УДК 517.925.7

A. Z. Mokhonko, A. A. Mokhonko

ON GROWTH ORDER OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A NEIGHBORHOOD OF A BRANCH POINT

A. Z. Mokhonko, A. A. Mokhonko. *On growth order of solutions of differential equations in a neighborhood of a branch point*, Mat. Stud. **40** (2013), 53–65.

Let M_k be the set of k-valued meromorphic in $G=\{z\colon r_0\leqslant |z|\}$ functions with a branch point of order $k-1$ at ∞; let E_* be a set of circles with finite sum of radii. Denote $M_*(r,f)=\max|f(z)|,\ z\in\{te^{i\theta}\colon 0\leqslant\theta\leqslant 2k\pi,\ r_0\leqslant t\leqslant r\}\setminus E_*,\ f\in M_k$; $m(r,f)=\frac{1}{2\pi k}\int_0^{2\pi k}\ln^+|f(re^{i\theta})|d\theta$. If $f\in M_k$ is a solution of the equation $P(z,f,f')=0$ and P is a polynomial in all variables then either $|f(re^{i\theta})|<r^{\nu},\ re^{i\theta}\in G\setminus E_*,\ \nu>0$ or $m(r,f)$ has growth order $\rho\geqslant\frac{1}{2k}$, and the following equality holds $\ln M_*(r,f)=(c+o(1))r^{\rho},\ c\neq 0,\ r\to+\infty$.

А. З. Мохонько, А. А. Мохонько. *О порядке роста решений дифференциальных уравнений в окрестности точки ветвления* // Мат. Студії. – 2013. – Т.40, №1. – С.53–65.

Пусть M_k — множество k-значных мероморфных в $G=\{z\colon r_0\leqslant|z|\}$ функций с точкой ветвления порядка $k-1$ в ∞; пусть E_* — некоторое множество кругов с конечной суммой радиусов. Обозначим $M_*(r,f)=\max|f(z)|,\ z\in\{te^{i\theta}\colon 0\leqslant\theta\leqslant 2k\pi,\ r_0\leqslant t\leqslant r\}\setminus E_*,\ f\in M_k$; $m(r,f)=\frac{1}{2\pi k}\int_0^{2\pi k}\ln^+|f(re^{i\theta})|d\theta$. Если $f\in M_k$ — решение уравнения $P(z,f,f')=0$, P — многочлен по всем переменным, то либо $|f(re^{i\theta})|<r^{\nu},\ re^{i\theta}\in G\setminus E_*,\ \nu>0$, либо $m(r,f)$ имеет порядок роста $\rho\geqslant\frac{1}{2k}$ и выполняется равенство $\ln M_*(r,f)=(c+o(1))r^{\rho}$, $c\neq 0$, $r\to+\infty$.

Differential equations $P(z,f,f',f'',f''')=0$ (where P is a polynomial in all variables) may have entire transcendental solutions of zero growth order (see [1, p. 224–226]). V. V. Zymoglyad showed in [2] that differential equations $P(z,f,f',f'')=0$, P is a polynomial in all variables, do not have entire transcendental solutions of zero growth order. In this paper we obtain asymptotic estimates for meromorphic solutions of first order algebraic equations. We show that this entails, in particular, the fact that entire transcendental solutions of $P(z,f,f')=0$, P is a polynomial in all variables, have the growth order ρ, $\frac{1}{2}\leqslant\rho<+\infty$.

Recall some definitions and properties.

Let (g,e_{z_0}), $g=\{z\colon |z-z_0|<\delta_{z_0}\}$ be a regular element or an element of the form $e_{z_0}(z)=\sum_{j=-s}^{+\infty}a_j(z-z_0)^j$, $z\in g=\{z\colon|z-z_0|<\delta_{z_0}\}$, $s\in\mathbb{N}$. Suppose the element (g,e_{z_0}) can be meromorphically continued along an arbitrary continuous curve $L\colon[0,1]\to G=\{z\colon r_0\leqslant|z|<+\infty\}$, $L(0)=z_0$, $L(1)=z_1$; outcome element is either a regular element $e_{z_1}(z)$, $z\in\{z\colon|z-z_1|<\delta_{z_1}\}$ or an element of the form $\tilde e_{z_1}(z)=\sum_{j=-m}^{+\infty}\tilde a_j(z-z_0)^j$, $z\in\{z\colon|z-z_1|<\tilde\delta_{z_1}\}$, $m\in\mathbb{N}$. It is possible that for an arbitrary $z_1\in G$ there exists

2010 *Mathematics Subject Classification:* 34M05, 34M35.

Keywords: algebraic differential equation; branch point of analytic function; meromorphic solution; order of growth.

an infinite set of distinct elements with the center z_1 that are continued from the element (g, e_{z_0}). The set of all such elements is denoted by $F(z),\ z \in G$ and say that $F(z),\ z \in G$, is *meromorphic in the domain G function, generated by the element* (g, e_{z_0}).

Let the curve L be closed: $L(0) = L(1) = z_0,\ \nu(L, 0) = s$ be the count of loops of the curve L around the point 0. The curve L is homotopic in G to the curve $c_s\colon z = |z_0|e^{i(\varphi_0+2\pi s\tau)},\ 0 \leqslant \tau \leqslant 1,\ c_s(0) = z_0 = c_s(1)$ that loops s times the circle of radius $|z_0|$ with the center at 0. If a meromorphic element (g, e_0) is continued along the curve L and along the curve c_s, the results coincide: this is an element centered at z_0.

Assume that the meromorphic extension of the element (g, e_0) along the circle $c_1\colon z = |z_0|e^{i(\varphi_0+2\pi\tau)},\ 0 \leqslant \tau \leqslant 1,\ c_1(0) = z_0 = c_1(1)$ that loops once around the circle with the radius $|z_0|$ and the center at 0, is different from the initial one. If there exists $k \in \mathbb{N}$ such that the meromorphic extension of the element (g, e_0) along the curve $c_k\colon z = |z_0|e^{i(\varphi_0+2\pi k\tau)},\ 0 \leqslant \tau \leqslant 1,\ c_k(0) = z_0 = c_k(1)$ that goes k times around the circle of radius $|z_0|$ centered at the origin, generates the initial element (g, e_0) then the point ∞ is a *finite order branch point.* Let $k,\ k \in \mathbb{N}$ be the least one with the property described above. Then the number $k-1$ is the *branching order* of the point ∞. In this case for any $z_1 \in G$ there exist exactly k distinct elements centered at z_1 that are extensions of the element (g, e_{z_0}). That means $F(z),\ z \in G$ is a *k-valued meromorphic in the domain G function generated by the element* (g, e_{z_0}).

Let M_k be the set of k-valued meromorphic in $G = \{r^{i\theta}\colon 0 \leqslant \theta \leqslant 2\pi k,\ r_0 \leqslant r\}$ functions with the branch point of order $k-1$ at ∞; other singular points of function $f \in M_k$ are poles with the only possible concentration point at ∞. Let A_k be the set of k-valued analytic in G functions which have unique singular point at infinity: algebraic branch point of order $k-1$; therefore $A_k \subset M_k$.

Rewrite $z,\ z \in \mathbb{C} \setminus \{0\}$ in the exponential form: $z = re^{i\theta}$. A function $f \in M_k$ can be considered as a single-valued $f(z),\ z \in G = \{re^{i\theta}\colon 0 \leqslant \theta \leqslant 2\pi k,\ r_0 \leqslant r\}$, on the Riemann surface $G = \{re^{i\theta}\colon 0 \leqslant \theta \leqslant 2\pi k,\ r_0 \leqslant r\}$.

