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INFINITE DIMENSIONAL LINEAR GROUPS WITH A SPACIOUS FAMILY OF G-INVARIANT SUBSPACES

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Let F be a field, A be a vector space over F, $GL(F, A)$ be the group of all automorphisms of the vector space A. If $B \leq A$ then denote by $\mathrm{Core}_G(B)$ the largest G-invariant subspace of B. A subspace B is called almost G-invariant if $\dim_F(B/\mathrm{Core}_G(B))$ is finite. In this paper we described the case where every subspace of A is almost G-invariant.

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Пусть F — поле, A — векторное пространство над F, $GL(F, A)$ — группа всех автоморфизмов векторного пространства A. Если $B \leq A$, тогда обозначим через $\mathrm{Core}_G(B)$ наибольшее G-инвариантное подпространство B. Подпространство B называется почти G-инвариантным, если $\dim_F(B/\mathrm{Core}_G(B))$ конечна. В этой работе описан случай, когда каждое подпространство пространства A является почти G-инвариантным.

1. Introduction. Let F be a field, A a vector space over F and $GL(F, A)$ a group of all F-automorphisms of A. If G is a subgroup of $GL(F, A)$ then, as usual, a subspace B of A is called G-invariant if $bx \in B$ for every $b \in B$ and every $x \in G$. If A has finite dimension then the group G is called finite dimensional. The theory of finite dimensional linear groups is one of the most developed group-theoretical branches (see, for example, the book [8]). However, in the case where A has infinite dimension over F, the situation becomes totally different. This case is much more complicated and its consideration requires some additional restrictions. Imposing classical finiteness conditions is one of the most efficient and natural approaches here. The study of infinite dimensional linear groups satisfying some finiteness conditions proved to be very promising. Many valuable results have been obtained in this way (see, for example, the surveys [4, 7]).

Recently another approach in studying of infinite dimensional groups appeared. This approach is based on the notion of invariance of action of a group G. We have the following simple fact: if every subspace of A is G-invariant then G must be abelian. Consequently, the study of infinite dimensional linear groups having very big family of G-invariant subspaces could be fruitful. This has been shown in the papers [1, 2, 5, 6]. In the present paper this approach continues to be implemented.

If B is a subspace of A then the sum of arbitrary family of G-invariant subspaces of B is a G-invariant subspace. It follows that B has the largest G-invariant subspace $\mathrm{Core}_G(B)$, which is called the *G-core* of B. We observe that the *G-core* of B can be zero.

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A subspace B is called almost G-invariant, if $\dim_F(B/\operatorname{Core}_G(B))$ is finite.

In the present paper we consider linear groups G for which every subspace of A is almost G-invariant.

2. Preliminary results. Let R be a ring and A be an R-module. Denote by $\operatorname{Soc}_R(A)$ the submodule generated by all simple R-submodules of A (if A has simple R-submodules). If A does not include simple R-submodules then put $\operatorname{Soc}_R(A) = \langle 0 \rangle$.

We begin our study with the case of a non-periodic group G.

Lemma 1. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is almost G-invariant. If g is an element of G having infinite order then A is periodic as an $F\langle g\rangle$-module.*

Proof. Suppose that the result is false. Then there exists an element $a \in A$ such that $\operatorname{Ann}_{F\langle g\rangle}(a) = \langle 0 \rangle$. It follows that

$$aF\langle g\rangle \cong F\langle g\rangle = \bigoplus_{n\in Z} Fg^n.$$

Put $x = \langle g^2\rangle$ and $D = aF\langle x\rangle$. Then $A = D \oplus Dg$. We remark that $\dim_F(D)$ is infinite. Put $C = \operatorname{Core}_{\langle g\rangle}(D)$. Then $\dim_F(D/C)$ is finite, which implies that $\dim_F(C)$ is infinite. In particular $C \neq \{0\}$. If d is an arbitrary non-zero element of C then $dg \in C$. On the other hand, since $d \in D$, $dg \in Dg$, that is $dg \in D \cap Dg = \{0\}$. Then $d = (dg)g^{-1} = 0$, and we obtain a contradiction. This contradiction shows that A is a periodic $F\langle g\rangle$-module. □

Corollary 1. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is almost G-invariant. If g is an arbitrary element of G then $aF\langle g\rangle$ has finite F-dimension for every element $a \in A$.*

Proof. Indeed, if g has infinite order then Lemma 1 shows that A is periodic as an $F\langle g\rangle$-module. Then $\operatorname{Ann}_{F\langle g\rangle}(a) \neq < 0 >$ for every element $a \in A$. We have

$$aF\langle g\rangle \cong F\langle g\rangle / \operatorname*{Ann}_{F\langle g\rangle}(a).$$

We recall that F-dimension of $F\langle g\rangle/I$ is finite for each non-zero ideal I of $F\langle g\rangle$. Hence $\dim_F(aF\langle g\rangle)$ is finite.

