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BAR AND COBAR CONSTRUCTIONS FOR CURVED ALGEBRAS AND COALGEBRAS

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We provide bar and cobar constructions as functors between some categories of curved algebras and curved augmented coalgebras over a graded commutative ring. These functors are adjoint to each other.

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Мы рассматриваем бар и кобар конструкции как функторы между некоторыми категориями кривых алгебр и кривых увеличенных коалгебр над градуированным коммутативным кольцом. Эти функторы сопряжены друг с другом.

In this paper we recall some notions and reproduce some results from Positselski [5, 6] in a modified form. Our exposition differs in two aspects: firstly, we work over a graded commutative ring $\Bbbk$ instead of a field or a topological local ring, secondly, we modify the definitions of categories of curved algebras and curved coalgebras.

The advantage of using graded commutative rings over usual commutative rings is that it allows to place (co)derivations of certain degree on equal footing with (co)algebra homomorphisms. Take note of the last condition in the following definition.

Definition 1. A *graded strongly commutative ring* is a graded ring $\Bbbk$ such that $ba = (-1)^{|a|\cdot|b|}ab$ for all homogeneous elements a, b and $c^2 = 0$ for all elements c of an odd degree.

The first condition implies only that $2c^2 = 0$ for elements c of an odd degree.

We give explicit formulae and detailed proofs. Motivations come from A_∞-algebras and A_∞-coalgebras.

For any graded $\Bbbk$-module M and an integer a denote by $M[a]$ the same module with the grading shifted by a: $M[a]^k = M^{a+k}$. Denote by $\sigma^a\colon M \to M[a]$, $M^k \ni x \mapsto x \in M[a]^{k-a}$ the "identity map" of degree $\deg\sigma^a = -a$. Write elements of $M[a]$ as $m\sigma^a$. Typically, a map is written on the right of its argument. The composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is denoted by $f \cdot g\colon X \to Z$ or simply by fg. If $f\colon V \to X$ is a homogeneous map of certain degree, the map $f[a]\colon V[a] \to X[a]$ is defined as $f[a] = (-1)^{a\deg f}\sigma^{-a}f\sigma^a = (-1)^{af}\sigma^{-a}f\sigma^a$. The tensor

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product of homogeneous maps f, g between graded $\Bbbk$-modules is defined at elements x, y of a certain degree as

$$(x \otimes y).(f \otimes g) = (-1)^{\deg y \cdot \deg f} x.f \otimes y.g.$$

Thus, the Koszul sign rule holds and we deal in the closed symmetric monoidal category **gr** of graded $\Bbbk$-modules with the symmetry $x \otimes y \mapsto (-1)^{\deg x \cdot \deg y} y \otimes x$.

1. Curved (co)algebras. We define curved algebras and curved coalgebras as well as their morphisms are suitable for our purposes.

1.1. Curved algebras. We begin with curved algebras of various kinds.

Definition 2. A *strict-unit-complemented curved A_∞-algebra* $(A, (b_n)_{n\geqslant 0}, \boldsymbol{\eta}, \mathbf{v})$ consists of a graded $\Bbbk$-module A, degree 1 maps $b_n \colon A[1]^{\otimes n} \to A[1]$ (operations) for $n \geqslant 0$, a degree -1 map $\boldsymbol{\eta} \colon \Bbbk \to A[1]$ (strict unit) and a degree 1 map $\mathbf{v} \colon A[1] \to \Bbbk$ (splitting of the unit) such that

$$\sum_{r+k+t=n} (1^{\otimes r} \otimes b_k \otimes 1^{\otimes t}) b_{r+1+t} = 0 \colon A[1]^{\otimes n} \to A[1], \quad \forall n \geqslant 0, \tag{1}$$

$$(1 \otimes \boldsymbol{\eta}) b_2 = 1_{A[1]}, \ (\boldsymbol{\eta} \otimes 1) b_2 = -1_{A[1]}, \ (1^{\otimes a} \otimes \boldsymbol{\eta} \otimes 1^{\otimes c}) b_{a+1+c} = 0 \ \text{ if } \ a + c \neq 1, \ \boldsymbol{\eta} \cdot \mathbf{v} = 1_\Bbbk.$$

For any graded $\Bbbk$-module X the tensor $\Bbbk$-module $XT^{\geqslant} = \oplus_{n\geqslant 0} X^{\otimes n}$ is equipped with the cut coproduct

$$(x_1 \cdots x_n)\Delta = \sum_{k=0}^{n} x_1 \cdots x_k \otimes x_{k+1} \cdots x_n.$$

The collection $\check{b} = (b_n)_{n\geqslant 0} \colon A[1]T^{\geqslant} \to A[1]$ amounts to a degree 1 coderivation $b \colon A[1]T^{\geqslant} \to A[1]T^{\geqslant}$ of the counital coassociative coalgebra $A[1]T^{\geqslant}$,

$$b| = \sum_{r+k+t=n} 1^{\otimes r} \otimes b_k \otimes 1^{\otimes t} \colon A[1]^{\otimes n} \to A[1]T^{\geqslant}.$$

Equation (1) is equivalent to $b^2 = 0$.

Getting rid of the shift [1] we rewrite the above operations as in [3, (0.7)]

$$m_n = (-1)^n \sigma^{\otimes n} \cdot b_n \cdot \sigma^{-1} \colon A^{\otimes n} \to A, \quad \deg m_n = 2 - n, \quad n \geqslant 0,$$

$$\eta = \bigl(\Bbbk \xrightarrow{\boldsymbol{\eta}} A[1] \xrightarrow{\sigma^{-1}} A\bigr), \quad \deg \eta = 0, \ \mathsf{v} = \bigl(A \xrightarrow{\sigma} A[1] \xrightarrow{\mathbf{v}} \Bbbk\bigr), \quad \deg \mathsf{v} = 0.$$

In these terms Definition 2 becomes the following one.

Definition 3. A *strict-unit-complemented curved* A_∞*-algebra* $(A, (m_n)_{n\geqslant 0}, \eta, \mathsf{v})$ consists of a graded $\Bbbk$-module A, maps $m_n \colon A^{\otimes n} \to A$ of degree $2 - n$ (operations) for $n \geqslant 0$, a degree 0 map $\eta \colon \Bbbk \to A$ (strict unit) and a degree 0 map $\mathsf{v} \colon A \to \Bbbk$ (splitting of the unit) such that

$$\sum_{j+p+q=n} (-1)^{jp+q} (1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}) \cdot m_{j+1+q} = 0 \colon A^{\otimes n} \to A, \quad \forall n \geqslant 0, \tag{2}$$

$$(1 \otimes \eta) m_2 = 1_A, \ (\eta \otimes 1) m_2 = 1_A, \ (1^{\otimes a} \otimes \eta \otimes 1^{\otimes c}) m_{a+1+c} = 0 \ \text{ if } \ a + c \neq 1, \ \eta \cdot \mathsf{v} = 1_\Bbbk.$$

Restricting the above notion we give the following definition.

Definition 4. A *unit-complemented curved algebra* $(A, m_2, m_1, m_0, \eta, \mathsf{v})$ is a strict-unit-complemented curved A_∞-algebra A with the strict unit η and with $m_n = 0$ for $n > 2$.

For such an algebra A equations (2) reduce to the system

$$(1\otimes m_2)m_2=(m_2\otimes 1)m_2,\ m_2m_1=(1\otimes m_1+m_1\otimes 1)m_2,\ m_1^2=(m_0\otimes 1-1\otimes m_0)m_2,$$
$$m_0m_1=0,\ (1\otimes\eta)m_2=1,\ (\eta\otimes 1)m_2=1,\ \eta m_1=0,\ \eta\mathsf{v}=1_{\Bbbk},$$

which tells that A is a unital associative graded algebra (A,m_2,η) of degree 1 derivation m_1, whose square is an inner derivation, that is, a commutator with an element m_0 (curvature) of degree 2 and $m_0m_1=0$. A direct complement $\overline{A}=\operatorname{Ker}\mathsf{v}$ to the $\Bbbk$-submodule $\eta\colon\Bbbk\hookrightarrow A$ is chosen.

The following example of a unit-complemented curved algebra was considered by Positselski in [5, Section 0.6], see also [4].

Example 1. Let M be a smooth manifold, let $E\to M$ be a smooth vector bundle, $\Bbbk=\mathbb{R}$. Denote $\Omega^k(E)=\Gamma(E\otimes\wedge^kT^*M)$, $k\in\mathbb{N}$. Let $\nabla\colon\Omega^0(E)\to\Omega^1(E)$ be a connection on E which is viewed as a covariant exterior derivative $\nabla\colon\Omega^k(E)\to\Omega^{k+1}(E)$ such that

$$\forall\,\tau\in\Omega^\bullet(E)\quad\forall\,\omega\in\Omega^\bullet(M)\quad(\tau\omega)\nabla=(-1)^\omega(\tau\nabla)\cdot\omega+\tau\cdot(\omega d).$$

The category of vector bundles on M is Cartesian closed. The evaluation map $\mathrm{ev}\colon E\times\operatorname{End}E\to E$ leads to the action $\Omega^k(E)\otimes\Omega^n(\operatorname{End}E)\to\Omega^{k+n}(E)$. Moreover, elements $h\in A^n=\Omega^n(\operatorname{End}E)$ can be identified with $\Omega^\bullet(M)$-linear maps $h\colon\Omega^k(E)\to\Omega^{k+n}(E)$, thus, $(\tau\omega)h=(-1)^{n|\omega|}(\tau h)\omega$. For instance, the curvature 2-form $-m_0=\nabla^2$ is a $\Omega^\bullet(M)$-linear map, hence an element of $\Omega^2(\operatorname{End}E)$.

The graded algebra $A^\bullet=\Omega^\bullet(\operatorname{End}E)$ equipped with the derivation $(h)d_A=h\cdot\nabla-(-1)^h\nabla\cdot h$ (which is a covariant exterior derivative on the vector bundle $\operatorname{End}E$) and with the curvature element $m_0\in A^2$ is a curved algebra since $(h)d_A^2=m_0h-hm_0$, $(m_0)d_A=0$. The latter equation is the Bianchi identity.

A morphism between curved A_∞-algebras A and B should be given by a family of components $f_n\colon A[1]^{\otimes n}\to B[1]$, $n\geqslant 0$. The obtained matrix entries

$$f_n^k=\sum_{i_1+\dots+i_k=n}f_{i_1}\otimes f_{i_2}\otimes\dots\otimes f_{i_k}\colon A[1]^{\otimes n}\to B[1]^{\otimes k}$$

define a map $f\colon A[1]T^{\geqslant}\to B[1]\hat{T}^{\geqslant}$, which in general does not factor through $B[1]T^{\geqslant}$. The equation $fb=bf$, which we write as

$$\sum_{i_1+\dots+i_k=n}(f_{i_1}\otimes f_{i_2}\otimes\dots\otimes f_{i_k})b_k^B=\sum_{r+k+t=n}(1^{\otimes r}\otimes b_k^A\otimes 1^{\otimes t})f_{r+1+t},$$

also makes sense under some additional assumptions (like extra filtration [2] or topological structure of $\Bbbk$ [6]). We shall consider only curved algebras B, which insures that the sum in the left hand side is finite. Moreover, we assume that components f_n vanish for $n>1$ and f_0 is of the form

$$f_0=\bigl(\Bbbk\xrightarrow{\underline{f}}\Bbbk\xrightarrow{\boldsymbol{\eta}}B[1]\bigr),\tag{3}$$

where $\deg\underline{f}=1$. The latter assumption was made in order to deal with augmented coalgebras in bar and cobar constructions, which does not exclude that similar results could be obtained under weaker assumptions.

Definition 5. A *morphism of unit-complemented curved algebras* $f\colon A \to B$ is a pair $(f_1, \underline{f})$ consisting of $\Bbbk$-linear maps $f_1\colon A[1] \to B[1]$ of degree 0 and $\underline{f}\colon \Bbbk \to \Bbbk$ of degree 1 such that

$$(f_1 \otimes f_1)b_2^B = b_2^A f_1, \quad f_1 b_1^B = b_1^A f_1, \quad b_0^B = b_0^A f_1, \quad \boldsymbol{\eta}_A f_1 = \boldsymbol{\eta}_B. \tag{4}$$

The composition $h\colon A \to C$ of morphisms $f\colon A \to B$ and $g\colon B \to C$ is $h_1 = f_1 g_1$, $\underline{h} = \underline{g} + \underline{f}$.

Under assumption (3) the expected conditions

$$f_1 b_1^B + (f_0 \otimes f_1)b_2^B + (f_1 \otimes f_0)b_2^B = b_1^A f_1,\ b_0^B + f_0 b_1^B + (f_0 \otimes f_0)b_2^B = b_0^A f_1,\ h_0 = g_0 + f_0 g_1$$

reduce to the given ones. In fact, $\underline{f} \in \Bbbk^1$ implies $\underline{f}^2 = 0$ due to graded commutativity of $\Bbbk$, see Definition 1.

