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WIMAN TYPE INEQUALITIES FOR ENTIRE DIRICHLET SERIES WITH ARBITRARY EXPONENTS

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We prove analogues of the classical Wiman inequality for entire Dirichlet series $f(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}$ with arbitrary positive exponents (λ_n) such that $\sup\{\lambda_n\colon n\geq 0\}=+\infty$.

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Для целых рядов Дирихле $f(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}$ с положительными показателями (λ_n) удовлетворяющеми условию $\sup\{\lambda_n\colon n\geq 0\}=+\infty$ получены аналоги классического неравенства Вимана.

It is well known ([1, 2, 3]) that for every nonconstant entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and every $\varepsilon > 0$ there exists an exceptional set $E = E(f, \varepsilon)$ of finite logarithmic measure, i.e. $\int_E \frac{dr}{r} < +\infty$, such that the inequality (*Wiman's inequality*)

$$M_f(r) \leq \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}$$

holds for all $r \in [1, +\infty) \setminus E$, where $M_f(r) = \max\{|f(z)|\colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n\colon n \geq 0\}$. Some analogues of Wiman's inequality for entire Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} F_n e^{z\lambda_n}, \quad z \in \mathbb{C}, \tag{1}$$

where $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leq n \uparrow +\infty)$, were obtained in [4, 5]. In particular, in the paper by M. M. Sheremeta ([4]) we find the following *statement: if*

$$(\exists \Delta > 0)(\exists \rho \in [1/2; 1])(\exists D > 0)\colon\ |n(t) - \Delta t^\rho| \leq D\ (t \geq t_0), \tag{2}$$

where $n(t) = \sum_{\lambda_n \leq t} 1$ *is the counting function of the sequence* (λ_n) *then for every entire Dirichlet series of form* (1) *there exists a set* $E \subset [0; +\infty)$ *of finite Lebesgue measure on* $\mathbb{R}$ *such that for all* $x \in [0; +\infty) \setminus E$ *one has*

$$M(x, F) \leq \mu(x, F)\big(\ln \mu(x, F)\big)^{(2\rho-1)/2+\varepsilon}, \tag{3}$$

where

$$M(x, F) = \sup\{|F(x + iy)|\colon y \in \mathbb{R}\}, \quad \mu(x, F) = \max\{|F_n|e^{x\lambda_n}\colon n \geq 0\}.$$

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If $\lambda_n \equiv n$ $(n \geq 0)$, then $\Delta = D = \rho = 1$ in (2) and (3) implies Wiman's inequality. In particular, Theorem 2 ([5]) yields that for every increasing to $+\infty$ sequence (λ_n) satisfying (2) there exists an entire Dirichlet series of form (1) for which

$$\frac{M(x,F)}{\mu(x,F)}\big(\ln\mu(x,F)\big)^{-(2\rho-1)/2} \to +\infty$$

as $x \to +\infty$, i.e. $\varepsilon > 0$ in (3) cannot be replaced with $\varepsilon = 0$.

Let $\mathcal{D}$ be the class of all absolutely convergent Dirichlet series in $\mathbb{C}$ of form (1) with a sequence of the exponents (λ_n) such that $\lambda_n \geq 0$ $(n \geq 0)$ and

$$\sup\{\lambda_n\colon\ n \geq 0\} = +\infty,$$

i.e. the sequence of exponents of a function $F \in \mathcal{D}$ need not be monotone and has arbitrarily many cluster points (in particular, can be everywhere dense). It worth be noted that some asymptotic properties of functions $F \in \mathcal{D}$ were investigated in the papers [6]–[10]. In this paper we consider analogues of Wiman's inequality for the class $\mathcal{D}$.

For a function $F \in \mathcal{D}$ of form (1) denote by (μ_n) the sequence $(-\ln|F_n|)_{n\geq 0}$ arranged by decreasing.

Let L be the class of positive continuous functions increasing to $+\infty$ on $[0;+\infty)$ and L_1 the class of functions $\Phi \in L$ such that $\varphi(2t) = O(\varphi(t))$ $(t \to +\infty)$, where φ is the inverse function to Φ.