If g has finite order k then $aF\langle g\rangle \leq aF + agF + ag^2F + \ldots + ag^{k-1}F$ and again $\dim_F(aF\langle g\rangle)$ is finite. □

Lemma 2. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is almost G-invariant. Let g be an arbitrary element of G. If C is a subspace of A such that $C \cap \operatorname{Soc}_{F\langle g\rangle}(A) = \{0\}$ then $\dim_F(C)$ is finite.*

Proof. Put $E = \operatorname{Core}_{\langle g\rangle}(C)$, then $\dim_F(C/E)$ is finite. Suppose that $E \neq \{0\}$ and choose in E a non-zero element a. Since E is $\langle g\rangle$-invariant, $aF\langle g\rangle \leq E$. By Corollary 1 $\dim_F(aF\langle g\rangle)$ is finite. Then $aF\langle g\rangle$ includes a minimal $F\langle g\rangle$-submodule. It follows that

$$aF\langle g\rangle \cap \operatorname*{Soc}_{F\langle g\rangle}(A) \neq \langle 0 \rangle.$$

However

$$\langle 0 \rangle \neq aF\langle g\rangle \cap \operatorname*{Soc}_{F\langle g\rangle}(A) \leq E \cap \operatorname*{Soc}_{F\langle g\rangle}(A) \leq C \cap \operatorname*{Soc}_{F\langle g\rangle}(A),$$

and we obtain a contradiction with the choice of C. This contradiction shows that $E = \langle 0 \rangle$, so that $\dim_F(C)$ is finite. □

Lemma 3. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is almost G-invariant. Let g be an arbitrary element of G. Then A includes an $F\langle g\rangle$-submodule C satisfying the following conditions:*

(i) $\dim_F(A/C)$ *is finite;*

(ii) every subspace of C is $\langle g\rangle$-invariant.

Proof. If $0 \neq a \in A$ then Corollary 1 shows that $\dim_F(aF\langle g\rangle)$ is finite. Then $aF\langle g\rangle$ includes a minimal $F\langle g\rangle$-submodule M. An inclusion $M \leq \mathrm{Soc}_{F\langle g\rangle}(A)$ yields that $\mathrm{Soc}_{F\langle g\rangle}(A) \neq \langle 0\rangle$. Put $S = \mathrm{Soc}_{F\langle g\rangle}(A)$. We have $A = S \oplus D$ for some F-subspace D of A. Using now Lemma 2 we obtain that $\dim_F(D)$ is finite. In turn out, it follows that S has finite codimension.

For $F\langle g\rangle$-submodule S we have a direct decomposition $S = \bigoplus_{\lambda\in\Lambda} A_\lambda$, where A_λ is a simple $F\langle g\rangle$-submodule for every $\lambda \in \Lambda$. Put $M = \{\lambda \in \Lambda \mid \dim_F(A_\lambda) > 1\}$ and $B = \bigoplus_{\lambda\in M} A_\lambda$. Since every subspace of A is almost G-invariant, it is likewise almost $\langle g\rangle$-invariant. An application of Lemma 2.1 of paper [5] shows that $\dim_F(B)$ is finite.

Put $\Delta = \Lambda \setminus M$. Since $\dim_F(A_\lambda) = 1$ for each $\lambda \in \Delta$, $A_\lambda = a_\lambda F$ for some elements $a_\lambda \in A_\lambda$, $\lambda \in \Delta$. It follows that $a_\lambda g = \alpha_\lambda a_\lambda$ for some elements $\alpha_\lambda \in F$, $\lambda \in \Delta$. With the help of arguments from the proof of Proposition 2.2 of paper [5], we obtain that there exists a subset $\Gamma \subseteq \Delta$ such that $\alpha_\lambda = \alpha_\mu = \alpha$ for all $\lambda, \mu \in \Gamma$, and a subset $\Delta \setminus \Gamma$ is finite. It follows that a subspace $C = \bigoplus_{\lambda\in\Gamma} A_\lambda$ has finite codimension. If $a \in C$, then $a = \sum_{1\leq j\leq n} \beta_{\lambda(j)} a_{\lambda(j)}$, where $\beta_{\lambda(j)} \in F$, $\lambda(j) \in \Gamma$, $1 \leq j \leq n$. We have