Consider the differential equation

$$\sum_{\varkappa+\varsigma=p} f^{\varkappa} f'^{\varsigma} v_{\varkappa\varsigma}(z) z^{\tau_{\varkappa\varsigma}} = \sum_{\varkappa+\varsigma<p} w_{\varkappa\varsigma}(z) f^{\varkappa} f'^{\varsigma}, \quad p \in \mathbb{N}, \quad \varkappa,\ \varsigma \in \mathbb{N} \cup \{0\}, \tag{1}$$
$$|w_{\varkappa\varsigma}(z)| < |z|^{a_{\varkappa\varsigma}},\ \ v_{\varkappa\varsigma}(z) = c_{\varkappa\varsigma} + o(1),\ z \to \infty,$$
$$z \in G = \{re^{i\theta}\colon 0 \leqslant \theta \leqslant 2k\pi,\ r_0 \leqslant r < +\infty\}, \quad k \in \mathbb{N}, \quad \tau_{\varkappa\varsigma},\ a_{\varkappa\varsigma} \in \mathbb{R}, \quad c_{\varkappa\varsigma} \in \mathbb{C},$$

$v_{\varkappa\varsigma}(z),\ w_{\varkappa\varsigma}(z),\ z \in G$ are analytic functions, for example, for some $\varkappa, \varsigma,\ \varkappa + \varsigma = p,\ v_{\varkappa\varsigma}(z) = \cos\frac{1}{\sqrt{z}}$, and for $\varkappa, \varsigma$ such that $\varkappa + \varsigma < p,\ w_{\varkappa\varsigma}(z) = z^{a_{\varkappa\varsigma}-\varepsilon}\operatorname{Ln} z,\ \varepsilon > 0$.
By definition, $v_{\varkappa\varsigma}(z) \equiv 0$ if $c_{\varkappa\varsigma} = 0$; let $\exists\, \varkappa_*, \varsigma_* \in \mathbb{N} \cup \{0\}$:

$$\varkappa_* + \varsigma_* = p, \qquad c_{\varkappa_*\varsigma_*} \neq 0. \tag{2}$$

As E_* we denote a set of circles in the domain G with a finite sum of radii.

Denote

$$m(r, f) = \frac{1}{2\pi k}\int_0^{2\pi k} \ln^+ |f(re^{i\theta})|d\theta, \tag{3}$$
$$M_*(r, f) = \max|f(z)|,\ \ z \in \{te^{i\theta}\colon 0 \leqslant \theta \leqslant 2k\pi,\ r_0 \leqslant t \leqslant r\} \setminus Q,$$

$Q \subset E_*$ the set of circles with centers at zeros and poles of f with a finite sum of radii (see (28)). By definition, $m(r, f) \leqslant \ln M_*(r, f),\ r \notin \Delta,\ \operatorname{mes}\Delta < +\infty$.

Theorem 1. *Let $f \in M_k$ be a solution of equation* (1)*. Then either*

$$\exists\, d = d(\varepsilon): \quad |f(re^{i\theta})| < r^{\nu+\varepsilon}, \quad re^{i\theta} \in G \setminus E_*, \quad r > d, \tag{4}$$

or $m(r, f)$ has growth order $\rho \geqslant \frac{1}{2k}$ and the following equality holds

$$\ln M_*(r, f) = (c + o(1))r^{\rho}, \quad c \neq 0, \quad c, \rho = \text{const}, \quad r \to +\infty. \tag{5}$$

Moreover,

a) *there exists a finite (maybe empty) set of angles $\{z = te^{i\theta} : r_0 \leqslant t < +\infty,\ \eta_\kappa < \theta < \gamma_\kappa\}$, such that $\forall\, \epsilon > 0\ \ \forall\, \varepsilon > 0\ \ \exists\, d$:*

$$\ln f(z) = (c_\kappa + g(z))z^{\rho_\kappa}, \quad |g(z)| < \varepsilon, \quad z \in \{te^{i\theta} : d \leqslant t,\ \eta_\kappa + \epsilon \leqslant \theta \leqslant \gamma_\kappa - \epsilon\}; \tag{6}$$

b) *there exists a finite (maybe empty) set of rays $\{z = te^{i\varphi_s} : r_0 \leqslant t < +\infty\}$, for which*

$$\ln |f(re^{i\varphi_s})| = o(r^{\rho}), \quad r \in [r_0, +\infty) \setminus \Delta, \quad \operatorname{mes} \Delta < +\infty; \tag{7}$$

c) *on the complement to these angles and rays (in the domain G)*

$$|f(re^{i\theta})| < r^{\nu+\varepsilon}, \quad \varepsilon > 0, \quad r > r(\theta), \quad re^{i\theta} \notin E_*. \tag{8}$$

Numbers $\eta_\kappa, \gamma_\kappa, c_\kappa, \rho_\kappa, \rho, c, \varphi_s, \nu$, are defined in view of equation (1); E_* is any set of circles with a finite sum of radii.

If a solution $f \in M_k$ of equation (1) *has the characteristic $m(r, f)$ of growth order $\rho = \frac{1}{2k}$, then for some $\eta \in \mathbb{R}\ \ \forall\, \epsilon > 0\ \ \forall\, \varepsilon > 0\ \ \exists\, d$:*

$$\ln f(z) = z^{\frac{1}{2k}}(c + g(z)), \quad |g(z)| < \varepsilon, \quad z \in \{te^{i\theta} : d \leqslant t,\ \eta + \epsilon \leqslant \theta \leqslant \eta + 2\pi k - \epsilon\}, \tag{9}$$

and on the ray $\{z = te^{i\eta} : r_0 \leqslant t < +\infty\}$

$$\ln |f(re^{i\eta})| = o(r^{\frac{1}{2k}}), \quad r \in [r_0, +\infty) \setminus \Delta, \quad \operatorname{mes} \Delta < +\infty. \tag{10}$$

Remark 1. If a solution (1) is an entire function $f(z), z \in \mathbb{C}$ (thus $k = 1$) then the characteristic $m(r, f) = T(r, f)$ (see [3]). If the estimate (4) holds, then f is a polynomial of a degree not grater than ν. Otherwise the characteristic $m(r, f)$ has growth order $\rho \geqslant \frac{1}{2k} = \frac{1}{2}$, and f is an entire transcendental function of order $\rho \geqslant \frac{1}{2}$ ([4]).

Example 1. The function $f(z) = \cos \sqrt[2k]{z}$, $z \in \mathbb{C}$, from the ring A_k is a solution of the equation $f^2 + f'^2 4k^2 z^{2-\frac{1}{k}} = 0$, has growth order $\rho = \frac{1}{2k}$. Moreover, $\ln f(z) = \sqrt[2k]{z}(-i + o(1))$, $z = re^{i\theta} \in \{te^{i\theta} : d \leqslant t,\ 0 < \theta < 2\pi k\}$, $\theta = \text{const}$, which means that (6) holds, and on the ray $\{z:\ z = r > 0\}$ statement (7) is valid.

Remark 2. It will be proved that if characteristic equation (14) does not depend on L then for a solution $f \in M_k$ of equation (1) the following inequality holds

$$|f(z)| < |z|^{\nu+\varepsilon}, \quad z \in \{z = re^{i\theta} :\ 0 \leqslant \theta \leqslant 2\pi k,\ r \geqslant d\} \setminus E_*. \tag{11}$$

Example 2. The Weierstrass elliptic function $\wp(z)$, $z \in \mathbb{C}$ is meromorphic with growth order $\rho = 2$ ([5, V.2, p. 422]). It is a solution of the equation $(f')^2 = 4f^3 - g_2 f - g_3$, $g_2, g_2 = \text{const}$, ([5, V.2, p. 362]). For the Weierstrass function estimates (11), (4) hold true.

Remark 3. If in characteristic equation (14) all degrees $d_j \leqslant 0$, then for angle coefficients of the Newton diagram of equation (1) we have $0 \geqslant \rho_1 \geqslant \ldots \geqslant \rho_T$. For this case it will be proved that (11) is valid.

Example 3. The function $f(z) = \sqrt{z},\ z \in \mathbb{C}$, is a solution of the equation $2ff' = 1$. For this function estimate (11) is valid.

Proof of Theorem 1. If $f \in M_k$ is a solution of (1) then f has growth order $\mu,\ 0 \leqslant \mu < +\infty$ ([6]). Let (see (1) and below)

$$n = \max\{\varsigma\colon\ \varkappa + \varsigma = p,\ c_{\varkappa\varsigma} \neq 0\}, \quad q = \min\{\varsigma\colon\ \varkappa + \varsigma = p,\ c_{\varkappa\varsigma} \neq 0\}. \tag{12}$$

Divide both parts of (1) by $f^p(z)$. After a simple transformation and coefficients reassigning this equation can be presented as

$$\Big(\frac{zf'(z)}{f(z)}\Big)^n + \sum_{j=1}^{n-q}\Big(\frac{zf'(z)}{f(z)}\Big)^{n-j} v_j(z) z^{d_j} = \omega(z), \ \ v_j(z) = c_j + o(1), \ \ c_{n-q} \neq 0, \tag{13}$$
$$\omega(z) = \sum_{\varkappa+\varsigma\leqslant p-1} w_{\varkappa\varsigma}(z) z^{n-\tau_{p-n,n}} \frac{(f'/f)^\varsigma}{f^{p-\varkappa-\varsigma}}, \ \ d_j \in \mathbb{R}, \ \ c_0 = 1, \ \ d_0 = 0.$$

Here $z \in G = \{r^{i\theta}\colon 0 \leqslant \theta \leqslant 2\pi k,\ r_0 \leqslant r\}$. Denote $\frac{zf'(z)}{f(z)} = L(z),\ c_0 = 1,\ d_0 = 0$, and rewrite equation (13) as

$$L^n(z) + \sum_{j=1}^{n-q} L^{n-j}(z) v_j(z) z^{d_j} = \omega(z), \ \ v_j(z) = c_j + o(1), \ \ c_{n-q} \neq 0. \tag{14}$$

Here $z \in G$, $q \geqslant 0$. This equation is called *characteristic* for (1).