By $\ln$-meas $(E) = \int_{E\cap[1,+\infty)} d\ln r$ denote the logarithmic measure of a set $E \subset \mathbb{R}$.

Theorem 1. *Let $F \in \mathcal{D}$, $\Phi_1 \in L_1$, $\Phi_1(x) \overset{def}{=} \frac{1}{x}\ln\mu(x,F)$. If*

$$(\exists\alpha > 0)\colon\quad \int_{t_0}^{+\infty} t^{-2}(n_1(t))^{\alpha}dt < +\infty, \quad n_1(t) \overset{def}{=} \sum_{\mu_n \leq t} 1, \quad t_0 > 0, \tag{4}$$

then there exists a set $E \subset \mathbb{R}$ such that $\ln$-meas$(E) < +\infty$ and the relation

$$M(x,F) = o(\mu(x,F)\ln^{1/\alpha}\mu(x,F)) \tag{5}$$

holds as $x \to +\infty$ $(x \notin E)$.

In order to prove Theorem 1 we need the following lemma.

Lemma 1 ([10])**.** *Let $F \in \mathcal{D}$ such that $\Phi_1 \in L_1$, and $v(t)$ be a nonnegative function on $[0,+\infty)$ for which $v(t) > 0$ for $t > t_0$ and $\int_0^{+\infty} v(t)dt < +\infty$. If $\ln n = o(\ln|a_n|)$ $(n \to +\infty)$, then there exists a function $c_1(t) \uparrow +\infty$ $(t \to +\infty)$ such that for all $n \geq 0$ and $x > 0$ $(x \notin E, \ln-\text{meas}(E) < +\infty)$ one has*

$$|a_n|e^{x\lambda_n} \leq \mu(x,F)\exp\Big\{-x\int_{\mu_\nu}^{\mu_n}(\mu_n - t)\frac{c_1(t)}{\varphi(t)}v(4t)dt\Big\},$$

where $\mu_n = -\ln|a_n|$, $\nu = \nu(x,F) = \max\{n\colon |a_n|e^{x\lambda_n} = \mu(x,F)\}$ is the central index of series (1).

Proof of Theorem 1. With no loss of generality we may and do assume that $\lambda_0 = 0 = \mu_0 \leq \mu_n = -\ln|a_n| \nearrow +\infty$ $(1 \leq n \to +\infty)$. It is easy to see that condition (4) implies $(n_1(t))^{\alpha} = o(t)$ $(t \to +\infty)$ (in particular $n^{\alpha} \leq \mu_n$ as $n \geq n_0$), and thus

$$\sum_{n=1}^{+\infty}\mu_n^{-2/\alpha} < +\infty, \quad \int_{t_0}^{\infty} t^{-2}N(t)dt < +\infty, \quad N(t) \overset{def}{=} \int_{t_0}^{t} u^{-1}(n_1(u))^{\alpha}du,\ t_0 > 0.$$

By Lemma 1 with the function $v(t) = 16t^{-2}(n_1(t))^\alpha$ $(t \ge t_0)$, $v(t) = 0$ $(t \in [0; t_0))$, as $n = 0$ we obtain for all $x > 0$ outside of some exceptional set E_1 of finite logarithmic measure

$$\ln \mu(x, F) \ge x \int_{t_0}^{\mu_\nu} \frac{c_1(t)(n_1(4t))^\alpha}{t\varphi(t)} dt.$$