The last equation of (4) tells that f_1 preserves the unit. These equations can be rewritten for conventional $\Bbbk$-linear maps

$$\mathsf{f}_1 = \bigl(A \xrightarrow{\sigma} A[1] \xrightarrow{f_1} B[1] \xrightarrow{\sigma^{-1}} B\bigr), \quad \deg \mathsf{f}_1 = 0,$$
$$\mathsf{f}_0 = \bigl(\Bbbk \xrightarrow{f_0} B[1] \xrightarrow{\sigma^{-1}} B\bigr) = \bigl(\Bbbk \xrightarrow{\underline{f}} \Bbbk \xrightarrow{\eta} B\bigr), \quad \deg \mathsf{f}_0 = 1,$$

as follows.

Definition 6. A *morphism of unit-complemented curved algebras* $f\colon A \to B$ is a pair $(\mathsf{f}_1, \underline{f})$ consisting of $\Bbbk$-linear maps $\mathsf{f}_1\colon A \to B$ of degree 0 and $\underline{f}\colon \Bbbk \to \Bbbk$ of degree 1 such that

$$(\mathsf{f}_1 \otimes \mathsf{f}_1)m_2^B = m_2^A \mathsf{f}_1, \quad \mathsf{f}_1 m_1^B = m_1^A \mathsf{f}_1, \quad m_0^B = m_0^A \mathsf{f}_1, \quad \eta^A \mathsf{f}_1 = \eta^B.$$

The composition $\mathsf{h}\colon A \to C$ of morphisms $\mathsf{f}\colon A \to B$ and $\mathsf{g}\colon B \to C$ is $\mathsf{h}_1 = \mathsf{f}_1 \mathsf{g}_1$, $\underline{h} = \underline{g} + \underline{f}$. The unit morphism is $(\mathrm{id}, 0)$. The category of unit-complemented curved algebras is denoted UCCAlg.

In particular, $\mathsf{f}_1\colon A \to B$ is a morphism of unital associative graded algebras.

1.2. Curved coalgebras. Now we define curved coalgebras of various kinds.

Definition 7. A *strict-counit-complemented curved A_∞-coalgebra* $(C, (\xi_n)_{n\geqslant 0}, \boldsymbol{\varepsilon}, \mathbf{w})$ consists of a graded $\Bbbk$-module C, degree 1 maps $\xi_n\colon C[-1] \to C[-1]^{\otimes n}$ (cooperations) for $n \geqslant 0$, a degree -1 map $\boldsymbol{\varepsilon}\colon C[-1] \to \Bbbk$ (strict counit) and a degree 1 map $\mathbf{w}\colon \Bbbk \to C[-1]$ (splitting of the counit) such that

$$\sum_{r+k+t=n} \xi_{r+1+t}(1^{\otimes r} \otimes \xi_k \otimes 1^{\otimes t}) = 0\colon C[-1] \to C[-1]^{\otimes n}, \quad \forall n \geqslant 0, \tag{5}$$
$$\xi_2(1 \otimes \boldsymbol{\varepsilon}) = -1_{C[-1]}, \quad \xi_2(\boldsymbol{\varepsilon} \otimes 1) = 1_{C[-1]}, \quad \xi_{a+1+c}(1^{\otimes a} \otimes \boldsymbol{\varepsilon} \otimes 1^{\otimes c}) = 0 \ \text{ if } \ a + c \neq 1,$$
$$\mathbf{w} \cdot \boldsymbol{\varepsilon} = 1_\Bbbk, \quad \mathbf{w}\xi_2 = -\mathbf{w} \otimes \mathbf{w}.$$

For any graded $\Bbbk$-module X its tensor algebra $XT^{\geqslant} = \oplus_{n\geqslant 0} X^{\otimes n}$ is naturally embedded into its completed tensor algebra $X\hat{T}^{\geqslant} = \prod_{n\geqslant 0} X^{\otimes n}$, $\iota\colon XT^{\geqslant} \hookrightarrow X\hat{T}^{\geqslant}$. An arbitrary ι-derivation $\xi\colon XT^{\geqslant} \to X\hat{T}^{\geqslant}$ is determined by its restriction to generators $\check{\xi}\colon X \to X\hat{T}^{\geqslant}$. In particular, the collection $(\xi_n)_{n\geqslant 0}$ amounts to a degree 1 ι-derivation $\xi\colon C[-1]T^{\geqslant} \to C[-1]\hat{T}^{\geqslant}$ and equations (5) can be interpreted as $\xi^2 = 0$.

Getting rid of the shift $[-1]$ we rewrite the above via maps

$$\delta_n = (-1)^n \sigma^{-1} \cdot \xi_n \cdot \sigma^{\otimes n} \colon C \to C^{\otimes n}, \quad \deg \delta_n = 2 - n, \quad n \geqslant 0,$$
$$\varepsilon = \big(C \xrightarrow{\sigma^{-1}} C[-1] \xrightarrow{\boldsymbol{\varepsilon}} \Bbbk\big), \quad \deg \varepsilon = 0, \quad \mathsf{w} = \big(\Bbbk \xrightarrow{\mathbf{w}} C[-1] \xrightarrow{\sigma} C\big), \quad \deg \mathsf{w} = 0.$$

In these terms Definition 7 becomes the following one.

Definition 8. A *strict-counit-complemented curved* A_∞*-coalgebra* $(C, (\delta_n)_{n\geqslant 0}, \varepsilon, \mathsf{w})$ consists of a graded $\Bbbk$-module C, maps $\delta_n \colon C \to C^{\otimes n}$ of degree $2-n$ (cooperations) for $n \geqslant 0$, a degree 0 map $\varepsilon \colon C \to \Bbbk$ (strict counit) and a degree 0 map $\mathsf{w} \colon \Bbbk \to C$ (splitting of the counit) such that

$$\sum_{r+k+t=n} (-1)^{r+kt} \delta_{r+1+t}(1^{\otimes r} \otimes \delta_k \otimes 1^{\otimes t}) = 0 \colon C \to C^{\otimes n}, \quad \forall n \geqslant 0, \tag{6}$$
$$\delta_2(1 \otimes \varepsilon) = 1_C, \quad \delta_2(\varepsilon \otimes 1) = 1_C, \quad \delta_{a+1+c}(1^{\otimes a} \otimes \varepsilon \otimes 1^{\otimes c}) = 0 \;\text{ if }\; a + c \neq 1,$$
$$\mathsf{w} \cdot \varepsilon = 1_\Bbbk, \quad \mathsf{w}\delta_2 = \mathsf{w} \otimes \mathsf{w}.$$

Restricting the above notion and adding a conilpotency condition we get the following definition.

Definition 9. A *curved augmented coalgebra* $(C, \delta_2, \delta_1, \delta_0, \varepsilon, \mathsf{w})$ is a strict-counit-complemented curved A_∞-coalgebra C with $\delta_n = 0$ for $n > 2$ such that $(\overline{C} = \operatorname{Ker} \varepsilon, \overline{\delta}_2)$ is conilpotent.

For such a coalgebra C equations (6) reduce to the system

$$\delta_2(1 \otimes \delta_2) = \delta_2(\delta_2 \otimes 1),\; \delta_1\delta_2 = \delta_2(1 \otimes \delta_1 + \delta_1 \otimes 1),\; \delta_1^2 = \delta_2(1 \otimes \delta_0 - \delta_0 \otimes 1),\; \delta_1\delta_0 = 0,$$
$$\delta_2(1 \otimes \varepsilon) = 1_C,\; \delta_2(\varepsilon \otimes 1) = 1_C,\; \delta_1\varepsilon = 0,\; \mathsf{w} \cdot \varepsilon = 1_\Bbbk,\; \mathsf{w}\delta_2 = \mathsf{w} \otimes \mathsf{w},$$

which tells that C is a counital coassociative graded coalgebra $(C, \delta_2, \varepsilon)$ of degree 1 coderivation δ_1, whose square is an inner coderivation determined by a functional $\delta_0 \colon C \to \Bbbk$ (curvature) of degree 2 and $\delta_1\delta_0 = 0$. The degree 0 map $\mathsf{w} \colon \Bbbk \to C$ is a homomorphism of graded coalgebras, the augmentation of C. In particular, $\Bbbk\mathsf{w} \hookrightarrow C$ is a direct complement to $\overline{C} = \operatorname{Ker} \varepsilon$. The non-counital graded coalgebra $\overline{C}$ equipped with the comultiplication $\overline{\delta}_2 = \delta_2 - 1 \otimes \mathsf{w} - \mathsf{w} \otimes 1 \colon \overline{C} \to \overline{C} \otimes \overline{C}$ is conilpotent by assumption, that is,

$$\bigcup_{n>1} \operatorname{Ker}(\overline{\Delta}^{(n)} \colon \overline{C} \to \overline{C}^{\otimes n}) = \overline{C}.$$

A morphism of curved A_∞-coalgebras $g \colon C \to D$ should be a **dg**-algebra morphism $g \colon C[-1]T^{\geqslant} \to D[-1]\hat{T}^{\geqslant}$, or, equivalently, a family of $\Bbbk$-linear degree 0 maps $g_n \colon C[-1] \to D[-1]^{\otimes n}$, $n \geqslant 0$, satisfying the equation $g\xi = \xi g$. However, to give sense to this equation in the form

$$\sum_{r+k+t=n} g_{r+1+t}(1^{\otimes r} \otimes \xi_k \otimes 1^{\otimes t}) = \sum_{i_1+\dots+i_k=n} \xi_k(g_{i_1} \otimes g_{i_2} \otimes \dots \otimes g_{i_k}) \colon C[-1] \to D[-1]^{\otimes n},$$

one has to make additional assumptions. We shall assume that C is a curved coalgebra and g_n vanish for $n > 1$. Moreover, we assume that g_1 preserves the splitting $\mathbf{w}$.

Definition 10. A *morphism of curved augmented coalgebras* $g\colon C \to D$ is a pair (g_1, g_0) consisting of $\Bbbk$-linear maps $g_1\colon C[-1] \to D[-1]$ and $g_0\colon C[-1] \to \Bbbk$ of degree 0 such that

$$\xi_2^C(g_1 \otimes g_1) = g_1\xi_2^D, \quad \xi_1^C g_1 + \xi_2^C(g_0 \otimes g_1 + g_1 \otimes g_0) = g_1\xi_1^D,$$
$$\xi_0^C + \xi_1^C g_0 + \xi_2^C(g_0 \otimes g_0) = g_1\xi_0^D, \quad g_1\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon}^C, \quad \mathbf{w}^C g_1 = \mathbf{w}^D.$$

The composition $h\colon C \to E$ of morphisms $f\colon C \to D$ and $g\colon D \to E$ is given by $h_1 = f_1g_1$, $h_0 = f_0 + f_1g_0$.

Rewriting this definition in terms of maps

$$\mathsf{g}_1 = \big(C \xrightarrow{\sigma^{-1}} C[-1] \xrightarrow{g_1} D[-1] \xrightarrow{\sigma} D\big), \quad \deg \mathsf{g}_1 = 0,$$
$$\mathsf{g}_0 = \big(C \xrightarrow{\sigma^{-1}} C[-1] \xrightarrow{g_0} \Bbbk\big), \quad \deg \mathsf{g}_0 = 1,$$

we give the following definition.

Definition 11. A *morphism of curved augmented coalgebras* $\mathsf{g}\colon C \to D$ is a pair $(\mathsf{g}_1, \mathsf{g}_0)$ consisting of $\Bbbk$-linear maps $\mathsf{g}_1\colon C \to D$ of degree 0 and $\mathsf{g}_0\colon C \to \Bbbk$ of degree 1 such that

$$\delta_2^C(\mathsf{g}_1 \otimes \mathsf{g}_1) = \mathsf{g}_1\delta_2^D, \quad \delta_1^C\mathsf{g}_1 + \delta_2^C(\mathsf{g}_0 \otimes \mathsf{g}_1 - \mathsf{g}_1 \otimes \mathsf{g}_0) = \mathsf{g}_1\delta_1^D,$$
$$\delta_0^C - \delta_1^C\mathsf{g}_0 - \delta_2^C(\mathsf{g}_0 \otimes \mathsf{g}_0) = \mathsf{g}_1\delta_0^D, \quad \mathsf{g}_1\varepsilon^D = \varepsilon^C, \quad \mathsf{w}^C\mathsf{g}_1 = \mathsf{w}^D.$$

The composition $\mathsf{h}\colon C \to E$ of morphisms $\mathsf{f}\colon C \to D$ and $\mathsf{g}\colon D \to E$ is given by $\mathsf{h}_1 = \mathsf{f}_1\mathsf{g}_1$, $\mathsf{h}_0 = \mathsf{f}_0 + \mathsf{f}_1\mathsf{g}_0$. The unit morphism is $(\mathrm{id}, 0)$. The category of curved augmented coalgebras is denoted CACoalg.