Consider the equation

$$x^n + \sum_{j=1}^{n-q} x^{n-j} v_j(z) z^{d_j} = \omega(z), \quad |\omega(z)| < |z|^{-A}, \ \ q \geqslant 0, \tag{15}$$

where A is a constant defined in (20), (23) by the form of equation (15). Coefficients $\omega(z),\ v_j(z),\ z \in \Phi,\ j \in \{1, 2, \ldots, n-q\}$ are some functions defined on an unbounded set $\Phi \subset g_{\alpha\beta} = \{z = re^{i\theta}\colon\ r \geqslant r_0,\ \alpha \leqslant \theta \leqslant \beta\}$ in the way that

$$\begin{gathered}\forall\,\delta > 0 \ \ \exists\, d = d(\delta) \ \ \forall z \in \Phi \cap \{z = re^{i\theta}\colon\ r \geqslant d,\ \alpha \leqslant \theta \leqslant \beta\} \Rightarrow \\ v_j(z) = (c_j + \mathrm{g}_j(z)), \quad c_j \in \mathbb{C}, \quad |\mathrm{g}_j(z)| < \delta.\end{gathered} \tag{16}$$

We set $\mathrm{g}_j(z) \equiv 0$ if $c_j = 0$. Let $F = \{j\colon\ v_j(z) \not\equiv 0,\ z \in \Phi,\ j \in \{1, 2, \ldots, n-q\}\}$.

1° If in equation (15) $q = 0\ \vee\ q \geqslant 1,\ d_{n-q} < \max_{j\in F} d_j$, then we denote $d_0 = 0$ and $H = \{(j, d_j)\colon j \in F \cup \{0\}\}$ a set of points on the plain.

2° If in equation (15) $q \geqslant 1,\ d_{n-q} = \max_{j\in F} d_j$, then we append to H another point $(n, d_n),\ d_n \overset{\text{def}}{=} -1$ and obtain the set $\widetilde{H} = \{(j, d_j)\colon j \in F \cup \{0\}\} \cup \{(n, d_n)\}$.

Let the terms in 1° hold. By the points of H, let us construct the *Newton diagram* of equation (15) (of the set H). Consider the convex hull of the set H ([7, v. 1, p. 788]). The boundary of this convex hull is a polygon divided by the points $(0, d_0)$ and $(n-q, d_{n-q})$

into two broken lines. The top line is the required Newton diagram. Let *vertices of Newton diagram* have abscissas

$$i_0, i_1, \dots, i_T, \quad 0 = i_0 < i_1 < \dots < i_T = n - q. \tag{17}$$

Denote

$$\rho_s = \frac{d_{i_s} - d_{i_{s-1}}}{i_s - i_{s-1}}, \quad s \in \{1, 2, \dots, T\}, \tag{18}$$

ρ_s are *angle coefficients of the Newton diagram segments*, $\rho_1 > \rho_2 > \dots > \rho_T$. Note that if in 1° the following term holds true $q \geqslant 1$, $d_{n-q} < \max_{j \in F} d_j$, then by the properties of the convex hull of the set H, the following inequality holds $\rho_T < 0$. Denote $\rho_s(n - j) + d_j \overset{\text{def}}{=} l_{j,s},\ s \in \{1, 2, \dots, T\},\ j \in F$;

$$\max_{j \in F} l_{j,s} \overset{\text{def}}{=} l_s. \tag{19}$$

In what follows we assume that in (15) the constant A is subject to the condition

$$A > \max(0, \max_{s \in \{1,2,\dots,T\}} -l_s). \tag{20}$$

Let the terms in 2° hold true. In this case the Newton diagram of equation (15) is constructed by the points $\widetilde{H}$. Newton's diagram vertices of the set $\widetilde{H}$ have abscissas $i_0, i_1, \dots, i_T, i_{T+1}$, $0 = i_0 < i_1 < \dots < i_T = n - q < i_{T+1} = n$, that are different from the abscissas of Newton's diagram vertices of the set H by just one additional point $i_{T+1} = n$. Angle coefficients of the Newton diagram segments of the set $\widetilde{H}$ are

$$\rho_s = \frac{d_{i_s} - d_{i_{s-1}}}{i_s - i_{s-1}}, \quad s \in \{1, 2, \dots, T, T+1\}, \quad \rho_1 > \rho_2 > \dots > \rho_T > \rho_{T+1}. \tag{21}$$

From the terms in 2° and convex hull properties [7, p. 788] we conclude: $\rho_{T+1} < 0$. Similarly in (19) denote by $\rho_s(n - j) + d_j = l_{j,s},\ s \in \{1, \dots, T, T+1\},\ j \in F$;

$$\max_{j \in F} l_{j,s} \overset{\text{def}}{=} l_s, \quad s \in \{1, \dots, T, T+1\}. \tag{22}$$

In the definition of the number A (20) make one extra assumption

$$A > \max(0, \max_{s \in \{1,2,\dots,T+1\}} -l_s). \tag{23}$$

In [8] the following lemma is proved.

Lemma 1. *Let in equation* (15) *conditions* (16) *hold true and the constant A is defined in* (20) *(or* (23)*) by the Newton diagram of this equation. Let $\Phi \subset g_{\alpha\beta}$, Φ be an unbounded closed (open) set. By Φ_0 we denote the connected component of the set Φ.*

If $\forall\, j \in F = \{j\colon\ \mathrm{g}_j(z) \not\equiv 0,\ z \in \Phi,\ j \in \{1, 2, \dots, n - q\}\}$ in equation (15) *the degrees $d_j \leqslant 0$ then all solutions of equation* (15) *are bounded in $\Phi \cap \{z = re^{i\theta}\colon\ r \geqslant d,\ \alpha \leqslant \theta \leqslant \beta\}$. Let $\exists\, j \in F\colon\ d_j > 0$. If a continuous function $x(z),\ z \in \Phi$ is a solution of equation* (15) *then only one of properties holds:*

1. *either $\forall\, \delta > 0\ \ \exists\, r_0$:*

$$\begin{gathered} x(z) = (y + u(z))z^{\rho}, \quad y \neq 0, \quad |u(z)| < \delta, \quad z \in \Phi_0, \quad |z| > r_0, \\ \rho \in \mathbb{R}, \quad y \in \mathbb{C}, \quad y = y(\Phi_0),\ \rho = \rho(\Phi_0), \end{gathered} \tag{24}$$

   y, ρ do not change if $z \in \Phi_0$, $|z| > r_0$; (y, ρ, correspondingly, one of finite set of y_j, ρ_s defined by the equation (15)*, ρ_s is an angle coefficient of Newton diagram).*

2. or

$$|x(z)| < |z|^{\zeta+\delta}, \quad \zeta+\delta<0, \quad z\in\Phi_0, \quad |z|>r_0, \tag{25}$$

$\zeta=\rho_T \vee \zeta=\rho_{T+1}$ *(see* (18), (21)*),* $\delta>0$ *is sufficiently small.*

Apply Lemma 1 to equation (14). Consider the set $F=\{j\colon\ v_j(z)=c_j+o(1)\not\equiv 0,\ z\in G,\ j\in\{1,2,\ldots,n-q\}\}$. By $\rho_s,\ \ s\in\{1,2,\ldots,T,T+1\}$ we denote angle coefficients of the Newton diagram segments of the set $H=\{(j,d_j)\colon j\in F\cup\{0\}\}$ (or the set $\tilde H$). Let

$$\mu_0=\max(\mu,\rho_1), \quad \rho_1>\rho_2>\ldots>\rho_T>\rho_{T+1}, \quad \mu\geqslant 0. \tag{26}$$

