ([8])**.** *Let R be a commutative Bezout domain. If $\overline{0}$ is an adequate element of the factor-ring R/aR, then a is an adequate element of the domain R.*

As an obvious consequence we obtain the following result.

Theorem 8. *A commutative Bezout domain is an adequate domain if and only if for each nonzero element $a \in R$ the factor-ring R/aR is a ring in which $\overline{0}$ is an adequate element.*

Theorem 9 ([9])**.** *A commutative Bezout domain is adequate if and only if for each nonzero element $a \in R$ the factor-ring R/aR is a semiregular ring.*

The following theorem is due to Warfield ([11]) and is presented here for completeness.

Theorem 10 ([11])**.** *If R is a semiregular ring then every projective module is isomorphic to a direct sum of ideals of the form Re, where $e^2 = e$.*

By Theorems 3 and 10, it is clear that any adequate domain is an elementary divisor ring. It is well known and also follows from Albrechts theorem, that over a Bezout domain every projective module is free. Thus the following characterization sense only for Bezout rings which are not domains.

Definition 3. A ring R is an f-ring if every pure ideal of R is generated by idempotents.

Proposition 2 ([12])**.** *Let R be a Bezout ring. The following statements are equivalent:*

1) *every projective R-module is a direct sum of a finitely generated modules;*
2) *every projective R-module is isomorphic to a direct sum of ideals e_iR, where $e_i^2 = e_i$, $e_i \in R$;*
3) *R is an f-ring.*

Denote by $\operatorname{rad} R$ be an ideal of the nilpotent elements in a ring R.

By Theorem 3 and Proposition 2 we obtain the following theorem.

Theorem 11. *Let R be a commutative Bezout ring of a stable range 2 and for each ideal I the factor-ring R/I is an f-ring. Then R is an elementary divisor ring.*

Proof. We have that R is a ring in which every finitely generated projective R/I-module is a direct sum of a principal ideal generated by idempotents. By Theorem 3, R is an elementary divisor ring. □

Obvious examples of such ring are adequate domains, fractionally semilocal Bezout rings ([14]), fractionally regular Hermite rings ([15]), fractionally IF Bezout rings ([14]).

In contrast, we give an example of a commutative Bezout ring R of stable range 2 in which every finitely generated projective R-module is a direct sum of principal ideals generated by idempotents, but in this ring not every projective module is a direct sum of principal ideals generated by idempotents ([15]). Let $H = [0; \infty)$ be the half-line and the set $H^* = \beta H \backslash H$. Then in $C(H^*)$ there are such examples ([16]).

Definition 4. A ring R is called *semi-cancellative* if it follows from $R^n = A \oplus B = C \oplus D$ and $A \cong C$ then $B \cong D$ for all (finitely generated) R-modules A, B, C and D over R and $n \in \mathbb{N}$ ([17]).

Proposition 3 ([11])**.** *A commutative Bezout domain R is an elementary divisor ring if and only if for each ideal I, the factor-ring R/I is a semi-cancellative ring.*

Proof. By Theorem 5, the factor-ring R/I is an ID-ring. By [8], the factor-ring R/I is a semi-cancellative ring. □

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