In particular, g_1 is a morphism of augmented graded coalgebras. Actually, g_0 occurs in the equations only as its restriction $\mathsf{g}_0' = \mathsf{g}_0|_{\overline{C}}$ and validity of the equations does not depend on $\underline{g} = \mathsf{w}\mathsf{g}_0 \in \Bbbk^1$. In fact, with the notation $\overline{\delta}_2^C = \big(\overline{C} \hookrightarrow C \xrightarrow{\delta_2} C \otimes C \xrightarrow{\overline{\mathrm{pr}}_C \otimes \overline{\mathrm{pr}}_C} \overline{C} \otimes \overline{C}\big)$, we have $\mathsf{w}\delta_2^C(\mathsf{g}_0 \otimes 1 - 1 \otimes \mathsf{g}_0) = (\mathsf{w}\mathsf{g}_0)\mathsf{w} - \mathsf{w}(\mathsf{w}\mathsf{g}_0) = 0$, which implies that

$$\delta_2^C(\mathsf{g}_0 \otimes 1 - 1 \otimes \mathsf{g}_0) = \overline{\mathrm{pr}}_C(\overline{\delta}_2^C + 1 \otimes \mathsf{w} + \mathsf{w} \otimes 1)(\mathsf{g}_0 \otimes 1 - 1 \otimes \mathsf{g}_0) = \overline{\mathrm{pr}}_C\, \overline{\delta}_2^C(\mathsf{g}_0 \otimes 1 - 1 \otimes \mathsf{g}_0). \tag{7}$$

Since $\mathsf{w}\delta_2^C(\mathsf{g}_0 \otimes \mathsf{g}_0) = (\mathsf{w}\mathsf{g}_0)^2 = 0$, we find that

$$\delta_2^C(\mathsf{g}_0 \otimes \mathsf{g}_0) = \overline{\mathrm{pr}}_C(\overline{\delta}_2^C + 1 \otimes \mathsf{w} + \mathsf{w} \otimes 1)(\mathsf{g}_0 \otimes \mathsf{g}_0) = \overline{\mathrm{pr}}_C\, \overline{\delta}_2^C(\mathsf{g}_0 \otimes \mathsf{g}_0).$$

Thus Definition 11 can be reformulated as follows.

Definition 12. A *morphism of curved augmented coalgebras* $\mathsf{g}\colon C \to D$ is a triple $(\mathsf{g}_1, \mathsf{g}_0', \underline{g})$ consisting of a homomorphism of augmented graded coalgebras $\mathsf{g}_1\colon C \to D$, a $\Bbbk$-linear map $\mathsf{g}_0'\colon \overline{C} \to \Bbbk$ of degree 1 and an element $\underline{g} \in \Bbbk^1$ (of degree 1) such that

$$\delta_1^C\mathsf{g}_1 + \overline{\mathrm{pr}}_C\, \overline{\delta}_2^C(\mathsf{g}_0' \otimes \mathsf{g}_1 - \mathsf{g}_1 \otimes \mathsf{g}_0') = \mathsf{g}_1\delta_1^D\colon C \to \overline{D},$$
$$\delta_0^C - \delta_1^C\mathsf{g}_0' - \overline{\mathrm{pr}}_C\, \overline{\delta}_2^C(\mathsf{g}_0' \otimes \mathsf{g}_0') = \mathsf{g}_1\delta_0^D\colon C \to \Bbbk.$$

The composition $\mathsf{h}\colon C \to E$ of morphisms $\mathsf{f}\colon C \to D$ and $\mathsf{g}\colon D \to E$ is given by $\mathsf{h}_1 = \mathsf{f}_1\mathsf{g}_1$, $\mathsf{h}_0' = \mathsf{f}_0' + \mathsf{f}_1\mathsf{g}_0'$, $\underline{h} = \underline{f} + \underline{g}$. The unit morphism is $(\mathrm{id}, 0, 0)$.

2. Bar and cobar constructions. We are going to prove the existence of two functors between categories of curved algebras and curved coalgebras, generalizing the well known bar and cobar constructions.

2.1. Bar-construction. Let us construct a functor $\mathrm{Bar}\colon \mathrm{UCCAlg} \to \mathrm{CACoalg}$, the bar-construction. Let $A = (A, (b_n)_{n\geqslant 0}, \boldsymbol{\eta}, \mathbf{v})$ be a strict-unit-complemented curved A_∞-algebra. The shift $\overline{A}[1]$ of the $\Bbbk$-submodule $\overline{A} = \operatorname{Ker} \mathsf{v} \subset A$ is the image of an idempotent $1 - \mathbf{v}\cdot \boldsymbol{\eta}\colon A[1] \to A[1]$, which we write as the projection $\overline{\mathrm{pr}} = 1 - \mathbf{v}\cdot\boldsymbol{\eta}\colon A[1] \to A[1]$. Define $\mathrm{Bar}\, A$ as $\overline{A}[1]T^{\geqslant}$ equipped with the cut comultiplication $\delta_2^{\mathrm{Bar}\,A}$, the counit $\varepsilon^{\mathrm{Bar}\,A} = \mathrm{pr}_0\colon \overline{A}[1]T^{\geqslant} \to \overline{A}[1]T^0 = \Bbbk$, the splitting $\mathsf{w}^{\mathrm{Bar}\,A} = \mathrm{in}_0\colon \Bbbk = \overline{A}[1]T^0 \hookrightarrow \overline{A}[1]T^{\geqslant}$, the degree 1 coderivation $\delta_1^{\mathrm{Bar}\,A} = \bar{b}\colon \overline{A}[1]T^{\geqslant} \to \overline{A}[1]T^{\geqslant}$ given by its components

$$\bar{b}_n = \big(\overline{A}[1]^{\otimes n} \hookrightarrow A[1]^{\otimes n} \xrightarrow{b_n} A[1] \xrightarrow{\overline{\mathrm{pr}}} \overline{A}[1]\big), \quad n \geqslant 0,$$

and a degree 2 functional $\delta_0^{\mathrm{Bar}\,A} = -\big(\overline{A}[1]T^{\geqslant} \hookrightarrow A[1]T^{\geqslant} \xrightarrow{\check{b}} A[1] \xrightarrow{\mathbf{v}} \Bbbk\big)$. Clearly, $\mathsf{w}^{\mathrm{Bar}\,A}$ is a graded coalgebra homomorphism and the coalgebra $\overline{\overline{A}[1]T^{\geqslant}} = \overline{A}[1]T^{>}$ with the cut comultiplication is conilpotent.

Let us verify the necessary identities. Both sides of the equation

$$(\delta_1^{\mathrm{Bar}\,A})^2 = \delta_2^{\mathrm{Bar}\,A}(1\otimes\delta_0^{\mathrm{Bar}\,A} - \delta_0^{\mathrm{Bar}\,A}\otimes 1)\colon \overline{A}[1]T^{\geqslant} \to \overline{A}[1]T^{\geqslant}$$

are coderivations. Hence, the equation is equivalent to its composition with $\mathrm{pr}_1\colon \overline{A}[1]T^{\geqslant} \to \overline{A}[1]$. That is, to

$$\sum_{r+k+t=n} (1^{\otimes r}\otimes \bar{b}_k \otimes 1^{\otimes t})\bar{b}_{r+1+t} = b_{n-1}\mathbf{v}\otimes 1 - 1\otimes b_{n-1}\mathbf{v}\colon \overline{A}[1]^{\otimes n} \to \overline{A}[1]$$

for all $n \geqslant 0$. This holds true due to computation

$$\sum_{r+k+t=n} (1^{\otimes r}\otimes b_k(1-\mathbf{v}\boldsymbol{\eta})\otimes 1^{\otimes t})b_{r+1+t}\,\overline{\mathrm{pr}} =$$
$$= -(1\otimes b_{n-1}\mathbf{v}\boldsymbol{\eta})b_2\,\overline{\mathrm{pr}} - (b_{n-1}\mathbf{v}\boldsymbol{\eta}\otimes 1)b_2\,\overline{\mathrm{pr}} = b_{n-1}\mathbf{v}\otimes 1 - 1\otimes b_{n-1}\mathbf{v}.$$

Furthermore, $\delta_1^{\mathrm{Bar}\,A}\delta_0^{\mathrm{Bar}\,A} = -\big(\overline{A}[1]T^{\geqslant} \xrightarrow{\bar{b}} \overline{A}[1]T^{\geqslant} \xrightarrow{\check{b}} A[1] \xrightarrow{\mathbf{v}} \Bbbk\big)$ vanishes due to

$$-\sum_{r+k+t=n} (1^{\otimes r}\otimes b_k(1-\mathbf{v}\boldsymbol{\eta})\otimes 1^{\otimes t})b_{r+1+t}\mathbf{v} =$$
$$= (1\otimes b_{n-1}\mathbf{v}\boldsymbol{\eta})b_2\mathbf{v} + (b_{n-1}\mathbf{v}\boldsymbol{\eta}\otimes 1)b_2\mathbf{v} = \mathbf{v}\otimes b_{n-1}\mathbf{v} - b_{n-1}\mathbf{v}\otimes\mathbf{v} = 0\colon \overline{A}[1]^{\otimes n} \to \Bbbk,$$

because $\overline{A}[1]\mathbf{v} = 0$. Thus the object $\mathrm{Bar}\, A$ of CACoalg is welldefined.

Let us describe the functor $\mathrm{Bar}\colon \mathrm{UCCAlg} \to \mathrm{CACoalg}$ on morphisms. It takes a morphism $f = (f_1, f_0)\colon A \to B$ to the morphism $\mathsf{Bar}\, f = \mathsf{g} = (\mathsf{g}_1, \mathsf{g}_0)\colon \overline{A}[1]T^{\geqslant} \to \overline{B}[1]T^{\geqslant}$, where the coalgebra homomorphism $\mathsf{Bar}_1 f = \mathsf{g}_1 = \overline{f}$ is specified by its components

$$\overline{f}_1 = \big(\overline{A}[1] \hookrightarrow A[1] \xrightarrow{f_1} B[1] \xrightarrow{1-\mathbf{v}\cdot\boldsymbol{\eta}} \overline{B}[1]\big),\ \overline{f}_0 = \big(\Bbbk \xrightarrow{f_0} B[1] \xrightarrow{1-\mathbf{v}\cdot\boldsymbol{\eta}} \overline{B}[1]\big) = 0, \qquad (8)$$

and the degree 1 functional is

$$\mathsf{Bar}_0 f = \mathsf{g}_0 = \big(\overline{A}[1]T^{\geqslant} \hookrightarrow A[1]T^{\geqslant} \xrightarrow{\check{f}} B[1] \xrightarrow{\mathbf{v}} \Bbbk\big). \qquad (9)$$

Notice that the coalgebra homomorphism $\overline{f}$ is strict, that is, it has only one non-vanishing component, the first one. Thus, $\overline{f}$ preserves the number of tensor factors, $\overline{f}| = \overline{f}_1^{\otimes n}: \overline{A}[1]^{\otimes n} \to \overline{B}[1]^{\otimes n}$, $n \geqslant 0$. In particular, $\mathsf{w}^{\operatorname{Bar} A}\overline{f} = \mathsf{w}^{\operatorname{Bar} B}$.