For the function $f\in M_k$ with growth order $\mu,\ \mu<+\infty$ the following statement is fulfilled, see [6, p. 208] (E_* is a set of circles with centers in poles and zeros of f with a finite sum of radii) $\forall\,\sigma>0\ \exists\, r_0$:

$$\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| < r^{2\mu+2+\sigma}\leqslant r^{2\mu_0+2+\sigma},\ \ re^{i\theta}\in G\setminus E_*, \tag{27}$$

$G=\{re^{i\theta}\colon 0\leqslant\theta\leqslant 2\pi k,\ r_0\leqslant r\}$. Thus the set

$$Q=\Big\{z\colon\ z\in G,\ \left|\frac{f'(z)}{f(z)}\right|>|z|^{2\mu_0+2+\sigma}\Big\}\subset E_*. \tag{28}$$

Let ∂Q be the boundary of Q. Since functions $|z|^{2\mu_0+2+\sigma}$, $\left|\frac{f'(z)}{f(z)}\right|$ are continuous, taking into account (27) and (28) we obtain ($\sigma>0$)

$$\left|\frac{f'(z)}{f(z)}\right|=|z|^{2\mu_0+2+\sigma},\ \ z\in\partial Q; \qquad \left|\frac{f'(z)}{f(z)}\right|\leqslant |z|^{2\mu_0+2+\sigma},\ \ z\in G\setminus Q. \tag{29}$$

Denote (see (23) and (20))

$$\begin{gathered} l=\max(0,\max_{s\in\{1,2,\ldots,T\}}-l_s) \ \ \vee \ \ l=\max(0,\max_{s\in\{1,2,\ldots,T+1\}}-l_s),\\ \nu=\max(\mathfrak{y},\ \max_{\varkappa+\varsigma\leqslant p-1}\{a_{\varkappa\varsigma}+n-\tau_{p-n,n}+(2\mu_0+2)p+l\}). \end{gathered} \tag{30}$$

Here the constant $\mathfrak{y}$ is defined by solutions of characteristic equation (14) (see (24)). Consider the sets

$$\begin{gathered} \Phi=\{z\colon\ z\in G\setminus Q,\ |f(z)|\geqslant|z|^{\nu+\varepsilon}\},\ \Phi_1=\{z\colon\ z\in G\setminus Q,\ |f(z)|<|z|^{\nu+\varepsilon}\},\\ \Phi^\circ=\{z\colon z\in G\setminus Q, |f(z)|>|z|^{\nu+\varepsilon}\}, \quad \Phi=\Phi^\circ\cup\partial\Phi. \end{gathered} \tag{31}$$

On the set Φ the conditions of Lemma 1 hold true ($|w_{\varkappa\varsigma}(z)|<|z|^{a_{\varkappa\varsigma}}$)

$$\begin{aligned} |\omega(z)| &\overset{(13)}{\leqslant} \sum_{\varkappa+\varsigma\leqslant p-1}|w_{\varkappa\varsigma}(z)z^{n-\tau_{p-n,n}}|\frac{|f'/f|^\varsigma}{|f|^{p-\varkappa-\varsigma}}\leqslant\\ &\leqslant\sum|z|^{n-\tau_{p-n,n}+a_{\varkappa\varsigma}}\frac{|f'/f|^\varsigma}{|f|}\overset{(29)}{\leqslant}\sum|z|^{n-\tau_{p-n,n}+a_{\varkappa\varsigma}+(2\mu_0+2+\sigma)\varsigma}\frac{1}{|f|}\overset{(31)}{\leqslant}\\ &\leqslant\sum|z|^{n-\tau_{p-n,n}+a_{\varkappa\varsigma}+(2\mu_0+2+\sigma)\varsigma-\nu-\varepsilon}\overset{(30)}{\leqslant}|z|^{-l+\sigma p+\frac{\sigma}{7}-\varepsilon}, \quad z\in\Phi, \end{aligned} \tag{32}$$

$-l+\sigma p+\frac{\sigma}{7}-\varepsilon<0$ (in (27), (32) assume that $\sigma<\frac{7\varepsilon}{7p+1}$).

If (14) does not depend on L then in the left-hand side of (1) only one summand $f^p v_{p0}(z) z^{\tau_{p0}}$, $v_{p0}(z) = c_{p0} + o(1)$, has the degree p in f and f'. Then there exists $d > 0$ such that $\Phi \cap \{z\colon |z| > d\} = \varnothing$. If we assume the contrary then taking into account (32) equation (14) has the form $c_{p0} + o(1) = o(1)$, $z \in \Phi$. From here we obtain that $c_{p0} = 0$ which contradicts the assumption (2). Thus in the case that is considered we have $\{z = re^{i\theta}\colon\ 0 \leqslant \theta \leqslant 2\pi k,\ r \geqslant d\} \setminus E_* \subset \Phi_1$ and from (31) it follows (11).

In particular, the Weierstrass elliptic function (see example 2) $\wp(z)$, $z \in \mathbb{C}$ is a meromorphic function of growth order $\mu = 2$. This function is a solution of the differential equation $(f')^2 = 4f^3 - g_2 f - g_3$, $g_2, g_2 = \text{const}$. Thus the equalities hold true: $4[\wp(z)]^3 \equiv (\frac{d\wp(z)}{dz})^2 + g_2\wp(z) + g_3$,

$$4 \equiv \frac{1}{\wp(z)} \left(\frac{\wp'(z)}{\wp(z)}\right)^2 + \frac{g_2\wp(z) + g_3}{[\wp(z)]^3}, \quad z \in \mathbb{C}. \tag{33}$$

This means that the characteristic equation does not depend on L (see (14)). For one-valued meromorphic function $\wp(z)$ of finite order μ the following inequality holds [4, p. 87] (E_* is a set of circles with a finite sum of radii)

$$\left|\frac{\wp'(z)}{\wp(z)}\right| < |z|^{2\mu+\sigma} = |z|^{4+\sigma}, \quad z \in \{z\colon d \leqslant |z| < +\infty\} \setminus E_*, \quad \sigma > 0. \tag{34}$$

Consider the set $\Phi_\wp = \{z\colon z \in \mathbb{C} \setminus E_*,\ \ |\wp(z)| > |z|^{8+3\sigma}\}$. There exists d, $d > 0$ such that $\Phi_\wp \cap \{z\colon |z| > d\} = \varnothing$. If this were wrong then, taking into account (34), equality (33) would take the following form $4 = o(1)$, $z \in \Phi_\wp$ which is obviously wrong. That is why $|\wp(z)| \leqslant |z|^{8+3\sigma}$, $z \in \mathbb{C} \setminus E_*$, $|z| > d$, $\sigma > 0$.

Let (14) depend on L. Assume for definiteness in (14) $q = 0$. Let Φ_0 be an arbitrary connected component of Φ, $\Phi_0 \subset \Phi$. According to Lemma 1 for the continuous function $L(z) = \frac{zf'(z)}{f(z)}$ which is a solution of (14) on the set Φ_0, one of the following assertions is true: either (24) or (25),

$$x(z) = L(z) = \frac{zf'(z)}{f(z)}. \tag{35}$$

Let $\mathfrak{y}$ be the greatest possible values of $|y|$ in (24) (see (30)).

If R is large enough then for the points $z \in \partial Q$ the following statement holds

$$z \in \partial Q,\ |z| > R \ \Rightarrow\ z \in \Phi_1 \ \overset{(31)}{\Rightarrow}\ |f(z)| < |z|^{\nu+\varepsilon}. \tag{36}$$

Indeed, if $z \in \Phi \cap \partial Q$ then from (24), (26) we have $\left|\frac{f'(z)}{f(z)}\right| < \frac{3}{2}|yz^{\rho_j - 1}| < |z|^{2\mu_0}$, which contradicts the first formula in (29). From (36) and (31) it follows that Φ° is an open set. From (36) and from the definition of Φ we conclude

$$z \in \partial\Phi,\ \ |z| > R \ \ \overset{(31)}{\Rightarrow}\ \ |f(z)| = |z|^{\nu+\varepsilon}. \tag{37}$$

It is possible to assume that for r_0 the following holds true: $f(r_0 e^{i\theta}) \neq 0, \infty$, $0 \leqslant \theta \leqslant 2\pi k$. Then

$$0 < c < |f(r_0 e^{i\theta})| < C, \quad 0 \leqslant \theta \leqslant 2\pi k, \quad c, C = \text{const}. \tag{38}$$