$$ag = \sum_{1\leq j\leq n} (\beta_{\lambda(j)} a_{\lambda(j)})g = \sum_{1\leq j\leq n} \beta_{\lambda(j)}(a_{\lambda(j)} g) = \sum_{1\leq j\leq n} \beta_{\lambda(j)}(\alpha a_{\lambda(j)}) = \sum_{1\leq j\leq n} \alpha\beta_{\lambda(j)} a_{\lambda(j)} =$$
$$= \alpha \sum_{1\leq j\leq n} \beta_{\lambda(j)} a_{\lambda(j)} = \alpha a.$$

This equation shows that every subspace of C is $\langle g\rangle$-invariant. □

3. Proof of the main results.

Theorem 1. *Let G be a subgroup of $GL(F, A)$. Suppose that every subspace of A is almost G-invariant. Then A includes an FG-submodule C satisfying the following conditions:*

(i) $\dim_F(A/C)$ *is finite;*

(ii) every subspace of C is G-invariant.

Proof. If every subspace of A is G-invariant then all is proved. Suppose now that there are elements $a_1 \in A$ and $g_1 \in G$ such that $a_1 g_1 \notin a_1 F$. Put $d_1 = a_1(g_1 - 1)$. It readily follows that $\dim_F(a_1 F + d_1 F) = 2$. By Lemma 3, A includes an $F\langle g_1\rangle$-submodule D_1 such that $\dim_F(A/D_1)$ is finite and every subspace of D_1 is $\langle g_1\rangle$-invariant. If $d_1 \notin D_1$, we put $E_1 = D_1$. If $d_1 \in D_1$ then we may find a complement E_1 to $d_1 F$ in D_1, $D_1 = d_1 F \oplus E_1$ say, and $d_1 \notin E_1$. In both cases, E_1 is an $F\langle g_1\rangle$-submodule such that every subspace of E_1 is $\langle g_1\rangle$-invariant, $\dim_F(A/E_1)$ is finite and $(a_1 F + d_1 F) \cap E_1 = \{0\}$. Put $L_1 = \mathrm{Core}_G(E_1)$. By our hypothesis, L_1 is an FG-submodule of A such that $\dim_F(A/L_1)$ is finite and every subspace of L_1 is $\langle g_1\rangle$-invariant.

If $ag \in aF$ for every elements $g \in G$ and $a \in L_1$ then it suffices to define $C = L_1$. Therefore, we suppose that there are elements $g_2 \in G$ and $a_2 \in L_1$ such that $a_2(g_2 - 1) = d_2 \notin a_2 F$. It readily follows that $\dim_F(a_2 F + d_2 F) = 2$. Using the above arguments, we construct an FG-submodule L_2 of L_1 such that $\dim_F(A/L_2)$ is finite, $(a_2F+d_2F)\cap L_2=\{0\}$ and every subspace of L_2 is $\langle g_2\rangle$-invariant. By the choice of L_1, every subspace of L_2 is also $\langle g_1\rangle$-invariant.

We continue proceeding in this way. If every subspace of L_2 is G-invariant then it suffices to define $C=L_2$. Otherwise, we iterate the same process. Here there appear the following two possibilities:

(1) this process will finish after finitely many steps;

and

(2) this process is infinite.

In the first case we obtain an FG-submodule C such that $\dim_F(A/C)$ is finite and every subspace of C is G-invariant.

Consider the second case. Then we can choose in A an infinite subset of elements $\{a_n \mid n\in\mathbb{N}\}$ and in a group G an infinite subset $\{g_n \mid n\in\mathbb{N}\}$ of elements such that the following conditions hold:

(a) $a_nF+d_nF=a_nF\oplus d_nF$, where $d_n=a_n(g_n-1)$, $n\in\mathbb{N}$;

(b) $(a_nF\oplus d_nF)\cap(\bigoplus_{1\le k\le n-1}(a_kF\oplus d_kF))=\{0\}$, $n\in\mathbb{N}$;

(c) $a_ng_k\in a_nF$, $d_ng_k\in d_nF$ whenever $k<n$, $n,k\in\mathbb{N}$.