Let us check that g is indeed a morphism of CACoalg. It is required that $\overline{b}^A \mathsf{g}_1 + \Delta(\mathsf{g}_0 \otimes \mathsf{g}_1 - \mathsf{g}_1 \otimes \mathsf{g}_0) = \mathsf{g}_1 \overline{b}^B$. All terms of this equation are $\overline{f}$-coderivations. Hence, the equation follows from its composition with pr_1. $\overline{b}^A \check{\overline{f}} + \Delta(\mathsf{g}_0 \otimes \check{\overline{f}} - \check{\overline{f}} \otimes \mathsf{g}_0) = \overline{f} \check{\overline{b}}^B : \overline{A}[1]T^{\geqslant} \to \overline{B}[1]$, that is, for all $n \geqslant 0$

$$\bar{b}_n \overline{f}_1 + f_0 \mathbf{v} \otimes \overline{f}_n + f_1 \mathbf{v} \otimes \overline{f}_{n-1} - \overline{f}_n \otimes f_0 \mathbf{v} - \overline{f}_{n-1} \otimes f_1 \mathbf{v} = \\ = \sum_{i_1+\dots+i_k=n} (\overline{f}_{i_1} \otimes \overline{f}_{i_2} \otimes \dots \otimes \overline{f}_{i_k}) \bar{b}_k : \overline{A}[1]^{\otimes n} \to \overline{B}[1].$$

In detail,

$$b_n(1 - \mathbf{v}\boldsymbol{\eta}) f_1 \,\overline{\mathrm{pr}} + \sum_{i_1+i_2=n} \bigl(f_{i_1}\mathbf{v} \otimes f_{i_2} \,\overline{\mathrm{pr}} - f_{i_1} \,\overline{\mathrm{pr}} \otimes f_{i_2}\mathbf{v}\bigr) = \\ = \sum_{i_1+\dots+i_k=n} [f_{i_1}(1 - \mathbf{v}\boldsymbol{\eta}) \otimes \dots \otimes f_{i_k}(1 - \mathbf{v}\boldsymbol{\eta})] b_k \,\overline{\mathrm{pr}}\,.$$

Cancelling the summands without $\mathbf{v}$ we reduce the equation to the valid identity

$$\sum_{i_1+i_2=n} \bigl(f_{i_1}\mathbf{v} \otimes f_{i_2} \,\overline{\mathrm{pr}} - f_{i_1} \,\overline{\mathrm{pr}} \otimes f_{i_2}\mathbf{v}\bigr) = - \sum_{i_1+i_2=n} \bigl[(f_{i_1}\mathbf{v}\boldsymbol{\eta} \otimes f_{i_2}) b_2 \,\overline{\mathrm{pr}} + (f_{i_1} \otimes f_{i_2}\mathbf{v}\boldsymbol{\eta}) b_2 \,\overline{\mathrm{pr}}\bigr].$$

Another equation to prove, $\check{b}^A \mathbf{v} + \overline{b}^A \check{f} \mathbf{v} + \Delta(\check{f}\mathbf{v} \otimes \check{f}\mathbf{v}) = \overline{f} \check{b}^B \mathbf{v} : \overline{A}[1]T^{\geqslant} \to \Bbbk$, is written explicitly as

$$b_n \mathbf{v} + b_n(1 - \mathbf{v}\boldsymbol{\eta}) f_1 \mathbf{v} + \sum_{i_1+i_2=n} f_{i_1}\mathbf{v} \otimes f_{i_2}\mathbf{v} = \\ = \sum_{i_1+\dots+i_k=n} [f_{i_1}(1 - \mathbf{v}\boldsymbol{\eta}) \otimes \dots \otimes f_{i_k}(1 - \mathbf{v}\boldsymbol{\eta})] b_k \mathbf{v} : \overline{A}[1]^{\otimes n} \to \Bbbk.$$

Cancelling the first and the third summands as well as summands that contain $\mathbf{v}$ only at the end, we obtain the valid equation

$$\sum_{i_1+i_2=n} f_{i_1}\mathbf{v} \otimes f_{i_2}\mathbf{v} = - \sum_{i_1+i_2=n} \bigl[(f_{i_1}\mathbf{v}\boldsymbol{\eta} \otimes f_{i_2}) b_2 \mathbf{v} + (f_{i_1} \otimes f_{i_2}\mathbf{v}\boldsymbol{\eta}) b_2 \mathbf{v} + (f_{i_1}\mathbf{v} \otimes f_{i_2}\mathbf{v})\boldsymbol{\eta}\mathbf{v}\bigr].$$

The identity morphism $f = (\mathrm{id}, 0)$ is mapped to the identity morphism $\mathsf{Bar}\, f = (\mathrm{id}, 0)$. Let us verify that Bar agrees with the composition. If $h = fg$ in UCCAlg, $h_1 = f_1 g_1$, $h_0 = g_0 + f_0 g_1$, then $\overline{h} = \overline{f}\overline{g}$. In fact, the equation

$$\sum_{i_1+\dots+i_k=n} (\overline{f}_{i_1} \otimes \overline{f}_{i_2} \otimes \dots \otimes \overline{f}_{i_k}) \overline{g}_k = \overline{h}_n$$

has the only non-vanishing realization $\overline{f}_1 \overline{g}_1 = \overline{h}_1$. Furthermore, $\mathsf{Bar}_0\, f + (\mathsf{Bar}_1\, f) \cdot \mathsf{Bar}_0\, g = \mathsf{Bar}_0\, h$ since $\check{f}\mathbf{v} + \overline{f}\check{g}\mathbf{v} = \check{h}\mathbf{v} : \overline{A}[1]T^{\geqslant} \to \Bbbk$. In fact, in arity n the left hand side is

$$f_n \mathbf{v} + f_n(1 - \mathbf{v}\boldsymbol{\eta}) g_1 \mathbf{v} + \delta_{n,0} g_0 \mathbf{v} = (f_n g_1 + \delta_{n,0} g_0)\mathbf{v} = h_n \mathbf{v}.$$

The functor Bar: UCCAlg $\to$ CACoalg (the bar-construction) is described.

2.2. Cobar-construction. Let us construct a functor Cobar: CACoalg $\to$ UCCAlg, the cobar-construction. Let $C = (C, (\xi_n)_{n\geqslant 0}, \boldsymbol{\varepsilon}, \mathbf{w})$ be a strict-counit-complemented curved A_∞-coalgebra. The shift $\overline{C}[-1]$ of the $\Bbbk$-submodule $\overline{C} = \operatorname{Ker}\varepsilon \subset C$ is the image of an idempotent $1 - \boldsymbol{\varepsilon}\cdot\mathbf{w}\colon C[-1] \to C[-1]$, which we write as the projection $\overline{\mathrm{pr}} = 1 - \boldsymbol{\varepsilon}\cdot\mathbf{w}\colon C[-1] \to \overline{C}[-1]$. Define $\operatorname{Cobar} C$ as $\overline{C}[-1]T^{\geqslant}$ equipped with the multiplication $m_2^{\operatorname{Cobar} C}$ in the tensor algebra, the unit $\eta^{\operatorname{Cobar} C} = \mathrm{in}_0\colon \Bbbk = \overline{C}[-1]T^0 \hookrightarrow \overline{C}[-1]T^{\geqslant}$, the splitting $\mathsf{v}^{\operatorname{Cobar} C} = \mathrm{pr}_0\colon \overline{C}[-1]T^{\geqslant} \to \overline{C}[-1]T^0 = \Bbbk$, the degree 1 derivation $m_1^{\operatorname{Cobar} C} = \bar{\xi}\colon \overline{C}[-1]T^{\geqslant} \to \overline{C}[-1]T^{\geqslant}$ given by its components $\bar{\xi}_n = \bigl(\overline{C}[-1] \hookrightarrow C[-1] \xrightarrow{\xi_n} C[-1]^{\otimes n} \xrightarrow{\overline{\mathrm{pr}}^{\otimes n}} \overline{C}[-1]^{\otimes n}\bigr)$, $n \geqslant 0$, and a degree 2 element

$$m_0^{\operatorname{Cobar} C} = -\mathbf{w}\otimes\mathbf{w} - \sum_{n\geqslant 0} \mathbf{w}\xi_n \in \overline{C}[-1]\hat{T}^{\geqslant}.$$

For general curved A_∞-coalgebra C the element $m_0^{\operatorname{Cobar} C}$ does not belong to $\overline{C}[-1]T^{\geqslant}$, however, if C is a curved augmented coalgebra, then it does. Conilpotency of $\overline{C}$ is not needed for existence of $\operatorname{Cobar} C$. Let us verify necessary identities.

If $n \neq 2$, then $\bar{\xi}_n = \xi_n\big|_{\overline{C}[-1]}$. Furthermore, $\bar{\xi}_2 = \xi_2\big|_{\overline{C}[-1]} \cdot [(1-\boldsymbol{\varepsilon}\mathbf{w}) \otimes (1-\boldsymbol{\varepsilon}\mathbf{w})] = \xi_2\big|_{\overline{C}[-1]} + 1\otimes\mathbf{w} - \mathbf{w}\otimes 1$. Extension of this map satisfies

$$\bar{\xi}_2 = \xi_2[(1-\boldsymbol{\varepsilon}\mathbf{w}) \otimes (1-\boldsymbol{\varepsilon}\mathbf{w})] = \xi_2 + 1\otimes\mathbf{w} - \mathbf{w}\otimes 1 - \boldsymbol{\varepsilon}(\mathbf{w}\otimes\mathbf{w})\colon C[-1] \to C[-1]^{\otimes 2}. \quad (10)$$

Both sides of the equation $(m_1^{\operatorname{Cobar} C})^2 = (m_0^{\operatorname{Cobar} C} \otimes 1 - 1 \otimes m_0^{\operatorname{Cobar} C}) m_2^{\operatorname{Cobar} C}$ are derivations. It is equivalent to its restriction to generators $\overline{C}[-1]$.

$$\sum_{r+k+t=n} \bar{\xi}_{r+1+t}(1^{\otimes r} \otimes \bar{\xi}_k \otimes 1^{\otimes t}) = (m_0^{\operatorname{Cobar} C})_{n-1} \otimes 1 - 1 \otimes (m_0^{\operatorname{Cobar} C})_{n-1}\colon \overline{C}[-1] \to \overline{C}[-1]^{\otimes n}. \quad (11)$$

Let us prove (11) for $(m_0^{\operatorname{Cobar} C})_2 = -\mathbf{w}\otimes\mathbf{w} - \mathbf{w}\xi_2 = 0$, $(m_0^{\operatorname{Cobar} C})_{n-1} = -\mathbf{w}\xi_{n-1}$ if $n \neq 3$. In fact, (11) is obvious for $n = 0$. It says for $n = 1$ that

$$\xi_1^2 + \bar{\xi}_2(1\otimes\xi_0 + \xi_0\otimes 1) = (1\otimes\mathbf{w} - \mathbf{w}\otimes 1)(1\otimes\xi_0 + \xi_0\otimes 1) = (\mathbf{w}\xi_0) - \xi_0\mathbf{w} + \xi_0\mathbf{w} - (\mathbf{w}\xi_0) = 0$$

as it has to be. If $n = 2$ or $n \geqslant 4$, then the left hand side of (11) is

$$\begin{aligned} \xi_1\xi_n + \bar{\xi}_2(\xi_{n-1}\otimes 1 + 1\otimes\xi_{n-1}) + \dots + \xi_{n-1} \sum_{r+2+t=n} 1^{\otimes r} \otimes \bar{\xi}_2 \otimes 1^{\otimes t} + \dots = \\ = (1\otimes\mathbf{w} - \mathbf{w}\otimes 1)(\xi_{n-1}\otimes 1 + 1\otimes\xi_{n-1}) + \\ + \xi_{n-1} \sum_{r+2+t=n} (1^{\otimes(r+1)} \otimes \mathbf{w} \otimes 1^{\otimes t} - 1^{\otimes r} \otimes \mathbf{w} \otimes 1^{\otimes(1+t)}) = -\xi_{n-1}\otimes\mathbf{w} + 1\otimes\mathbf{w}\xi_{n-1} - \\ -\mathbf{w}\xi_{n-1}\otimes 1 - \mathbf{w}\otimes\xi_{n-1} + \xi_{n-1}(1^{\otimes(n-1)}\otimes\mathbf{w} - \mathbf{w}\otimes 1^{\otimes(n-1)}) = 1\otimes\mathbf{w}\xi_{n-1} - \mathbf{w}\xi_{n-1}\otimes 1, \end{aligned}$$

as claimed. If $n = 3$, then the left hand side of (11) is

$$\begin{aligned} \xi_1\xi_3 + \bar{\xi}_2(\bar{\xi}_2\otimes 1 + 1\otimes\bar{\xi}_2) + \dots = (\xi_2 + 1\otimes\mathbf{w} - \mathbf{w}\otimes 1)[(\xi_2 - \mathbf{w}\otimes 1)\otimes 1 + 1\otimes(\xi_2 + 1\otimes\mathbf{w})] - \\ -\xi_2(\xi_2\otimes 1 + 1\otimes\xi_2) = 1\otimes(\mathbf{w}\xi_2 + \mathbf{w}\otimes\mathbf{w}) - (\mathbf{w}\xi_2 + \mathbf{w}\otimes\mathbf{w})\otimes 1 = 0, \end{aligned}$$

as claimed.