Hence, using the inequality $x \ge \frac{1}{2}\varphi(\mu_\nu)$ $(x \ge x_0)$, $\nu = \nu(x - 0, F)$ (inequality (10) from [10]), we get the following inequalities

$$\ln \mu(x, F) \ge x \int_{3\mu_\nu/4}^{\mu_\nu} \frac{c_1(t)(n_1(4t))^\alpha}{t\varphi(t)} dt \ge (n_1(3\mu_\nu))^\alpha c_2(\mu_\nu), \quad c_2(t) \overset{def}{=} \frac{1}{2} c_1\left(\frac{3}{4}t\right) \ln \frac{4}{3} \tag{6}$$

which hold for all $x \in [x_1, +\infty \setminus E_1$, where $x_1 \ge x_0$. Let $\sigma(x) \overset{def}{=} \sum_{\mu_n > 3\mu_\nu} |a_n| e^{x\lambda_n}$. Then Lemma 1 with the function $v(t) = 16t^{-2}N(t)$ $(t \ge t_0)$, $v(t) = 0$ $(t \in [0; t_0))$, implies that

$$\sigma(x)/\mu(x, F) \le \sum_{\mu_n > 3\mu_\nu} \mu_n^{-2/\alpha} \exp\left(\max\{\psi(y) \colon y \ge 3\mu_\nu\}\right), \tag{7}$$

as $x \to +\infty$ outside some exceptional set E_2 of finite logarithmic measure, where $\psi(y) = -xc_3(\mu_\nu) \int_{\mu_\nu}^{y} \frac{y-t}{t^2} \cdot \frac{N(4t)}{\varphi(t)} dt + \frac{2}{\alpha} \ln y$, and c_3 is the function c_1 from Lemma 1 associated with the function $v(t) = 16t^{-2}N(t)$.

Since $\psi'(y) = -xc_3(\mu_\nu) \int_{\mu_\nu}^{y} \frac{N(4t)}{t^2\varphi(t)} dt + \frac{2}{\alpha y}$ decreases on $[3\mu_\nu, +\infty)$, for all $y \ge 3\mu_\nu$ and for all large enough ν using the monotonicity of the function $t/\varphi(t)$ and the inequality $x \ge \frac{1}{2}\varphi(\mu_\nu)$ $(x \ge x_0)$, $\nu = \nu(x - 0, F)$, we obtain

$$\psi'(y) \le \psi'(3\mu_\nu) = -xc_3(\mu_\nu) \int_{\mu_\nu}^{3\mu_\nu} \frac{N(4t)}{t^2\varphi(t)} dt + \frac{2}{3\alpha\mu_\nu} \le$$
$$\le \frac{1}{2} c_3(\mu_\nu)\mu_\nu \int_{\mu_\nu}^{3\mu_\nu} \frac{N(4t)}{t^3} dt + \frac{2}{3\alpha\mu_\nu} \le -\frac{1}{27} c_3(\mu_\nu) \frac{N(4\mu_\nu)}{\mu_\nu} + \frac{2}{3\alpha\mu_\nu} < 0.$$

Therefore, the function $\psi(y)$ decreases on $[3\mu_\nu, +\infty)$ for $\nu \ge \nu_0$ and thus

$$\max\{\psi(y) \colon y \ge 3\mu_\nu\} = \psi(3\mu_\nu) \le -xc_3(\mu_\nu) \frac{\mu_\nu}{\varphi(\mu_\nu)} N(4\mu_\nu) \int_{\mu_\nu}^{2\mu_\nu} \frac{3\mu_\nu - t}{t^3} dt +$$
$$+ \frac{2}{\alpha} \ln(3\mu_\nu) \le -\frac{1}{16} c_3(\mu_\nu) N(4\mu_\nu) + \frac{2}{\alpha} \ln 3\mu_\nu,$$

as $\nu \to +\infty$. Hence, the relation $N(t)/\ln t \to +\infty$ $(t \to +\infty)$ (the function $N(t)$ is logarithmically convex) implies $\max\{\psi(y) \colon y \ge 3\mu_\nu\} \le -\frac{1}{17} c_3(\mu_\nu) N(4\mu_\nu)$ as $\nu \ge \nu_1$ for some $\nu_1 \ge \nu_0$. Therefore, from (7) passing $x \to +\infty$ we deduce

$$\sigma(x)/\mu(x, F) = o\Big(\exp\big\{-0.05 \cdot c_3(\mu_\nu) N(4\mu_\nu)\big\}\Big) \tag{8}$$

outside a set $E_1 \cup E_2$ of finite logarithmic measure. Applying relations (8) and (6) we complete the proof of Theorem 1. □

The following assertion shows that relation (5) under condition (4) in general can not be improved.