Take some φ, $0 \leqslant \varphi < 2\pi k$. Consider a ray $S(\varphi) = \{re^{i\varphi}\colon\ r \geqslant r_0,\ \varphi = \text{const}\}$. If $\Phi \cap S(\varphi) = \varnothing$ then from (31) it follows

$$|f(z)| < |z|^{\nu+\varepsilon}, \quad z \in S(\varphi). \tag{39}$$

Let $\Phi \cap S(\varphi) \neq \varnothing$, $\Phi = \Phi^\circ \cup \partial\Phi$. From (37) we have $|f(z)| = |z|^{\nu+\varepsilon}$, $z \in \partial\Phi$. The set $\Phi^\circ \cap S(\varphi)$ is a union of finite or countable set of disjunctive maximal connected components $\omega_t^\circ = \{z\colon z = re^{i\varphi}, r_{1t} < r < r_{2t}\}$, see [9, p. 58] such that $|f(z)| > |z|^{\nu+\varepsilon}$, $z \in \omega_t^\circ$. Moreover if $z_{1t} = r_{1t}e^{i\varphi}$ is the starting point, $z_{2t} = r_{2t}e^{i\varphi}$ is the ending point of ω_t° and $|z_{1t}| > r_0$, $|z_{2t}| < +\infty$ then from (37) it follows

$$|f(z_{1t})| = |z_{1t}|^{\nu+\varepsilon}, \quad |f(z_{2t})| = |z_{2t}|^{\nu+\varepsilon}. \tag{40}$$

Add the start and end points to the open interval ω_t° and obtain the closed interval $\omega_t = \{z\colon z = re^{i\varphi}, r_{1t} \leqslant r \leqslant r_{2t}\}$, $\Phi \supset \omega_t \supset \omega_t^\circ$, such that

$$|f(z)| \geqslant |z|^{\nu+\varepsilon}, \quad z \in \omega_t. \tag{41}$$

The connected set ω_t is contained in the connected component $\Phi_0 \subset \Phi$. That is why $\forall\, z \in \omega_t$ either equality (24) holds true ($y = y(t)$, $\rho = \rho(t)$) or inequality (25).

Let $\{\omega_t\}$ be the set of all segments ω_t on $S(\varphi)$; by ω_t^+ we denote such segments $\omega_t \in \{\omega_t\}$ for which equality (24) holds with $\rho > 0$; let ω_t^- be the segments ω_t for which equality (24) with $\rho \leqslant 0$ or (25) holds true. Let $\omega_{[z_{1t},z]}$ be an arc (a piece of) the segment ω_t from the point z_{1t} to the point $z \in \omega_t$.

Suppose there exists a section $\omega_t^- \subset S(\varphi)$. According to (24), (25), (35), for $s \in \omega_t^-$ the following inequality holds $\left|\frac{f'(s)}{f(s)}\right| \leqslant \frac{(|y|+\frac{\delta}{3})}{|s|}$. Thus by integrating $\frac{f'(z)}{f(z)}$ along ω_t^-, we obtain ($|z| = r$, $|z_{1t}| = r_{1t}$)

$$\ln\left|\frac{f(z)}{f(z_1)}\right| \leqslant \left|\int_{\omega_{[z_{1t},z]}} \frac{f'(s)}{f(s)} ds\right| \leqslant \left(|y| + \frac{\delta}{3}\right) \int_{r_{1t}}^{r} \frac{dx}{x} = \left(|y| + \frac{\delta}{3}\right) \ln\frac{|z|}{|z_{1t}|}. \tag{42}$$

If ω_t^- has an infinite length then from (42) it follows ($|y| \leqslant \nu$, $\delta < \varepsilon$) $\ln|f(z)| \leqslant \left(|y| + \frac{\delta}{2}\right) \times$ $\times \ln|z| + \ln|f(z_1)| < (\nu + \frac{\varepsilon}{2})\ln|z| + \ln|f(z_1)|$, $z \in \omega_t^- \subset \Phi$, $z \to +\infty$ that contradicts the first equality in (31). Therefore $r_{2t} < +\infty$.

If $r_{1t} > r_0$ then taking into account (40) at $z = z_{2t}$, inequality (42) has the form $(\nu + \varepsilon)\ln\frac{|z_{2t}|}{|z_{1t}|} \leqslant \left(|y| + \frac{\delta}{3}\right)\ln\frac{|z_{2t}|}{|z_{1t}|}$, $|y| \leqslant \nu$, $\delta < \varepsilon$, that is possible only if $z_{1t} = z_{2t}$, where the segment ω_t^- is actually a point $z_{1t} = z_{2t}$ if (40) holds true.

If $r_{1t} = r_0$ then (38) and (40) yield $\ln|f(z_{1t})| < \ln C$, $\ln|f(z_{2t})| = (\nu + \varepsilon)\ln|z_{2t}|$, thus at $z = z_{2t}$ inequality (42) has the form $\ln|f(z_{2t})| = (\nu + \varepsilon)\ln|z_{2t}| < \left(|y| + \frac{\delta}{3}\right)\ln\frac{r_{2t}}{r_0} + \ln C$, $r_0 < r_{2t}$, $|y| \leqslant \nu$, $\delta < \varepsilon$, that is possible only if $|z_{2t}| < R = \text{const}$ which does not depend on φ. Finally

$$\{z\colon\ z = re^{i\varphi},\ \varphi = \text{const},\ r > R\} \cap \omega_t^- = \varnothing. \tag{43}$$

In particular, if in characteristic equation (14) all degrees $d_j \leqslant 0$ then for angle coefficients of the Newton diagram segments of equation (14) we obtain $0 \geqslant \rho_1 \geqslant \ldots \geqslant \rho_T$. Thus, on an arbitrary connected component Φ_0 equality (24) holds true with $\rho \leqslant 0$. That is why segments $\omega_t^+ \subset S(\varphi)$ do not actually exist. This and (43) implies $\{z\colon z = re^{i\varphi},\ r \geqslant R,\ \varphi = \text{const}\} \cap \Phi^\circ = \varnothing$. Thus the ray $\{z = re^{i\varphi}\colon\ r \geqslant R,\ \varphi = \text{const}\} \overset{(31)}{\subset} \Phi_1 \cup Q \cup \partial\Phi$ and (11) holds true (see example 3).

Let $\exists\omega_t^+ \subset S(\varphi) = \{z\colon z = re^{i\varphi},\ r \geqslant r_0,\ \varphi = \text{const}\}$. For $z = re^{i\varphi} \in \omega_t^+$ equalities (24) ($\rho > 0$) and (40) hold.

If we integrate (24) on the set ω_t^+ and remove real parts then obtain (see (35)) ($y = |y|e^{i\beta},\ \rho > 0;\ z = re^{i\varphi},\ z_{1t} = r_{1t}e^{i\varphi} \in \omega_t^+$)

$$\ln\frac{f(z)}{f(z_{1t})} = \frac{y}{\rho}(z^\rho - z_{1t}^\rho) + \int_{\omega_{[z_{1t},z]}} u(\zeta)\zeta^{\rho-1}d\zeta,\ \left|\int_{\omega_{[z_{1t},z]}} u(\zeta)\zeta^{\rho-1}d\zeta\right| \leqslant \frac{\delta}{9}\int_{r_{1t}}^{r} x^{\rho-1}dx \leqslant$$
$$\leqslant \frac{\delta}{9\rho}(r^\rho - r_{1t}^\rho),\ \ln\left|\frac{f(z)}{f(z_{1t})}\right| = \frac{|y|}{\rho}(r^\rho - r_{1t}^\rho)(\cos(\rho\varphi+\beta) + \tilde{q}(z)),\quad |\tilde{q}(z)| < \frac{\delta}{9|y|}, \tag{44}$$

$z = re^{i\varphi} \in \omega_t^+,\ r_{1t} \leqslant r \leqslant r_{2t} \leqslant +\infty,\ \ \delta > 0,\ \delta$ is small.