The expression $m_0^{\mathrm{Cobar}\,C} m_1^{\mathrm{Cobar}\,C}$ is a well-defined element of $\overline{C}[-1]\hat{T}^{\geqslant}$. Its n-th component is

$$\begin{aligned} m_0^{\mathrm{Cobar}\,C} m_1^{\mathrm{Cobar}\,C} \mathrm{pr}_n &= -\mathbf{w}\otimes\mathbf{w}\overline{\xi}_{n-1} + \mathbf{w}\overline{\xi}_{n-1}\otimes\mathbf{w} - \sum_{r+k+t=n} \mathbf{w}\xi_{r+1+t}(1^{\otimes r}\otimes\overline{\xi}_k\otimes 1^{\otimes t}) = \\ = -\mathbf{w}\otimes\mathbf{w}\overline{\xi}_{n-1} &+ \mathbf{w}\overline{\xi}_{n-1}\otimes\mathbf{w} - \mathbf{w}\xi_{n-1}\sum_{r+2+t=n}(1^{\otimes(r+1)}\otimes\mathbf{w}\otimes 1^{\otimes t} - 1^{\otimes r}\otimes\mathbf{w}\otimes 1^{\otimes(1+t)}) = \\ &= -\mathbf{w}\otimes\mathbf{w}\overline{\xi}_{n-1} + \mathbf{w}\overline{\xi}_{n-1}\otimes\mathbf{w} - \mathbf{w}\xi_{n-1}(1^{\otimes(n-1)}\otimes\mathbf{w} - \mathbf{w}\otimes 1^{\otimes(n-1)}). \end{aligned} \tag{12}$$

If $n \neq 3$, then $\overline{\xi}_{n-1} = \xi_{n-1}$ and the obtained expression is equivalent to

$$-\mathbf{w}\otimes\mathbf{w}\xi_{n-1} + \mathbf{w}\xi_{n-1}\otimes\mathbf{w} - \mathbf{w}\xi_{n-1}\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}\xi_{n-1} = 0.$$

If $n = 3$, then (12) is equivalent to

$$-\mathbf{w}\otimes[\mathbf{w}(1\otimes\mathbf{w}-\mathbf{w}\otimes 1)] + [\mathbf{w}(1\otimes\mathbf{w}-\mathbf{w}\otimes 1)]\otimes\mathbf{w} = \mathbf{w}\otimes\mathbf{w}\otimes\mathbf{w}(-1-1+1+1) = 0.$$

Thus $m_0^{\mathrm{Cobar}\,C} m_1^{\mathrm{Cobar}\,C} = 0$. We obtain a map $\mathrm{Ob}\,\mathrm{CACoalg} \to \mathrm{Ob}\,\mathrm{UCCAlg}$.

Let us describe the functor $\mathrm{Cobar}\colon \mathrm{CACoalg} \to \mathrm{UCCAlg}$ on morphisms. It takes a morphism $g = (g_1, g_0)\colon C \to D$ to the morphism $\mathsf{Cobar}\,g = \mathsf{f} = (\mathsf{f}_1, \mathsf{f}_0)\colon \overline{C}[-1]T^{\geqslant} \to \overline{D}[-1]T^{\geqslant}$, where the algebra homomorphism $\mathsf{Cobar}_1\,g = \mathsf{f}_1 = \overline{g}$ is specified by its components

$$\begin{aligned} \overline{g}_1 = g_1 &= \big(\overline{C}[-1] \hookrightarrow C[-1] \xrightarrow{g_1} D[-1] \xrightarrow{\overline{\mathrm{pr}}} \overline{D}[-1]\big), \\ \overline{g}_0 = g_0' &= \big(\overline{C}[-1] \hookrightarrow C[-1] \xrightarrow{g_0} \Bbbk\big), \end{aligned} \tag{13}$$

and the degree 1 element is $\mathsf{Cobar}_0\,g = \mathsf{f}_0 = \big(\Bbbk \xrightarrow{\mathbf{w}} C[-1] \xrightarrow{\check{g}} D[-1]T^{\geqslant} \xrightarrow{\overline{\mathrm{pr}}\,T^{\geqslant}} \overline{D}[-1]T^{\geqslant}\big)$, which we write as $\mathbf{w}\check{\overline{g}}$ extending the notation. This element has the only non-vanishing component $\mathsf{f}_{00} = \big(\Bbbk \xrightarrow{\mathbf{w}} C[-1] \xrightarrow{g_0} \Bbbk\big) = \mathbf{w}g_0$. In fact,

$$\mathsf{f}_{01} = \big(\Bbbk \xrightarrow{\mathbf{w}} C[-1] \xrightarrow{g_1} D[-1] \xrightarrow{\overline{\mathrm{pr}}} \overline{D}[-1]\big) = \mathbf{w}g_1\,\overline{\mathrm{pr}} = 0.$$

Thus,

$$\begin{aligned} \mathsf{Cobar}_0\,g = \mathsf{f}_0 &= \big(\Bbbk \xrightarrow{\mathbf{w}} C[-1] \xrightarrow{g_0} \Bbbk \xhookrightarrow{\mathrm{in}_0} \overline{D}[-1]T^{\geqslant}\big), \\ \underline{\mathrm{Cobar}\,g} = \underline{f} &= \big(\Bbbk \xrightarrow{\mathbf{w}} C \xrightarrow{\mathrm{g}_0} \Bbbk\big) = \underline{g}. \end{aligned} \tag{14}$$

Let us check that f is indeed a morphism of UCCAlg. It is required that $\overline{g}\overline{\xi} + (\overline{g}\otimes\mathsf{f}_0 - \mathsf{f}_0\otimes\overline{g})m_2 = \overline{\xi}\overline{g}$. The second term vanishes, but this form of equation is easier to deal with. All terms of this equation are $\overline{g}$-derivations. Hence, the equation is equivalent to its restriction to $\overline{C}[-1]$: $\check{\overline{g}}\overline{\xi} + (\check{\overline{g}}\otimes\mathsf{f}_0 - \mathsf{f}_0\otimes\check{\overline{g}})m_2 = \check{\overline{\xi}}\overline{g}\colon \overline{C}[-1] \to \overline{D}[-1]T^{\geqslant}$, which means that for all $n \geqslant 0$

$$\begin{aligned} \overline{g}_1\overline{\xi}_n + (\overline{g}_n\otimes\mathbf{w}g_0 + \overline{g}_{n-1}\otimes\mathbf{w}g_1\,\overline{\mathrm{pr}} - \mathbf{w}g_0\otimes\overline{g}_n - \mathbf{w}g_1\,\overline{\mathrm{pr}}\otimes\overline{g}_{n-1})m_2 = \\ = \sum_{i_1+\dots+i_k=n} \overline{\xi}_k(\overline{g}_{i_1}\otimes\overline{g}_{i_2}\otimes\dots\otimes\overline{g}_{i_k})\colon \overline{C}[-1] \to \overline{D}[-1]^{\otimes n}. \end{aligned}$$

When written explicitly,

$$g_1(1-\boldsymbol{\varepsilon}\mathbf{w})\xi_n\,\overline{\mathrm{pr}}^{\otimes n} + (g_n\,\overline{\mathrm{pr}}^{\otimes n}\otimes\mathbf{w}g_0 + g_{n-1}\,\overline{\mathrm{pr}}^{\otimes(n-1)}\otimes\mathbf{w}g_1\,\overline{\mathrm{pr}} - \\ -\mathbf{w}g_0\otimes g_n\,\overline{\mathrm{pr}}^{\otimes n} - \mathbf{w}g_1\,\overline{\mathrm{pr}}\otimes g_{n-1}\,\overline{\mathrm{pr}}^{\otimes(n-1)})m_2 = \sum_{i_1+\dots+i_k=n}\xi_k(g_{i_1}\otimes\dots\otimes g_{i_k})\,\overline{\mathrm{pr}}^{\otimes n} + \\ + \sum_{i_1+i_2=n}\big(g_{i_1}\otimes\mathbf{w}g_{i_2} - \mathbf{w}g_{i_1}\otimes g_{i_2}\big)\,\overline{\mathrm{pr}}^{\otimes n}\colon \overline{C}[-1]\to\overline{D}[-1]^{\otimes n},$$

it becomes obvious.

Another equation must hold,

$$-\mathbf{w}\otimes\mathbf{w} - \mathbf{w}\check{\xi} - \mathbf{w}\check{\overline{g}}\overline{\xi} - (\mathbf{w}\check{\overline{g}}\otimes\mathbf{w}\check{\overline{g}})m_2 = -\mathbf{w}\check{\overline{g}}\otimes\mathbf{w}\check{\overline{g}} - \sum_{k\geqslant 0}\mathbf{w}\xi_k\check{\overline{g}}^{\otimes k}\colon \Bbbk\to\overline{D}[-1]T^{\geqslant}.$$

After cancelling two summands and changing the sign the equation is written as

$$\delta_{n,2}\mathbf{w}\otimes\mathbf{w} + \mathbf{w}\xi_n + \mathbf{w}\overline{g}_1\overline{\xi}_n = \sum_{i_1+\dots+i_k=n}\mathbf{w}\xi_k(\overline{g}_{i_1}\otimes\overline{g}_{i_2}\otimes\dots\otimes\overline{g}_{i_k})\colon \Bbbk\to\overline{D}[-1]^{\otimes n}.$$

Explicitly:

$$\delta_{n,2}\mathbf{w}\otimes\mathbf{w} + \mathbf{w}\xi_n + \mathbf{w}g_1(1-\boldsymbol{\varepsilon}\mathbf{w})\overline{\xi}_n = \sum_{i_1+\dots+i_k=n}\mathbf{w}\xi_k(g_{i_1}\otimes g_{i_2}\otimes\dots\otimes g_{i_k})\,\overline{\mathrm{pr}}^{\otimes n}.$$

Cancelling $\mathbf{w}g_1\overline{\xi}_n$ against the right hand side we come to the valid equation $\delta_{n,2}\mathbf{w}\otimes\mathbf{w} + \mathbf{w}\xi_n - \mathbf{w}\overline{\xi}_n = 0\colon \Bbbk\to\overline{D}[-1]^{\otimes n}$. In fact, $\overline{\xi}_n = \xi_n$ for $n\neq 2$ and the equation is obvious. For $n=2$ we have by (10) $\mathbf{w}\otimes\mathbf{w} + \mathbf{w}\xi_2 - \mathbf{w}\xi_2 - \mathbf{w}(1\otimes\mathbf{w}) + \mathbf{w}(\mathbf{w}\otimes 1) + \mathbf{w}\boldsymbol{\varepsilon}(\mathbf{w}\otimes\mathbf{w}) = 0$.

The identity morphism $g = (\mathrm{id},0)$ is mapped to the identity morphism $\mathsf{Cobar}\,g = (\mathrm{id},0)$. Let us verify that Cobar agrees with the composition. If $h = \big(C\xrightarrow{f}D\xrightarrow{g}E\big)$ in CACoalg, $h_1 = f_1g_1$, $h_0 = f_0 + f_1g_0$, then $(\mathsf{Cobar}_1 f)\cdot\mathsf{Cobar}_1 g = \overline{f}\overline{g} = \overline{h} = \mathsf{Cobar}_1 h$. In fact, the equation

$$\sum_{i_1+\dots+i_k=n}\overline{f}_k(\overline{g}_{i_1}\otimes\overline{g}_{i_2}\otimes\dots\otimes\overline{g}_{i_k}) = \overline{h}_n\colon \overline{C}[-1]\to\overline{E}[-1]^{\otimes n}$$

for $n=1$ holds due to $\overline{f}_1\overline{g}_1 = f_1(1-\boldsymbol{\varepsilon}\mathbf{w})g_1\,\overline{\mathrm{pr}} = f_1g_1\,\overline{\mathrm{pr}} = h_1\,\overline{\mathrm{pr}} = \overline{h}_1\colon \overline{C}[-1]\to\overline{E}[-1]$, and for $n=0$ it holds due to $\overline{f}_0 + \overline{f}_1\overline{g}_0 = f_0 + f_1(1-\boldsymbol{\varepsilon}\mathbf{w})g_0 = f_0 + f_1g_0 = h_0 = \overline{h}_0\colon \overline{C}[-1]\to\Bbbk$.

Furthermore, $\mathsf{Cobar}_0\,g + (\mathsf{Cobar}_0\,f)\cdot\mathsf{Cobar}_1\,g = \mathsf{Cobar}_0\,h$ since $\mathbf{w}\check{\overline{g}} + \mathbf{w}\check{\overline{f}}\overline{g} = \mathbf{w}\check{\overline{h}}\colon \Bbbk\to\overline{E}[-1]T^{\geqslant}$. In fact, the n-th component of the left hand side is

$$\mathbf{w}\overline{g}_n + \sum_{i_1+\dots+i_k=n}\mathbf{w}\overline{f}_k(\overline{g}_{i_1}\otimes\overline{g}_{i_2}\otimes\dots\otimes\overline{g}_{i_k})\colon \Bbbk\to\overline{E}[-1]^{\otimes n},$$

which for $n=1$ transforms to $\mathbf{w}\overline{g}_1 + \mathbf{w}\overline{f}_1\overline{g}_1 = \mathbf{w}\overline{g}_1 + \mathbf{w}f_1(1-\boldsymbol{\varepsilon}\mathbf{w})\overline{g}_1 = \mathbf{w}f_1g_1\,\overline{\mathrm{pr}} = \mathbf{w}\overline{h}_1\colon \Bbbk\to\overline{E}[-1]$, and for $n=0$ equals

$$\mathbf{w}\overline{g}_0 + \mathbf{w}\overline{f}_0 + \mathbf{w}\overline{f}_1\overline{g}_0 = \mathbf{w}\overline{g}_0 + \mathbf{w}\overline{f}_0 + \mathbf{w}f_1(1-\boldsymbol{\varepsilon}\mathbf{w})\overline{g}_0 = \mathbf{w}(f_0 + f_1g_0)\,\overline{\mathrm{pr}} = \mathbf{w}\overline{h}_0.$$

The functor Cobar: CACoalg → UCCAlg (the cobar-construction) is described.