Theorem 2. *For every $\alpha > 0$ there exists a function $F \in \mathcal{D}$ such that condition* (4) *and the relation*

$$(\forall \varepsilon > 0) \colon \quad \int_{t_0}^{+\infty} t^{-2}(n_1(t))^{\alpha+\varepsilon} dt = +\infty \tag{9}$$

hold and

$$\big(\forall\ \varepsilon\in(0;1/\alpha)\big):\quad \frac{F(x)}{\mu(x,F)(\ln\mu(x,F))^{1/\alpha-\varepsilon}}\to+\infty$$

as $x\to+\infty$.

Proof of Theorem 2. Let $\lambda_0=0,\ n_0=0$, and $\lambda_k=e^k$, $n_k=\big[\frac{1}{k}(e^k\mathrm{ln}^{-2}(k+1))^{1/\alpha}\big]+1\in\mathbb{N}$ $(k\geq 1)$. Consider first an entire Dirichlet series

$$f(z)=\sum_{k=1}^{+\infty}a_ke^{z\lambda_k},\quad a_k=\exp\{-\lambda_k\ln\lambda_k\}.$$

Taking into account that $\varkappa_k\overset{def}{=}(\ln a_{k-1}-\ln a_k)/(\lambda_k-\lambda_{k-1})=k+\frac{1}{e-1}\uparrow+\infty\ (1\leq k\uparrow+\infty)$, by [11, p.19] we obtain

$$\mu(x,f)=\exp\{-\lambda_k\ln\lambda_k+x\lambda_k\}\quad(x\in[\varkappa_k,\varkappa_{k+1}]).$$

We note now that $\varkappa_{k+1}-\varkappa_k=1$ and

$$\frac{e^k}{e-1}\leq\ln\mu(x,f)\leq\frac{e}{e-1}e^k,\quad x-\frac{e}{e-1}\leq k\leq x-\frac{1}{e-1}$$

for $x\in[\varkappa_k,\varkappa_{k+1}]$. Thus

$$n_k+1\geq d_0\frac{(\ln\mu(x,f))^{1/\alpha}}{\ln_2\mu(x,f)\ln_3^{2/\alpha}\mu(x,f)},\tag{10}$$

for $x\in[\varkappa_k,\varkappa_{k+1}]$, where $d_0>0$ is some constant, $\ln_k t\overset{def}{=}\ln\ln_{k-1}t\ (k\geq 2),\ \ln_1 t=\ln t$. We set

$$\lambda_k^{(s)}=\lambda_k+\frac{s}{n_k}\ln(3/2)\quad(1\leq s\leq n_k,\ k\geq 1).$$

Then $\lambda_k<\lambda_k^{(s)}<\lambda_k^{(s+1)}<\lambda_k+\ln(3/2)$ $(1\leq s\leq n_k-1)$, therefore

$$\Delta_k^{(s)}\overset{def}{=}\frac{1}{2}\exp\{\varkappa_{k+1}(\lambda_k-\lambda_k^{(s)})\}>\frac{1}{3}\exp\{\varkappa_k(\lambda_k-\lambda_k^{(s)})\}\overset{def}{=}\delta_k^{(s)}\quad(1\leq s\leq n_k,\ k\geq 1).$$

We put $a_k^{(s)}=(\Delta_k^{(s)}+\delta_k^{(s)})a_k/2$ and consider the Dirichlet series of the form

$$F(z)=\sum_{k=1}^{+\infty}\Big(a_ke^{z\lambda_k}+\sum_{s=1}^{n_k}a_k^{(s)}e^{z\lambda_k^{(s)}}\Big).$$