First suppose that all segments ω_t^+ have a finite length. For all segments ω_t^+, where $r_{1t} \geqslant r(\varphi)$, in (44) $\cos(\rho\varphi+\beta) = 0$. To prove this let us assume that $\cos(\rho\varphi+\beta) \neq 0$. Substitute (40) into (44) and obtain ($z = z_{2t}$)

$$(\nu+\varepsilon)\ln\frac{r_{2t}}{r_{1t}} = (r_{2t}^\rho - r_{1t}^\rho)\frac{|y|}{\rho}(\cos(\rho\varphi+\beta) + \tilde{q}(z_{2t})),\quad |\tilde{q}(z_{2t})| < \frac{\delta}{9|y|}. \tag{45}$$

Let in (45) $\cos(\rho\varphi+\beta) < 0$. If $\delta > 0$ is small and $r(\varphi)$ is large enough then in (45) the following inequality holds $|\tilde{q}(z)| < \frac{\delta}{9|y|} < -\cos(\rho\varphi+\beta),\ r(\varphi) \leqslant r_{1t} \leqslant |z|$. Then in (45) the left and right-hand side have distinct signs that presents the required contradiction.

If in (45) $\cos(\rho\varphi+\beta) > 0$, and $r(\varphi)$ is large enough then in (45) the following inequality holds $|\tilde{q}(z)| < \frac{\delta}{9|y|} < \frac{1}{2}\cos(\rho\varphi+\beta),\ r(\varphi) \leqslant r_{1t} \leqslant |z|$, and from (45) we obtain $(\nu+\varepsilon)\ln\frac{r_{2t}}{r_{1t}} \geqslant (r_{2t}^\rho - r_{1t}^\rho)\frac{|y|}{2\rho}\cos(\rho\varphi+\beta)$. Finally

$$c(\ln x_2 - \ln x_1) > x_2 - x_1,\quad x_1 = r_{1t}^\rho < x_2 = r_{2t}^\rho,\quad c = \frac{2(\nu+\varepsilon)}{|y|\cos(\rho\varphi+\beta)}. \tag{46}$$

Since the function $x - c\ln x$ grows on $(c, +\infty)$, (46) does not hold if r_1 (and, x_1) is large enough, i.e. $r_1 > r(\varphi)$. Thus

$$\omega_t^+ \cap \{z\colon z = re^{i\varphi},\ \cos(\rho\varphi+\beta) \neq 0,\ |z| > r(\varphi),\ r_{2t} < +\infty\} = \varnothing. \tag{47}$$

If $\cos(\rho\varphi+\beta) = 0$ on the segment ω_t^+ in (44) then $\varphi = \varphi_j$,

$$\varphi_j = \frac{\pi + 2\pi j}{2\rho} - \frac{\beta}{\rho},\quad 0 \leqslant \varphi_j \leqslant 2k\pi, \tag{48}$$

j is an integer; ρ, β, j take a finite amount of values ($0 \leqslant \varphi_j \leqslant 2k\pi$). In this case from (44) and (40) the inequality is inferred

$$\ln\left|\frac{f(re^{i\varphi})}{r_{1t}^{\nu+\varepsilon}}\right| < \frac{\delta}{9\rho}(r^\rho - r_{1t}^\rho),\quad re^{i\varphi} \in \omega_t^+,\quad \varphi = \varphi_j,\quad r_{1t} \leqslant r \leqslant r_{2t}, \tag{49}$$

$\delta > 0,\ \delta$ is arbitrarily small if r_{1t} is large enough. Then taking into account (43), (47), (49) and the fact that $|f(z)| < |z|^{\nu+\varepsilon}$, $z \in \Phi_1$ (see (31)) we obtain

$$\ln|f(re^{i\varphi})| < \frac{\delta}{9\rho}r^\rho,\quad r > r(\varphi),\quad \varphi = \varphi_j,\quad \rho = \max\rho_t. \tag{50}$$

Maximum is taken over all ρ_t that correspond to the segments $\omega_t^+ \subset S$.

Assume that the segment ω_t^+ has an infinite length ($r_{2t} = +\infty$). Take $d \in [r_{1t}, +\infty)$. Integrate (24) on the ray ω_t^+ from the point $de^{i\varphi}$ to the point $re^{i\varphi}$, $r > d$; similar to (44) by removing real parts we obtain

$$\ln \frac{f(z)}{f(de^{i\varphi})} = \frac{y}{\rho}(z^\rho - (de^{i\varphi})^\rho) + \int_{\omega_{[de^{i\varphi},z]}} u(\zeta)\zeta^{\rho-1}d\zeta, \quad y = |y|e^{i\beta},$$

$$\left|\int_{\omega_{[de^{i\varphi},z]}} u(\zeta)\zeta^{\rho-1}d\zeta\right| \leqslant \frac{\delta}{9}\int_d^r x^{\rho-1}dx \leqslant \frac{\delta}{9\rho}(r^\rho - d^\rho), \tag{51}$$

$$\ln\left|\frac{f(z)}{f(de^{i\varphi})}\right| = \frac{|y|}{\rho}(r^\rho - d^\rho)(\cos(\rho\varphi+\beta) + \tilde{q}(z)), \ \ |\tilde{q}(z)| < \frac{\delta}{9|y|}.$$

Here $z = re^{i\varphi} \in \omega_t^+$, $r_{1t} \leqslant d \leqslant r < +\infty$, $\delta > 0$, δ is small and $\omega_{[de^{i\varphi},z]}$ is a part of the segment ω_t from the point $de^{i\varphi} \in \omega_t$ to the point $z \in \omega_t$.

If in (51) $\cos(\rho\varphi+\beta) = 0$ then (see (48)) $\varphi = \varphi_j$ and $\ln|f(re^{i\varphi_j})| = o(r^\rho)$, $r \to +\infty$. From this and from (50), (48) statement (7) follows.

Let $\cos(\rho\varphi+\beta) < 0$ on the segment ω_t^+ in (51). If d is large enough, then in (51) one has $|\tilde{q}(z)| < -\cos(\rho\varphi+\beta)$. Thus, the right-hand side in (51) is negative. This infers $|f(z)| < |f(de^{i\varphi})|$, $\forall\, z \in \omega_t^+$ for which $|z| > d$. This contradicts (41).

Let $\cos(\rho\varphi+\beta) > 0$ on the segment ω_t^+ of an infinite length in (51). Then there exists an integer b, $b \in \mathbb{Z}$ such that $|\rho\varphi+\beta-2\pi b| < \frac{\pi}{2}$. Take $\tilde{\varphi}$ that satisfies $\frac{\pi}{2} > \rho\tilde{\varphi}+\beta-2\pi b > |\rho\varphi+\beta-2\pi b|$. Then

$$\cos(\rho\psi+\beta) \geqslant \cos(\rho\tilde{\varphi}+\beta) > 0, \qquad \varphi < \psi \leqslant \tilde{\varphi}. \tag{52}$$

Here $\omega_t^+ \subset \Phi_0$ is a segment of the connected component Φ. There exists $d_1 \geqslant d$, such that

$$\{z = re^{i\theta} \colon \varphi \leqslant \theta \leqslant \tilde{\varphi},\ r \geqslant d_1\} \subset \Phi_0. \tag{53}$$

Indeed, let ψ be the greatest value such that the arc $H_r = \{re^{i\theta} \colon \varphi \leqslant \theta \leqslant \psi,\ r = \text{const} \geqslant d\} \subset \Phi_0$. Assume that $\psi < \tilde{\varphi}$. Recall that as soon as $z \in \partial Q$, then (36) holds true and $z \in \Phi_1$ (see (31)); taking into account the definition of the point $re^{i\psi}$ and the definition of the connected component Φ_0 of the set Φ we infer that $re^{i\psi} \in \partial\Phi$. Then from (37) it follows

$$|f(re^{i\psi})| = r^{\nu+\varepsilon}. \tag{54}$$

If $z \in H_r$ then statement (24) holds true ($\rho = \rho(\Phi_0) > 0$). By integrating (24) along H_r and extracting the real parts we obtain (see (35))

$$\ln\left|\frac{f(re^{i\psi})}{f(z)}\right| = \frac{|y|}{\rho}\big(r^\rho\cos(\rho\psi+\beta) - r^\rho\cos(\rho\varphi+\beta)\big) + \operatorname{Re}\int_{H_r} u(\zeta)\zeta^{\rho-1}d\zeta \geqslant$$
$$\geqslant \frac{|y|}{\rho}\big(r^\rho\cos(\rho\psi+\beta) - r^\rho\cos(\rho\varphi+\beta)\big) - \frac{\delta}{9}r^\rho(\psi-\varphi).$$