3. Adjunction. We are showing that the two (bar and cobar) constructions are functors, adjoint to each other. The adjunction bijection will be the top row of the following diagram.

The two middle rows are natural transformations defined so that the two lower squares commute.

$$\begin{array}{ccc}
\mathrm{UCCAlg}(\overline{C}[-1]T^{\geqslant}, A) & \dashrightarrow & \mathrm{CACoalg}(C, \overline{A}[1]T^{\geqslant}) \\
\big\downarrow & & \big\downarrow \\
\mathbf{gr}\text{-alg}(\overline{C}[-1]T^{\geqslant}, A) \times \mathbf{gr}(\mathbb{k}[-1],\mathbb{k}) & \xrightarrow[\sim]{} & \mathbf{gr}\text{-nuCoalg}(\overline{C}, \overline{A}[1]T^{>}) \times \mathbf{gr}(C,\mathbb{k}[1]) \\
\wr\big\downarrow & & \wr\big\downarrow {\scriptstyle _\cdot\mathrm{pr}_1 \times \mathrm{id}} \\
\mathbf{gr}(\overline{C}[-1], A) \times \mathbf{gr}(\mathbb{k}[-1],\mathbb{k}) & \xrightarrow[\sim]{} & \mathbf{gr}(\overline{C}, \overline{A}[1]) \times \mathbf{gr}(C,\mathbb{k}[1]) \\
\big\| & & \big\| \\
\mathbf{gr}(\overline{C}[-1], \overline{A}) \times \mathbf{gr}(\overline{C}[-1],\mathbb{k}) \times \mathbf{gr}(\mathbb{k}[-1],\mathbb{k}) & \xrightarrow[\sim]{[1]} & \mathbf{gr}(\overline{C}, \overline{A}[1]) \times \mathbf{gr}(\overline{C},\mathbb{k}[1]) \times \mathbf{gr}(\mathbb{k},\mathbb{k}[1])
\end{array}$$

Notice that the set of morphisms of augmented graded coalgebras $C \to \overline{A}[1]T^{\geqslant}$ is in bijection with the set of morphisms of graded non-counital coalgebras $\mathbf{gr}$ - nuCoalg$(\overline{C}, \overline{A}[1]T^{>})$. The functor $X \mapsto XT^{>} = \oplus_{n>0}X^{\otimes n}$ has the structure of a comonad and $T^{>}$-coalgebras are precisely conilpotent non-counital coalgebras ([1, Section 6.7]). Since $\overline{C}$ is conilpotent, the arrow $\_ \cdot \mathrm{pr}_1 \oplus \mathrm{id}$ is a bijection by the well known lemma on Kleisli categories (generalized to multicategories in [1, Lemma 5.3]). Thus the second horizontal map is a bijection as well. Morphisms $\mathsf{f}\colon \overline{C}[-1]T^{\geqslant} \to A \in \mathrm{UCCAlg}$ and $\mathsf{g}\colon C \to \overline{A}[1]T^{\geqslant} \in \mathrm{CACoalg}$ are related as the following diagram shows. It consists of elements (morphisms of degree 0) of vertices of the previous diagram. For instance, $g_1^1 = \check{\mathsf{g}}_1 = (\check{\mathsf{f}}_1\,\overline{\mathrm{pr}})[1] = f_1^1[1]$, etc. Equivalently,

$$\sigma^{-1}\check{\mathsf{f}}_1\,\overline{\mathrm{pr}} = \check{\mathsf{g}}_1\sigma^{-1}\colon \overline{C} \to \overline{A}, \tag{15a}$$

$$\sigma^{-1}\check{\mathsf{f}}_1\mathsf{v} = \mathsf{g}_0|_{\overline{C}} \quad\colon \overline{C} \to \mathbb{k}, \tag{15b}$$

$$\underline{f} = \mathsf{w}\mathsf{g}_0 \quad\colon \mathbb{k} \to \mathbb{k}, \tag{15c}$$

where all components are listed in

$$\begin{array}{ccc}
\mathsf{f} & \longmapsto & \mathsf{g} \\
\big\downarrow & & \big\downarrow \\
(\mathsf{f}_1, \sigma\underline{f}) & \longmapsto & (\mathsf{g}_1, \mathsf{g}_0\sigma) \\
\big\downarrow & & \big\downarrow \\
(\check{\mathsf{f}}_1, \sigma\underline{f}) & \longmapsto & (\check{\mathsf{g}}_1, \mathsf{g}_0\sigma) \\
\big\downarrow & & \big\downarrow \\
(\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}, \check{\mathsf{f}}_1\mathsf{v}, \sigma\underline{f}) & \overset{[1]}{\longmapsto} & (\check{\mathsf{g}}_1, \mathsf{g}_0|_{\overline{C}}\sigma, \mathsf{w}\mathsf{g}_0\sigma)
\end{array} \tag{16}$$

We are going to show that systems of equations on pairs $(\mathsf{f}_1, \sigma\underline{f})$ and $(\mathsf{g}_1, \mathsf{g}_0\sigma)$ saying that these pairs are morphisms of UCCAlg and CACoalg are equivalent. In fact, these systems are

$$\check{\mathsf{f}}_1 m_1^A = \check{m}_1^{\mathrm{Cobar}\,C}\mathsf{f}_1\colon \overline{C}[-1] \to A, \; m_0^A = m_0^{\mathrm{Cobar}\,C}\mathsf{f}_1\colon \mathbb{k} \to A, \tag{17}$$

$$\delta_1^C\check{\mathsf{g}}_1 + \delta_2^C(\mathsf{g}_0\otimes 1 - 1\otimes\mathsf{g}_0)\check{\mathsf{g}}_1 = \mathsf{g}_1\check{\delta}_1^{\mathrm{Bar}\,A}\colon C \to \overline{A}[1], \; \delta_0^C - \delta_1^C\mathsf{g}_0 - \delta_2^C(\mathsf{g}_0\otimes\mathsf{g}_0) = \mathsf{g}_1\delta_0^{\mathrm{Bar}\,A}\colon C \to \mathbb{k}. \tag{18}$$

Note that the image of any coderivation $C \to C$ is contained in $\overline{C} = \operatorname{Ker}\varepsilon$. In more detailed equations (17) and (18) read

$$\check{\mathsf f}_1 m_1^A = \xi_0\eta + \xi_1\check{\mathsf f}_1 + \overline{\xi}_2(\check{\mathsf f}_1 \otimes \check{\mathsf f}_1)m_2^A \colon \overline{C}[-1] \to A,\ m_0^A = -\mathbf{w}\xi_0\eta^A - \mathbf{w}\xi_1\check{\mathsf f}_1 \colon \Bbbk \to A, \tag{19}$$

$$\delta_1^C \check{\mathsf g}_1 + \delta_2^C(\mathsf g_0 \otimes 1 - 1 \otimes \mathsf g_0)\check{\mathsf g}_1 = \varepsilon^C b_0^A \,\overline{\mathrm{pr}}_A + \overline{\mathrm{pr}}_C\,\check{\mathsf g}_1 b_1^A \,\overline{\mathrm{pr}}_A + \overline{\mathrm{pr}}_C\,\overline{\delta}_2^C(\check{\mathsf g}_1 \otimes \check{\mathsf g}_1)b_2^A \,\overline{\mathrm{pr}}_A \colon C \to \overline{A}[1],$$

$$\delta_0^C - \delta_1^C \mathsf g_0 - \delta_2^C(\mathsf g_0 \otimes \mathsf g_0) = -\varepsilon^C b_0^A \mathbf{v} - \overline{\mathrm{pr}}_C\,\check{\mathsf g}_1 b_1^A \mathbf{v} - \overline{\mathrm{pr}}_C\,\overline{\delta}_2^C(\check{\mathsf g}_1 \otimes \check{\mathsf g}_1)b_2^A \mathbf{v} \colon C \to \Bbbk. \tag{20}$$

Let us rewrite systems (19) and (20) splitting each equation in two accordingly to splitting the target A or the source C in two summands

$$\check{\mathsf f}_1 m_1^A \,\overline{\mathrm{pr}}_A = \xi_1\check{\mathsf f}_1\,\overline{\mathrm{pr}}_A + \overline{\xi}_2(\check{\mathsf f}_1 \otimes \check{\mathsf f}_1)m_2^A \,\overline{\mathrm{pr}}_A \colon \overline{C}[-1] \to \overline{A}, \tag{21a}$$

$$\check{\mathsf f}_1 m_1^A \mathsf v = \xi_0 + \xi_1\check{\mathsf f}_1\mathsf v + \overline{\xi}_2(\check{\mathsf f}_1 \otimes \check{\mathsf f}_1)m_2^A \mathsf v \colon \overline{C}[-1] \to \Bbbk, \tag{21b}$$

$$m_0^A \,\overline{\mathrm{pr}}_A = -\mathbf{w}\xi_1\check{\mathsf f}_1\,\overline{\mathrm{pr}}_A \colon \Bbbk \to \overline{A}, \tag{21c}$$

$$m_0^A \mathsf v = -\mathbf{w}\xi_0 - \mathbf{w}\xi_1\check{\mathsf f}_1\mathsf v \colon \Bbbk \to \Bbbk, \tag{21d}$$

$$\delta_1^C \check{\mathsf g}_1 + \delta_2^C(\mathsf g_0 \otimes 1 - 1 \otimes \mathsf g_0)\check{\mathsf g}_1 = \check{\mathsf g}_1 b_1^A \,\overline{\mathrm{pr}}_A + \overline{\delta}_2^C(\check{\mathsf g}_1 \otimes \check{\mathsf g}_1)b_2^A \,\overline{\mathrm{pr}}_A \colon \overline{C} \to \overline{A}[1], \tag{22a}$$

$$\delta_0^C - \delta_1^C \mathsf g_0 - \overline{\delta}_2^C(\mathsf g_0 \otimes \mathsf g_0) = -\check{\mathsf g}_1 b_1^A \mathbf{v} - \overline{\delta}_2^C(\check{\mathsf g}_1 \otimes \check{\mathsf g}_1)b_2^A \mathbf{v} \colon \overline{C} \to \Bbbk, \tag{22b}$$

$$\mathsf w\delta_1^C \check{\mathsf g}_1 = b_0^A \,\overline{\mathrm{pr}}_A \colon \Bbbk \to \overline{A}[1], \tag{22c}$$

$$\mathsf w\delta_0^C - \mathsf w\delta_1^C \mathsf g_0 = -b_0^A \mathbf{v} \colon \Bbbk \to \Bbbk. \tag{22d}$$

We claim that equations (21x) and (22x) are equivalent for $\mathrm{x} \in \{\mathrm{a,b,c,d}\}$. In fact, let us rewrite the equations once again in the same order replacing m and δ with their definitions and composing with σ^{-1} wherever appropriate:

$$\sigma^{-1}\check{\mathsf f}_1\sigma b_1^A\sigma^{-1}\,\overline{\mathrm{pr}}_A + \sigma^{-1}\xi_1\check{\mathsf f}_1\,\overline{\mathrm{pr}}_A + \sigma^{-1}\overline{\xi}_2(\check{\mathsf f}_1\sigma \otimes \check{\mathsf f}_1\sigma)b_2^A\sigma^{-1}\,\overline{\mathrm{pr}}_A = 0 \colon \overline{C} \to \overline{A}, \tag{23a}$$

$$\sigma^{-1}\check{\mathsf f}_1\sigma b_1^A\sigma^{-1}\mathsf v + \sigma^{-1}\xi_0 + \sigma^{-1}\xi_1\check{\mathsf f}_1\mathsf v + \sigma^{-1}\overline{\xi}_2(\check{\mathsf f}_1\sigma \otimes \check{\mathsf f}_1\sigma)b_2^A\sigma^{-1}\mathsf v = 0 \colon \overline{C} \to \Bbbk, \tag{23b}$$

$$b_0^A\sigma^{-1}\,\overline{\mathrm{pr}}_A + \mathbf{w}\xi_1\check{\mathsf f}_1\,\overline{\mathrm{pr}}_A = 0 \colon\ \Bbbk \to \overline{A}, \tag{23c}$$

$$b_0^A\sigma^{-1}\mathsf v + \mathbf{w}\xi_0 + \mathbf{w}\xi_1\check{\mathsf f}_1\mathsf v = 0 \colon\ \Bbbk \to \Bbbk. \tag{23d}$$