Put $B=\bigoplus_{j\in\mathbb{N}}a_jF$, $D=\bigoplus_{j\in\mathbb{N}}d_jF$. Then $B\cap D=\langle 0\rangle$. Let $Z=\mathrm{Core}_G(B)$. Then $\dim_F(B/Z)$ is finite, in particular, Z is non-zero. The inclusion $Z\subseteq B$ implies that $Z\cap D=\langle 0\rangle$. Let a be a non-zero element of Z. Then

$$a=\alpha_1a_{k(1)}+\alpha_2a_{k(2)}+\ldots+\alpha_ta_{k(t)}$$

for some positive integers $k(1)<k(2)<\ldots<k(t)$, and non-zero elements $\alpha_1,\alpha_2,\ldots,\alpha_t\in F$. We have now

$$\begin{aligned}a(g_{k(1)}-1)&=(\alpha_1a_{k(1)}+\alpha_2a_{k(2)}+\ldots+\alpha_ta_{k(t)})(g_{k(1)}-1)=\\&=\alpha_1a_{k(1)}(g_{k(1)}-1)+\alpha_2a_{k(2)}(g_{k(1)}-1)+\ldots+\alpha_ta_{k(t)}(g_{k(1)}-1).\end{aligned}$$

By (c) if $j>1$, then $a_{k(j)}g_{k(1)}=v_ja_{k(j)}$ for some $v_j\in F$, so that $a_{k(j)}(g_{k(1)}-1)=v_ja_{k(j)}-a_{k(j)}=(v_j-1)a_{k(j)}$ and $\alpha_ja_{k(j)}(g_{k(1)}-1)=\alpha_j(v_j-1)a_{k(j)}$, $2\le j\le t$. Thus

$$a(g_{k(1)}-1)=\alpha_1d_{k(1)}+\alpha_2(v_2-1)a_{k(2)}+\ldots+\alpha_t(v_t-1)a_{k(t)}.$$

Since $\alpha_1\neq 0$, $\alpha_1d_{k(1)}$ is a non-zero element of D. On the other hand, $\alpha_2(v_2-1)a_{k(2)}+\ldots+\alpha_t(v_t-1)a_{k(t)}\in B$, so that

$$\alpha_1d_{k(1)}+\alpha_2(v_2-1)a_{k(2)}+\ldots+\alpha_t(v_t-1)a_{k(t)}\notin B\ge Z.$$

Hence is the case (2) we obtain a contradiction, which proves the result. □

Theorem 2. *Let G be a subgroup of $GL(F,A)$. Suppose that every subspace of A is almost G-invariant.*

(i) if $\mathrm{char}(F)=0$ then G includes a normal abelian torsion-free subgroup Z such that G/Z is isomorphic to subgroup of $L\times V$, where V is a subgroup of multiplicative group of F and L is a subgroup of $GL_n(F)$ for some positive integer n.

(ii) if $\mathrm{char}(F)=p$ is a prime then G includes a normal abelian elementary p-subgroup Z such that G/Z is isomorphic to a subgroup of $L\times V$, where V is a subgroup of a multiplicative group of F, and L is a subgroup of $GL_n(F)$ for some positive integer n.

Proof. Theorem 1 shows that A includes an FG-submodule C such that $\dim_F(A/C)$ is finite and every subspace of C is G-invariant. Put $K = C_G(C)$. By Lemma 3.4 of paper [5] $V = G/K$ is isomorphic to a subgroup of a multiplicative group of a field F. Put now $T = C_G(A/C)$. Since A/C has finite dimension, say n, $L = G/T$ is isomorphic to a subgroup of finite dimensional linear group $GL_n(F)$. Finally, let $Z = T \cap K$, then Z stabilizes the series of $\{0\} \leq C \leq A$. By a classical result due to Kaluznin (see, for example, [3, Theorem 1.C.1 and Proposition 1.C.3]) Z is either an elementary abelian p-subgroup if $\mathrm{char}(F) = p > 0$, or a torsion-free abelian subgroup if $\mathrm{char}(F) = 0$. Finally, by Remak's Theorem, we obtain a new embedding of G/Z in the direct product $G/K \times G/T = V \times L$.

□

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