The latter statement together with (54) and (51) imply

$$\begin{aligned}\ln r^{\nu+\varepsilon} &\geqslant \frac{|y|}{\rho}\big(r^\rho\cos(\rho\psi+\beta) - r^\rho\cos(\rho\varphi+\beta)\big) - \frac{\delta}{9}r^\rho(\psi-\varphi)+\\ &+\frac{|y|}{\rho}(r^\rho - d^\rho)(\cos(\rho\varphi+\beta)+\tilde{q}(z)) + \ln|f(de^{i\varphi})| \geqslant \frac{|y|}{\rho}r^\rho\cos(\rho\psi+\beta)-\\ &-\frac{\delta}{9}r^\rho(\psi-\varphi) - \frac{|y|}{\rho}d^\rho\cos(\rho\varphi+\beta) - \frac{\delta}{9\rho}(r^\rho - d^\rho) + \ln|f(de^{i\varphi})|, \quad r \geqslant d.\end{aligned} \tag{55}$$

In (52) $\cos(\rho\tilde{\varphi}+\beta)>0$; thus choose $\delta>0$ so that

$$\frac{|y|}{\rho}\cos(\rho\tilde{\varphi}+\beta)-\frac{\delta}{9}(\psi-\varphi)-\frac{\delta}{9\rho}>0. \tag{56}$$

The segment ω_t^+ has an infinite length. Let d be such that for $|z|=r>d$ the relations (24), (51), (55) hold true with δ satisfying (56). If r is large enough, $r\geqslant d_1\geqslant d$, then from (52), (56) it follows that inequality (55) is false. Thus $\psi\geqslant\tilde{\varphi}$. This completes the proof of (53).

Denote by

$$\eta=\frac{2\pi b-\beta}{\rho}-\frac{\pi}{2\rho},\qquad \vartheta=\frac{2\pi b-\beta}{\rho}+\frac{\pi}{2\rho},\qquad b\in\mathbb{Z}, \tag{57}$$

where b is the number defined above. Similar to the proof of (53) one can prove that $\forall\,\epsilon>0\ \ \exists d:$

$$\tilde{P}=\{z=re^{i\theta}\colon \eta+\epsilon\leqslant\theta\leqslant\vartheta-\epsilon,\ r\geqslant d\}\subset\Phi_0. \tag{58}$$

Therefore, for an arbitrary $\theta\in[\eta+\epsilon,\vartheta-\epsilon]$ on the ray $S(\theta)=\{re^{i\theta}\colon\ r\geqslant d\}\subset\Phi_0$ the same conditions hold as those that gave the possibility to prove on $\{re^{i\varphi}\colon\ r\geqslant d\}\subset\Phi_0$ equality (51). Thus, $\forall\,\epsilon>0\ \ \forall\,\delta>0\ \ \exists\,d=d(\epsilon,\delta):$

$$\begin{gathered}\ln\frac{f(z)}{f(de^{i\theta})}=\frac{y}{\rho}(z^\rho-(de^{i\theta})^\rho)+q(re^{i\theta}),\quad |q(re^{i\theta})|<\frac{\delta}{9\rho}(r^\rho-d^\rho),\\ \ln\left|\frac{f(re^{i\theta})}{f(de^{i\theta})}\right|=\frac{|y|}{\rho}(r^\rho-d^\rho)(\cos(\rho\theta+\beta)+\tilde{q}(re^{i\theta})),\quad |\tilde{q}(re^{i\theta})|<\frac{\delta}{9|y|},\\ z=re^{i\theta}\in\tilde{P},\quad \cos(\rho\theta+\beta)>\tilde{c}>0,\quad \eta+\epsilon\leqslant\theta\leqslant\vartheta-\epsilon,\end{gathered} \tag{59}$$

$$\ln|f(re^{i\theta})|=\Big(\frac{|y|}{\rho}\cos(\rho\theta+\beta)+o(1)\Big)r^\rho,\ \ \cos(\rho\theta+\beta)>0,\ \ \theta\in(\eta,\vartheta), \tag{60}$$

$r\to+\infty$. Then (6) follows from the first equality in (59).

As it was mentioned above ρ, y,β, b,φ_j (see (24), (57)) take a finite amount of possible values. Thus in (57) for $\eta=\eta_j$, $\vartheta=\vartheta_j$ there also exists just a finite amount of possible values. Conclude that there exists only a finite number of intervals $(\eta_j,\,\vartheta_j)$, where the estimates similar to (59), (60) hold true.

For an arbitrary $\varphi,\ 0\leqslant\varphi<2\pi k$ consider a ray $S(\varphi)=\{re^{i\varphi}\colon\ r\geqslant r_0,\ \varphi=\text{const}\}\subset\Phi\cup\Phi_1\cup Q$, $Q\overset{(28)}{\subset}E_*$ a set of circles with a finite sum of radii, $\Phi=\Phi^\circ\cup\partial\Phi$ (see (31)). The set $\Phi^\circ\cap S(\varphi)$ is a union of disjunctive segments ω_t°. According to (43) $\{re^{i\varphi}\colon\ \varphi=\text{const},\ r>R\}\cap\omega_t^-=\varnothing$. If the ray $S(\varphi)$ contains the segment ω_t^+ of an infinite length then (51) holds where $\cos(\rho\varphi+\beta)\geqslant0$. If $\cos(\rho\varphi+\beta)=0$ then $\varphi=\varphi_j$ (see (48)) and (7) holds true. If $\cos(\rho\varphi+\beta)>0$ then (59) and (6) hold.

Suppose $S(\varphi)$ does not contain any segment ω_t^+ of infinite length. If φ takes one of finite amounts of values $\varphi=\varphi_j$ (48) then $|f(z)|<|z|^{\nu+\varepsilon}$ holds on the intersection of the ray $S=\{re^{i\varphi}\colon\ r\geqslant r(\varphi),\ \varphi=\text{const}\}\subset S(\varphi)$ with the set Φ_1 defined by (31), and (47) and (50) hold on the segments ω_t^+. This means that (7) holds true, $\rho=\max\rho_t$ where the maximum is taken over those ρ_j that corresponds to the segments $\omega_t\subset S(\varphi)$. If $\varphi\neq\varphi_j$ then $|f(z)|<|z|^{\nu+\varepsilon}$ and (47) are valid on the intersection of the ray S with the sets Φ_1, ω_t^+. This completes the proof of (8).

Denote

$$\begin{gathered}G(r)=\{se^{i\theta}\colon 0\leqslant\theta\leqslant2k\pi,\ r_0\leqslant s\leqslant r\}\setminus Q,\quad Q\subset E_*,\\ M_*(r,f)=\max|f(z)|,\ \ z\in G(r),\ \ |f(\varpi)|=M_*(r,f),\ \ \varpi\in G(r),\end{gathered} \tag{61}$$

where the open set Q is defined in (28), the set $G(r)$ is closed. If for some a, $M_*(r,f) < r^{\nu+2\varepsilon}$, $\forall\, r > a$, then we have (4). Suppose that

$$\forall\, a > 0 \ \ \exists\, r > a\colon \ M_*(r,f) = |f(\varpi)| \geqslant r^{\nu+2\varepsilon}. \tag{62}$$

The maximum of the absolute value of $f(z)$ on the closed set $G(r)$ is reached on the edge. Taking into account (36), $|f(z)| < |z|^{\nu+\varepsilon}$, $z \in \partial Q$, $|z| > R \ \Rightarrow \varpi \notin \partial Q$. Keeping in mind (38), $|\varpi| \neq r_0$. Thus if (62) holds then for the point ϖ at which the maximum of $|f(z)|$, $z \in G(r)$ is reached we have

$$|f(\varpi)| = M_*(r,f) \geqslant r^{\nu+2\varepsilon} \quad \Rightarrow \quad |\varpi| = r, \ \ \varpi \notin \partial Q. \tag{63}$$

That is why Macintyre formula holds ([4, p. 59–62], [10])

$$\frac{\varpi f'(\varpi)}{f(\varpi)} = \frac{rM_*'(r,f)}{M_*(r,f)} \geqslant 0, \quad |\varpi| = r, \tag{64}$$

$M_*'(r,f)$ is the right-side derivative of $M_*(r,f)$.