In transforming (22a) use that $\delta_2^C(\mathsf g_0 \otimes 1 - 1 \otimes \mathsf g_0) = \delta_2^C(\overline{\mathrm{pr}}_C \otimes \overline{\mathrm{pr}}_C)(\mathsf g_0 \otimes 1 - 1 \otimes \mathsf g_0) \colon \overline{C} \to C$ actually takes values in $\overline{C}$, see (7). The second system is

$$\begin{gathered}\sigma^{-1}\xi_1^C\sigma\check{\mathsf g}_1\sigma^{-1} + \sigma^{-1}\overline{\xi}_2^C(\sigma\mathsf g_0 \otimes \sigma + \sigma \otimes \sigma\mathsf g_0)\check{\mathsf g}_1\sigma^{-1} + \\ + \check{\mathsf g}_1 b_1^A \,\overline{\mathrm{pr}}_A\,\sigma^{-1} + \sigma^{-1}\overline{\xi}_2^C(\sigma\check{\mathsf g}_1 \otimes \sigma\check{\mathsf g}_1)b_2^A \,\overline{\mathrm{pr}}_A\,\sigma^{-1} = 0 \colon \overline{C} \to \overline{A},\end{gathered} \tag{24a}$$

$$\begin{gathered}\sigma^{-1}\xi_0^C + \sigma^{-1}\xi_1^C\sigma\mathsf g_0 + \sigma^{-1}\overline{\xi}_2^C(\sigma\mathsf g_0 \otimes \sigma\mathsf g_0) + \check{\mathsf g}_1 b_1^A\mathbf{v} + \\ + \sigma^{-1}\overline{\xi}_2^C(\sigma\check{\mathsf g}_1 \otimes \sigma\check{\mathsf g}_1)b_2^A\mathbf{v} = 0 \colon \overline{C} \to \Bbbk,\end{gathered} \tag{24b}$$

$$\mathsf w\sigma^{-1}\xi_1^C\sigma\check{\mathsf g}_1\sigma^{-1} + b_0^A \,\overline{\mathrm{pr}}_A\,\sigma^{-1} = 0 \colon \Bbbk \to \overline{A}, \tag{24c}$$

$$\mathsf w\sigma^{-1}\xi_0^C + \mathsf w\sigma^{-1}\xi_1^C\sigma\mathsf g_0 + b_0^A\mathbf{v} = 0 \colon \Bbbk \to \Bbbk. \tag{24d}$$

According to our system of notation $\sigma\,\overline{\mathrm{pr}} = \overline{\mathrm{pr}}\,\sigma$. We shall use that $\check{\mathsf f}_1 = \check{\mathsf f}_1\,\overline{\mathrm{pr}}_A + \check{\mathsf f}_1\mathsf v\eta$. Substituting relations (15a) and (15b) into the above equations we find that the latter are pairwise equivalent.

In fact, (23c) is equivalent to (24c) and (23d) is equivalent to (24d). Equivalence of (23a) and (24a) follows from the identity

$$\sigma^{-1}\overline{\xi}_2(\check{\mathsf{f}}_1\mathsf{v}\eta\otimes\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A+\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\otimes\check{\mathsf{f}}_1\mathsf{v}\eta)m_2^A\,\overline{\mathrm{pr}}_A=\sigma^{-1}\overline{\xi}_2(\sigma\mathsf{g}_0|_{\overline{C}}\otimes\sigma+\sigma\otimes\sigma\mathsf{g}_0|_{\overline{C}})\check{\mathsf{g}}_1\sigma^{-1}.$$

Equivalence of (23b) and (24b) follows from the identity

$$\sigma^{-1}\overline{\xi}_2^C[(\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A+\check{\mathsf{f}}_1\mathsf{v}\eta)\otimes(\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A+\check{\mathsf{f}}_1\mathsf{v}\eta)]m_2^A\mathsf{v}=$$
$$=\sigma^{-1}\overline{\xi}_2^C(\sigma\mathsf{g}_0\otimes\sigma\mathsf{g}_0)+\sigma^{-1}\overline{\xi}_2^C(\sigma\check{\mathsf{g}}_1\otimes\sigma\check{\mathsf{g}}_1)b_2^A\sigma^{-1}\mathsf{v}\colon\overline{C}\to\Bbbk,$$

which can be expanded to

$$\sigma^{-1}\overline{\xi}_2^C(\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\otimes\check{\mathsf{f}}_1\mathsf{v}\eta+\check{\mathsf{f}}_1\mathsf{v}\eta\otimes\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A+\check{\mathsf{f}}_1\mathsf{v}\eta\otimes\check{\mathsf{f}}_1\mathsf{v}\eta)]m_2^A\mathsf{v}=\sigma^{-1}\overline{\xi}_2^C(\sigma\mathsf{g}_0|_{\overline{C}}\otimes\sigma\mathsf{g}_0|_{\overline{C}})\colon\overline{C}\to\Bbbk.$$

The latter equation follows from the obvious one $\sigma^{-1}\overline{\xi}_2^C(\check{\mathsf{f}}_1\mathsf{v}\otimes\check{\mathsf{f}}_1\mathsf{v})=\sigma^{-1}\overline{\xi}_2^C(\sigma\mathsf{g}_0|_{\overline{C}}\otimes\sigma\mathsf{g}_0|_{\overline{C}})\colon$ $\overline{C}\to\Bbbk$. Hence, the bijection

$$\mathrm{UCCAlg}(\overline{C}[-1]T^{\geqslant},A)\xrightarrow{\sim}\mathrm{CACoalg}(C,\overline{A}[1]T^{\geqslant})\tag{25}$$

is constructed.

Theorem 1. *The functors* Cobar: CACoalg $\leftrightarrows$ UCCAlg: Bar *are adjoint to each other.*

Proof. We have to prove naturality of bijection (25) with respect to A and C. The bijection takes

$$(\mathsf{f}_1,\underline{f})\mapsto(\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A,\check{\mathsf{f}}_1\mathsf{v}_A,\sigma\underline{f})\xmapsto{[1]}(\sigma^{-1}\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma,\sigma^{-1}\check{\mathsf{f}}_1\mathsf{v}_A\sigma,\underline{f}\sigma)=(\check{\mathsf{g}}_1,\mathsf{g}_0|_{\overline{C}}\sigma,\mathsf{w}\mathsf{g}_0\sigma)\mapsto(\mathsf{g}_1,\mathsf{g}_0\sigma),\tag{26}$$

where $\mathsf{g}_0=(\mathsf{g}_0|_{\overline{C}},\mathsf{w}\mathsf{g}_0)=(\sigma^{-1}\check{\mathsf{f}}_1\mathsf{v}_A,\underline{f})\colon C=\overline{C}\oplus\Bbbk\to\Bbbk$, $\mathsf{g}_1=\overline{\Delta}_C^{(k)}\cdot\check{\mathsf{g}}_1^{\otimes k}=\overline{\Delta}_C^{(k)}\cdot(\sigma^{-1}\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma)^{\otimes k}\colon\overline{C}\to\overline{A}[1]^{\otimes k}$, $k>0$.

Naturality of (25) with respect to A means that for each $\mathsf{h}\colon A\to B\in\mathrm{UCCAlg}$

$$\begin{array}{ccc}\mathrm{UCCAlg}(\overline{C}[-1]T^{\geqslant},A) & \xrightarrow{\sim} & \mathrm{CACoalg}(C,\overline{A}[1]T^{\geqslant})\\ {\scriptstyle\mathrm{UCCAlg}(1,\mathsf{h})}\downarrow & = & \downarrow{\scriptstyle\mathrm{CACoalg}(C,\mathsf{Bar}\,h)}\\ \mathrm{UCCAlg}(\overline{C}[-1]T^{\geqslant},B) & \xrightarrow{\sim} & \mathrm{CACoalg}(C,\overline{B}[1]T^{\geqslant})\end{array}\tag{27}$$

The left-bottom path takes $(\mathsf{f}_1,\underline{f})$ to

$$(\mathsf{f}_1\mathsf{h}_1,\underline{f}+\underline{h})\mapsto(\check{\mathsf{f}}_1\mathsf{h}_1\,\overline{\mathrm{pr}}_B,\check{\mathsf{f}}_1\mathsf{h}_1\mathsf{v}_B,\sigma(\underline{f}+\underline{h}))$$
$$\xmapsto{[1]}(\sigma^{-1}\check{\mathsf{f}}_1\mathsf{h}_1\,\overline{\mathrm{pr}}_B\,\sigma,\sigma^{-1}\check{\mathsf{f}}_1\mathsf{h}_1\mathsf{v}_B\sigma,(\underline{f}+\underline{h})\sigma)\mapsto(\mathsf{q}_1,\mathsf{q}_0\sigma),$$

where

$$\mathsf{q}_1=\overline{\Delta}_C^{(k)}\cdot(\sigma^{-1}\check{\mathsf{f}}_1\mathsf{h}_1\,\overline{\mathrm{pr}}_B\,\sigma)^{\otimes k}\colon\overline{C}\to\overline{B}[1]^{\otimes k},\quad\mathsf{q}_0=(\sigma^{-1}\check{\mathsf{f}}_1\mathsf{h}_1\mathsf{v}_B,\underline{f}+\underline{h})\colon C=\overline{C}\oplus\Bbbk\to\Bbbk.$$

The top bijection takes $(\mathsf{f}_1,\underline{f})$ to (26) and the right morphism takes it to $\bigl(\mathsf{g}_1\cdot\mathrm{Bar}_1\,h,(\mathsf{g}_0+(\varepsilon_C\oplus\mathsf{g}_1)\,\mathrm{Bar}_0\,h)\sigma\bigr)$. We have

$$\mathsf{g}_1\cdot\mathrm{Bar}_1\,h=\overline{\Delta}_C^{(k)}\cdot\check{\mathsf{g}}_1^{\otimes k}\cdot\overline{h}_1^{\otimes k}=\overline{\Delta}_C^{(k)}(\sigma^{-1}\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma\overline{h}_1)^{\otimes k}=\overline{\Delta}_C^{(k)}(\sigma^{-1}\check{\mathsf{f}}_1(1-\mathsf{v}\eta)\mathsf{h}_1\sigma\,\overline{\mathrm{pr}}_B)^{\otimes k}=$$
$$=\overline{\Delta}_C^{(k)}(\sigma^{-1}\check{\mathsf{f}}_1\mathsf{h}_1\,\overline{\mathrm{pr}}_B\,\sigma)^{\otimes k}=\mathsf{q}_1\colon\overline{C}\to B[1]^{\otimes k},$$
$$\mathsf{g}_0+\mathsf{g}_1\,\mathrm{Bar}_0\,h=\sigma^{-1}\check{\mathsf{f}}_1\mathsf{v}_A+\sigma^{-1}\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma h_1\mathbf{v}_B=\sigma^{-1}\check{\mathsf{f}}_1\mathsf{h}_1\mathsf{v}_B=\mathsf{q}_0\colon\overline{C}\to\Bbbk$$

due to obvious identity $\mathsf{v}_A + \overline{\mathrm{pr}}_A \mathsf{h}_1\mathsf{v}_B = \mathsf{h}_1\mathsf{v}_B \colon A \to \Bbbk$. Furthermore,

$$\mathsf{w}\mathsf{g}_0 + (\mathsf{w}, 0)(\varepsilon_C \oplus \mathsf{g}_1)\operatorname{Bar}_0 h = \underline{f} + h_0\mathbf{v}_B = \underline{f} + \underline{h} = \mathsf{w}\mathsf{q}_0 \colon \Bbbk \to \Bbbk$$

due to computation $h_0\mathbf{v}_B = \mathsf{h}_0\mathsf{v}_B = \underline{h}\eta_B\mathsf{v}_B = \underline{h}$. Therefore, equation (27) is proven.