It is easy to verify that $F\in\mathcal{D}$ and

$$\mu(x,F)/3\leq a_k^{(s)}\exp\{x\lambda_k^{(s)}\}\leq\mu(x,F)/2,\quad\mu(x,F)=\mu(x,f)\tag{11}$$

for $x\in[\varkappa_k;\varkappa_{k+1}]$, since $a_k^{(s)}\in(\delta_k^{(s)}a_k;\Delta_k^{(s)}a_k)$. Indeed, for $x\in[\varkappa_k;\varkappa_{k+1}]$ we have

$$a_k^{(s)}e^{x\lambda_k^{(s)}}\leq\Delta_k^{(s)}a_ke^{x\lambda_k^{(s)}}=\frac{1}{2}\cdot a_k\exp\{x\lambda_k^{(s)}+\varkappa_{k+1}(\lambda_k-\lambda_k^{(s)})\}\leq\frac{1}{2}\cdot a_ke^{x\lambda_k}=\frac{\mu(x,F)}{2},$$

and on the other hand

$$a_k^{(s)}e^{x\lambda_k^{(s)}}\geq\delta_k^{(s)}a_ke^{x\lambda_k^{(s)}}=\frac{1}{3}\cdot a_k\exp\{x\lambda_k^{(s)}+\varkappa_k(\lambda_k-\lambda_k^{(s)})\}\geq\frac{1}{3}\cdot a_ke^{x\lambda_k}=\frac{\mu(x,F)}{3}.$$

In addition, for $1\leq s\leq n_k$ and $x>0$ we have

$$a_k^{(s)}e^{x\lambda_k^{(s)}}\leq\Delta_k^{(s)}\Big(\frac{3}{2}\Big)^xa_ke^{x\lambda_k}\leq\frac{1}{2}\Big(\frac{3}{2}\Big)^xa_ke^{x\lambda_k}$$

and thus

$$a_ke^{x\lambda_k}+\sum_{s=1}^{n_k}a_k^{(s)}e^{x\lambda_k^{(s)}}\leq\Big(1+\frac{1}{2}n_k\Big(\frac{3}{2}\Big)^x\Big)a_ke^{x\lambda_k},$$

which easily yields that $F\in\mathcal{D}$.

Let $n_1(t)$ be the counting function of the sequence $\{\mu_k^*\} = \{\mu_k\} \cup \{\mu_k^{(s)}\}$, where $\mu_k \overset{def}{=} -\ln a_k = ke^k$, $\mu_k^{(s)} \overset{def}{=} -\ln a_k^{(s)}$. Direct calculations verify that conditions (4) and (9) are satisfied. Indeed, $\mu_k^{(s)} = (1+o(1))\mu_k$ as $k \to +\infty$ $(1 \le s \le n_k)$, thus for all $q = \alpha + \varepsilon$, $\varepsilon > 0$, we get

$$\int_{2\mu_k}^{4\mu_k} \frac{(n_1(t))^q}{t^2} dt \ge \frac{(n_1(2\mu_k))^q}{4\mu_k} \ge \frac{1}{4\mu_k} \bigg(\sum_{s=1}^{k} (n_s+1) \bigg)^q \ge \frac{(n_k)^q}{4\mu_k} \to +\infty \ (k \to +\infty),$$

hence condition (9) holds. Similarly, putting $b_k = (\mu_k + \mu_{k-1})/2$ as $k \to +\infty$ we obtain

$$\int_{b_k}^{b_{k+1}} \frac{(n_1(t))^\alpha}{t^2} dt \le d \frac{(n_1(b_{k+1}))^\alpha}{\mu_k} = d \frac{1}{\mu_k} \bigg(\sum_{s=1}^{k} (n_s+1) \bigg)^\alpha \le d \frac{1}{\mu_k} \big(k(n_k+1) \big)^\alpha = \frac{(d+o(1))}{k \ln^2(k+1)},$$

where $d > 0$ is some constant, hence condition (4) holds.

Using conditions (10) and (11) we complete the proof of Theorem 2

$$F(x) \ge (n_k+1)\mu(x,F)/3 \ge \frac{d_0 \mu(x,F)(\ln \mu(x,F))^{1/\alpha}}{3 \ln_2 \mu(x,F) \ln_3^{2/\alpha} \mu(x,F)}. \qquad \square$$

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