Taking into account (63), (31) the point $\varpi \in \Phi_0 \subset \Phi$ and in this point (24) holds. Thus

$$\frac{\varpi f'(\varpi)}{f(\varpi)} = (y + u(\varpi))\varpi^\rho \geqslant 0, \quad |u(\varpi)| < \frac{\delta}{9}, \quad \varpi = re^{i\varphi(r)} \in \Phi_0, \quad r > a \geqslant r_0,$$

$y = |y|e^{i\beta} \neq 0$, $\delta > 0$ is small. By the reasonings described right above, the following asymptotic relation for the argument holds

$$\rho\varphi(r) + \beta + o(1) = 2m\pi, \ m \in \mathbb{Z}, \ r \to \infty, \ \cos(\rho\varphi(r) + \beta) = 1 + o(1), \ \cos(\rho\varphi(r) + \beta) > \tfrac{1}{2}, \tag{65}$$

m takes a finite amount of values ($0 \leqslant \varphi(r) \leqslant 2k\pi$). Since $\varpi \in \Phi_0$, we have that $\varpi \in \omega_t^-$ or $\varpi \in \omega_t^+$. From (43) for $|\varpi| = r > R$, it follows that $\varpi \notin \omega_t^-$. Thus $\varpi \in \omega_t^+$. If ω_t^+ is a segment of finite length then (45) is valid for $\varphi = \varphi(r)$. According to (65) we have $\cos(\rho\varphi(r) + \beta) > \frac{1}{2}$. Thus from (45) we have $(\nu + \varepsilon)\ln\frac{r_{2t}}{r_{1t}} = (r_{2t}^\rho - r_{1t}^\rho)\frac{|y|}{\rho}(\cos(\rho\varphi(r) + \beta) + \tilde{q}(z_{2t})) > (r_{2t}^\rho - r_{1t}^\rho)\frac{|y|}{\rho}(\frac{1}{2} + \tilde{q}(z_{2t})) > (r_{2t}^\rho - r_{1t}^\rho)\frac{|y|}{4\rho}$, $|\tilde{q}(z_{2t})| < \frac{\delta}{9|y|} < \frac{1}{4}$, or

$$c(\ln x_2 - \ln x_1) > x_2 - x_1, \quad x_1 = r_{1t}^\rho < x_2 = r_{2t}^\rho, \quad c = \frac{2(\nu + \varepsilon)}{|y|}. \tag{66}$$

The function $x - c\ln x$ increases on $(c, +\infty)$. That is why (66) does not hold if $x_1 = r_{1t}^\rho > c = \frac{2(\nu+\varepsilon)}{|y|}$. If $x_1 = r_{1t}^\rho < c$, $x_2 = r_{2t}^\rho > c$ then from (66) it follows that ($1 < r_0 \leq r_{1t}$) $c\ln x_2 > x_2 - c$, $c = \frac{2(\nu+\varepsilon)}{|y|}$; this is true if $x_2 < x_* = \text{const}$. Thus if $|\omega| = r > \max\bigl(R, \sqrt[\rho]{2(\nu+\varepsilon)/|y|}, \sqrt[\rho]{x_*}\bigr)$, $\omega \in \omega_t^+$, then this segment ω_t^+ cannot have finite length. Thus if $|\varpi| = r > \max\bigl(R, \sqrt[\rho]{2(\nu+\varepsilon)/|y|}, \sqrt[\rho]{x_*}\bigr)$, $\omega \in \omega_t^+$, then $\varpi \in \omega_t^+$ a segment of an infinite length where (51) holds. Then (60) is also true. Thus, for the point $\varpi = re^{i\varphi(r)}$ at which the maximum of the absolute value is reached, the argument $\varphi(r)$ belongs to the union of a finite amount of segments (η_j, ϑ_j), where estimates similar to (59), (60) hold true. Thus after substituting $\theta = \varphi(r)$ into (60) (or (59)) we obtain as $r \to \infty$

$$M_*(r,f) = |f(\varpi)| = |f(re^{i\varphi(r)})| \overset{(60)}{=} \left(\frac{|y|}{\rho}\cos(\rho\varphi(r) + \beta) + o(1)\right)r^\rho \overset{(65)}{=} \left(\frac{|y|}{\rho} + o(1)\right)r^\rho.$$

From here we get (5).

Take arbitrary $\phi,\ \psi,\ \eta < \phi < \psi < \vartheta$ (see (57)). From (59) it follows that the estimate (60) is uniform in θ, $\phi \leqslant \theta \leqslant \psi$ and

$$\cos(\rho\theta + \beta) > \tilde{c} > 0, \quad \phi \leqslant \theta \leqslant \psi. \tag{67}$$

Assume that $\rho < \frac{1}{2k}$. Then taking into account (57), $\vartheta - \eta = \frac{\pi}{\rho} > 2k\pi$ and numbers $\phi,\ \psi,\ \eta < \phi < \psi < \vartheta$ can be taken in the way that

$$\psi = \phi + 2k\pi, \quad \sin(\rho\phi + \beta + k\pi\rho) \neq 0. \tag{68}$$

Since $f(re^{i\phi}) = f(re^{i(\phi+2k\pi)})$, from (60), (68) if follows

$$\begin{aligned} 0 = \ln|f(re^{i\psi})| - \ln|f(re^{i\phi})| &= \big(\cos(\rho\phi + \rho 2k\pi + \beta) - \cos(\rho\phi + \beta) + o(1)\big)\tfrac{|y|}{\rho}r^\rho = \\ &= \big(-2(\sin\rho k\pi)\sin(\rho\phi + \beta + k\pi\rho) + o(1)\big)\tfrac{|y|}{\rho}r^\rho. \end{aligned} \tag{69}$$

By assumption $\rho < \frac{1}{2k}$ so $0 < \rho k\pi < \frac{\pi}{2}$, thus $\sin\rho k\pi \neq 0$. From this statement and from (68) it follows $(\sin\rho k\pi)\sin(\rho\phi + \beta + k\pi\rho) \neq 0$ which contradicts (69). Thus

$$\rho \geqslant 1/2k. \tag{70}$$

From the condition $\eta < \phi < \psi < \vartheta$ and from (57), we have $\psi - \phi < \vartheta - \eta = \frac{\pi}{\rho} \overset{(70)}{\leqslant} 2k\pi$. Thus

$$m(r,f) \geqslant \frac{1}{2\pi k}\int_\phi^\psi \ln^+|f(re^{i\theta})|d\theta \overset{(60)}{=} \frac{1}{2\pi k}\int_\phi^\psi\left(\frac{|y|}{\rho}\cos(\rho\theta + \beta) + o(1)\right)r^\rho d\theta \overset{(67)}{\geqslant} \frac{\tilde{c}(\psi - \phi)|y|}{\rho 4\pi k}r^\rho,$$

$r > d$. This together with (70) gives the fact that the characteristic $m(r,f)$ has the growth order $\rho \geqslant \frac{1}{2k}$. If in (70) $\rho = \frac{1}{2k}$ then $\vartheta - \eta = \frac{\pi}{\rho} = 2k\pi$ and from (59), (60) we obtain (9). Estimate (10) on the ray follows from (7). $\square$

REFERENCES

1. Valiron G. Analytical functions. – Nauka, Moscow, 1957. – 235 p. (in Russian)
2. Zimoglyad V.V. *On the order of growth of transcendental entire solutions of algebraic differential equations of second order*// Mat. Sb. – V.85(127), №2(6). – P. 286–302. (in Russian)
3. Goldber A.A., Ostrovsky I.V., Value distribution of meromorphic functions. – Nauka, Moscow, 1970. (in Russian)
4. Strelitz Sh.I. Asymptotic properties of analytic solutions of differential equations. – Mintis, Vilnius, 468 p. (in Russian)
5. Markushevich A.I. Theory of analytic functions. – Nauka, Moskow. (in Russian)
6. Mokhon'ko A.Z., Mokhon'ko V.D. *On order of growth of analytic solutions for algebraic differential equations having logarithmic singularity*// Mat. Stud. – 2000. – V.13, №2. – P. 203–218.
7. Mathematical encyclopedia in five volumes. – Soviet encyclopedia, Moscow, 1977, V.1. (in Russian)
8. Mokhon'ko A.Z. *On the growth of meromorphic solutions of algebraic differential equations in an angular domain*// Sib. Math. J. – 1990. – V.31, №2. – P. 123–130. (in Russian)
9. Kolmogorov A.N., Fomin S.V. Elements of the theory of functions and functional analysis. – Nauka, Moscow, 1972. – 496 p. (in Russian)
10. Macintyre A.J. *Wimans method and "flat regions" of integral functions*// Quart. J. Math. – 1938. – V.9. – P. 81–88.
11. Mokhon'ko A.Z. *Estimates of absolute value of the logarithmic derivative of the function meromorphic in an anngular domain and its application*// Ukr. Math. J. – 1989. – V.41, №6. – P. 839–843. (in Russian)

National University "Lvivska Politechnika"
mohon@yandex.ru

Received 11.04.2013