Naturality of (25) with respect to C means that for each $\mathsf{j}\colon C \to D \in \mathrm{CACoalg}$

$$\begin{array}{ccc}
\mathrm{UCCAlg}(\overline{D}[-1]T^{\geqslant}, A) & \xrightarrow{\sim} & \mathrm{CACoalg}(D, \overline{A}[1]T^{\geqslant}) \\
{\scriptstyle \mathrm{UCCAlg}(\mathsf{Cobar}\, j, A)}\downarrow & = & \downarrow{\scriptstyle \mathrm{CACoalg}(\mathsf{j},1)} \\
\mathrm{UCCAlg}(\overline{C}[-1]T^{\geqslant}, A) & \xrightarrow{\sim} & \mathrm{CACoalg}(C, \overline{A}[1]T^{\geqslant})
\end{array} \tag{28}$$

The left-bottom path takes $(\mathsf{f}_1, \underline{f})$ to

$$\begin{aligned}
((\mathsf{Cobar}_1 j)\cdot \mathsf{f}_1, \underline{\operatorname{Cobar} j} + \underline{f}) &\mapsto (j_1\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A, j_0 + j_1\check{\mathsf{f}}_1\mathsf{v}_A, \sigma(\underline{\operatorname{Cobar} j} + \underline{f})) \\
&\xmapsto{[1]} (\sigma^{-1}j_1\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma, \sigma^{-1}(j_0 + j_1\check{\mathsf{f}}_1\mathsf{v}_A)\sigma, (\underline{\operatorname{Cobar} j} + \underline{f})\sigma) \mapsto (\mathsf{r}_1, \mathsf{r}_0\sigma),
\end{aligned}$$

which takes into an account that $((\mathsf{Cobar}_1 j)\cdot\mathsf{f}_1)^\vee = j_0\eta_A + j_1\check{\mathsf{f}}_1 \colon \overline{C}[-1] \to A$. The top bijection takes $(\mathsf{f}_1, \underline{f})$ to $(\mathsf{g}_1, \mathsf{g}_0\sigma)$ from (26) and the right morphism takes it to $(\mathsf{j}_1\mathsf{g}_1, (\mathsf{j}_0 + \mathsf{j}_1\mathsf{g}_0)\sigma)$. This coincides with $(\mathsf{r}_1, \mathsf{r}_0\sigma)$. In fact,

$$\begin{gathered}
\mathsf{j}_1\mathsf{g}_1 = \mathsf{j}_1\overline{\Delta}_D^{(k)}(\sigma^{-1}\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma)^{\otimes k} = \overline{\Delta}_C^{(k)}(\sigma^{-1}j_1\check{\mathsf{f}}_1\,\overline{\mathrm{pr}}_A\,\sigma)^{\otimes k} = \mathsf{r}_1 \colon \overline{C} \to \overline{A}[1]^{\otimes k}, \\
\mathsf{j}_0 + \mathsf{j}_1\mathsf{g}_0 = \sigma^{-1}j_0 + \mathsf{j}_1\sigma^{-1}\check{\mathsf{f}}_1\mathsf{v}_A = \sigma^{-1}(j_0 + j_1\check{\mathsf{f}}_1\mathsf{v}_A) = \mathsf{r}_0 \colon \overline{C} \to \Bbbk, \\
\mathsf{w}_C\mathsf{j}_0 + \mathsf{w}_C\mathsf{j}_1\mathsf{g}_0 = \mathsf{w}_C\mathsf{j}_0 + \mathsf{w}_D\mathsf{g}_0 = \underline{\operatorname{Cobar} j} + \underline{f} = \mathsf{w}\mathsf{r}_0 \colon \Bbbk \to \Bbbk.
\end{gathered}$$

Therefore equation (28) holds and the theorem is proven. □

Notice that both sides of (15c) do not appear in the equations at all. One may assume that the components of morphisms of curved algebras and coalgebras belonging to $\Bbbk^1$ are all 0. Then one gets subcategories uccAlg ⊂ UCCAlg and caCoalg ⊂ CACoalg with smaller sets of morphisms.

Definition 13. Objects of the category uccAlg are unit-complemented curved algebras and morphisms are graded algebra homomorphisms $\mathsf{f}\colon A \to B$ such that $\mathsf{f}m_1^B = m_1^A\mathsf{f}$, $m_0^B = m_0^A\mathsf{f}$. The composition and the identity morphisms are inherited from **gr**-alg.

Definition 14. Objects of the category caCoalg are curved augmented coalgebras and morphisms $\mathsf{g}\colon C \to D$ are pairs $(\mathsf{g}_1, \mathsf{g}_0')$ consisting of a homomorphism of augmented graded coalgebras $\mathsf{g}_1\colon C \to D$ and a $\Bbbk$-linear map $\mathsf{g}_0'\colon \overline{C} \to \Bbbk$ of degree 1 such that

$$\begin{gathered}
\delta_1^C\mathsf{g}_1 + \overline{\mathrm{pr}}_C\,\overline{\delta}_2^C(\mathsf{g}_0' \otimes \mathsf{g}_1 - \mathsf{g}_1 \otimes \mathsf{g}_0') = \mathsf{g}_1\delta_1^D \colon C \to \overline{D}, \\
\delta_0^C - \delta_1^C\mathsf{g}_0' - \overline{\mathrm{pr}}_C\,\overline{\delta}_2^C(\mathsf{g}_0' \otimes \mathsf{g}_0') = \mathsf{g}_1\delta_0^D \colon C \to \Bbbk.
\end{gathered}$$

The composition $\mathsf{h}\colon C \to E$ of morphisms $\mathsf{f}\colon C \to D$ and $\mathsf{g}\colon D \to E$ is given by $\mathsf{h}_1 = \mathsf{f}_1\mathsf{g}_1$, $\mathsf{h}_0' = \mathsf{f}_0' + \mathsf{f}_1\mathsf{g}_0'$. The unit morphism is $(\mathrm{id}, 0)$.

Notice that there is a functor Bar: uccAlg → caCoalg, making the diagram of functors

$$\begin{array}{ccc} \text{uccAlg} & \xrightarrow{\text{Bar}} & \text{caCoalg} \\ \downarrow & = & \downarrow \\ \text{UCCAlg} & \xrightarrow{\text{Bar}} & \text{CACoalg} \end{array}$$

commute on the nose. In view of (8) Bar takes a morphism $f\colon A \to B \in$ uccAlg to the strict coalgebra morphism $\mathsf{Bar}_1 f = \overline{f}_1 T^{\geqslant}\colon \overline{A}[1]T^{\geqslant} \to \overline{B}[1]T^{\geqslant}$, and the degree 1 functional is $\mathsf{Bar}'_0 f = \big(\overline{A}[1]T^{>} \xrightarrow{\mathrm{pr}_1} \overline{A}[1] \hookrightarrow A[1] \xrightarrow{f} B[1] \xrightarrow{\mathsf{v}} \Bbbk\big)$. The restriction of (9) to $\Bbbk$ vanishes.

Also there is a functor Cobar: caCoalg → uccAlg, making commutative the diagram of functors

$$\begin{array}{ccc} \text{caCoalg} & \xrightarrow{\text{Cobar}} & \text{uccAlg} \\ \downarrow & = & \downarrow \\ \text{CACoalg} & \xrightarrow{\text{Cobar}} & \text{UCCAlg} \end{array}$$

It takes a morphism $\underline{\mathsf{g}} = (\mathsf{g}_1, \mathsf{g}'_0)\colon C \to D \in$ caCoalg to the algebra homomorphism $\mathsf{Cobar}\, g\colon \overline{C}[-1]T^{\geqslant} \to \overline{D}[-1]T^{\geqslant}$, specified by its components $\overline{g}_1 = g_1|\colon \overline{C}[-1] \to \overline{D}[-1]$, $g'_0\colon \overline{C}[-1] \to \Bbbk$, which coincides with (13). If $\underline{g} = 0$, then $\underline{\text{Cobar}\, g}$ given by (14) vanishes as well. Since equations (17), (18) distinguishing morphisms in diagram (16) do not involve $\underline{f}$, $\underline{g}$, we have the following consequence.

Corollary 1 (to Theorem 1)**.** *The functors* Cobar: caCoalg ⇆ uccAlg: Bar *are adjoint to each other.*

Now let us describe the full subcategories of the above categories.

Definition 15. A *unit-complemented* **dg**-*algebra* is a unit-complemented curved algebra $(A, m_2, m_1, 0, \eta, \mathsf{v})$ with $m_0 = 0$. Equivalently, it is a **dg**-algebra (A, m_2, m_1, η) with a degree 0 map $\mathsf{v}\colon A \to \Bbbk$ (splitting of the unit) such that $\eta \cdot \mathsf{v} = 1_\Bbbk$. Morphisms of such algebras are morphisms of **dg**-algebras. Their full subcategory is denoted uc**dg**Alg ⊂ uccAlg.

Definition 16. *Augmented curved coalgebras* are defined as curved augmented coalgebras $(C, \delta_2, \delta_1, \delta_0, \varepsilon, \mathsf{w})$ with

$$\mathsf{w}\delta_1 = 0, \quad \mathsf{w}\delta_0 = 0. \tag{29}$$

The full subcategory of such coalgebras is denoted acCoalg ⊂ caCoalg.

L. Positselski ([5]) formulates equations (29) as $(\mathsf{w}, 0)\colon \Bbbk \to C$ being a morphism in caCoalg. Clearly, Cobar(Ob acCoalg) ⊂ Ob uc**dg**Alg.

Proposition 1. *The functor* Bar *restricts to a functor* Bar: uc**dg**Alg → acCoalg, *which has a left adjoint. The adjunction is* Cobar: acCoalg ⇆ uc**dg**Alg: Bar.

Proof. We have to prove that Bar(Ob uc**dg**Alg) ⊂ Ob acCoalg. This follows from two remarks. First, $\mathsf{w}^{\mathrm{Bar}\,A}\delta_1^{\mathrm{Bar}\,A} = \big(\Bbbk \xrightarrow{\mathrm{in}_0} \overline{A}[1]T^{\geqslant} \xrightarrow{\overline{b}} \overline{A}[1]T^{\geqslant}\big) = 0$ since

$$\overline{b} = \sum_{a+k+c=n}^{k>0} \big(1^{\otimes a} \otimes \overline{b}_k \otimes 1^{\otimes c}\colon \overline{A}[1]^{\otimes n} \to \overline{A}[1]^{\otimes(a+1+c)}\big).$$

Second, $\mathsf{w}^{\mathrm{Bar}\,A}\delta_0^{\mathrm{Bar}\,A} = -\big(\Bbbk \xrightarrow{\mathrm{in}_0} A[1]T^{\geqslant} \xrightarrow{\check{b}} A[1] \xrightarrow{\mathsf{v}} \Bbbk\big) = 0$ since $b_0 = 0$. □

4.1. Twisting cochains. Let us consider a unit-complemented curved algebra A and a curved augmented coalgebra C. A morphism $f \in \operatorname{uccAlg}(\overline{C}[-1]T^{\geqslant}, A)$ is identified with a degree 0 map $\check{\mathsf{f}}_1 : \overline{C}[-1] \to A$ which satisfies equations (19). Equivalently, the degree 1 map $\theta : C \to A$ satisfies the equations

$$\mathsf{w}\theta = 0 : \Bbbk \to A, \tag{30a}$$

$$\theta m_1^A + \delta_1\theta = \delta_0\eta^A + \varepsilon^C m_0^A - \delta_2(\theta\otimes\theta)m_2^A : C \to A. \tag{30b}$$

In fact, each solution of (30a) has the form $\theta = \langle C \xrightarrow{\overline{\mathrm{pr}}_C} \overline{C} \xrightarrow{\sigma^{-1}} \overline{C}[-1] \xrightarrow{\check{\mathsf{f}}_1} A\rangle$ for a unique $\check{\mathsf{f}}_1$. Restricting (30b) to $\overline{C}$ gives the top equation from (19), while restricting to the image of $\mathsf{w} : \Bbbk \to C$ gives the bottom equation from (19).

Definition 17. The degree 1 map $\theta : C \to A$ that satisfies (30) is called a *twisting cochain.*

The set $\operatorname{Tw}(C, A) = \{\text{twisting cochains } \theta : C \to A\}$ is in bijection with the homomorphism sets

$$\operatorname{uccAlg}(\overline{C}[-1]T^{\geqslant}, A) \xrightarrow{\sim} \operatorname{Tw}(C, A) \xrightarrow{\sim} \operatorname{caCoalg}(C, \overline{A}[1]T^{\geqslant}),$$

$$\check{\mathsf{f}}_1 \longmapsto \overline{\mathrm{pr}}_C\,\sigma^{-1}\check{\mathsf{f}}_1 = \theta \longmapsto (\theta\big|_{\overline{C}}\sigma, \theta\big|_{\overline{C}}\mathsf{v}) = (\check{\mathsf{g}}_1, \check{\mathsf{g}}_0').$$

When A is a unit-complemented **dg**-algebra and C is an augmented curved coalgebra the notion of a twisting cochain simplifies to a degree 1 map $\theta : \overline{C} \to A$ which satisfies the equation $\theta m_1^A + \delta_1\theta = \delta_0\eta^A - \delta_2(\overline{\mathrm{pr}}_C \otimes \overline{\mathrm{pr}}_C)(\theta\otimes\theta)m_2^A : \overline{C} \to A$.

4.2. Conclusion. The results of the paper indicate that a dual notion to differential graded algebra is the augmented curved coalgebra, and not a differential graded coalgebra as one might think a priori.

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