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LIMITS OF SEQUENCES OF DARBOUX-LIKE MAPPINGS

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We call a mapping $f\colon X \to Y$ an *l-Darboux mapping* if the image of any arcwise connected subset of X is connected. We prove that the class of l-Darboux F_σ-measurable mappings of a topological space to a metric space is closed with respect to uniform limits.

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Отображение $f\colon X \to Y$ мы называем *l-Дарбу отображением*, если образ любого дугообразно связного подмножества пространства X связный. Доказано, что класс l-Дарбу F_σ-измеримых отображений между топологическим и метрическим пространствами замкнут относительно равномерных пределов.

1. Introduction. A mapping f between topological spaces X and Y has *the Darboux property* if the image of any connected subset of X is connected. A mapping f is *a connectivity mapping* if the graph of the restriction $f|_C$ is connected for every connected subset C of X. Notice that each connectivity mapping has the Darboux property. We say that f is *(weakly) Gibson* ([4]) if $f(\overline{U}) \subseteq \overline{f(U)}$ for any open (connected) subset U of X. According to [3] each Darboux mapping is weakly Gibson.

It is well known that the class of all Darboux Baire-one mappings $f\colon \mathbb{R} \to \mathbb{R}$ is closed with respect to uniform limits ([1, Theorem 3.4]). It is naturally to ask whether the same is true for mappings defined on $\mathbb{R}^n$, $n > 1$. Observe that according to [2, Corollary 9.16] a uniform limit of connectivity mappings $f_m\colon \mathbb{R}^n \to \mathbb{R}$, $n > 1$, is a connectivity mapping.

In the paper we consider *l-Darboux mappings* or *Darboux mappings in the sense of Pawlak* (see definitions in Section 3, [7]) and prove that the class of all l-Darboux F_σ-measurable mappings $f\colon X \to Y$, where X is a topological space and Y is a metric space, is closed with respect to uniform limits. To do this we firstly prove the auxiliary fact that each weakly Gibson F_σ-measurable mapping $f\colon X \to Y$ is a connectivity mapping if X is a connected subset of the real line and Y is an arbitrary space (see Section 2).

It was proved in [6] that any function $f\colon \mathbb{R} \to \mathbb{R}$ is a pointwise limit of Darboux mappings. In Section 4 we show that any mapping between an ω-resolvable space X and a separable space Y is a pointwise limit of Gibson mappings.

2. Connectivity functions and $\mathcal{G}$-closed sets. For a topological space X we denote by $\mathcal{T}(X)$ the topology of X, by $\mathcal{C}(X)$ we denote the collection of all connected subsets of X and by $\mathcal{G}(X)$ we denote the family of all open connected subsets of X.

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A mapping $f\colon X \to Y$ is

- a *Baire-one mapping*, $f \in \mathrm{B}_1(X,Y)$, if there exists a sequence of continuous mappings $f_n\colon X \to Y$ which is pointwise convergent to f on X;
- *F_σ-measurable* if $f^{-1}(V)$ is an F_σ-subset of X for any open set $V \subseteq Y$;
- *(weakly) Gibson* if for an arbitrary $U \in \mathcal{T}(X)$ $(U \in \mathcal{G}(X))$ we have $f(\overline{U}) \subseteq \overline{f(U)}$;
- *Darboux*, $f \in D(X,Y)$, if for any $C \in \mathcal{C}(X)$ the set $f(C)$ is connected;
- a *connectivity mapping* if the graph of $f|_C$ is connected for every connected subset C of X.

A subset A of X is called *ambiguous* if it is both F_σ and G_δ in X.

If $f\colon X \to Y$ is a mapping we define $\gamma_f\colon X \to X \times Y$ by $\gamma_f(x) = (x, f(x))$ for all $x \in X$.

If a space X is homeomorphic to a space Y we write $X \simeq Y$.

Let $\mathcal{A}$ be a system of subsets of X. A set $E \subseteq X$ is called *closed with respect to* $\mathcal{A}$ or, briefly, *$\mathcal{A}$-closed*, if the inclusion $A \subseteq E$ implies $\overline{A} \subseteq E$ for any $A \in \mathcal{A}$.

Here and throughout the paper we consider $\mathcal{G}$-closed subsets of X, i.e. such sets which are closed with respect to the system of all open connected subsets of X.

Let $E \subseteq X$. Clearly, if E is closed or $\mathrm{int}E = \varnothing$ then E is $\mathcal{G}$-closed in X. Remark that the converse is not true. Indeed, let $E = \bigcup_{n=1}^{\infty}[\frac{1}{2n}, \frac{1}{2n-1}]$. Then E is a $\mathcal{G}$-closed subset of $\mathbb{R}$ with non-empty interior which is not closed.

Theorem 1. *Let X be a connected and locally connected hereditarily Baire space, X_1 be a $\mathcal{G}$-closed ambiguous subset of X such that its complement X_2 is also $\mathcal{G}$-closed. Then either $X_1 = X$ or $X_2 = X$.*

Proof. Suppose on the contrary that $X_1 \neq X$ and $X_2 \neq X$. Set $F = \overline{X}_1 \cap \overline{X}_2$. Then $F \neq \varnothing$, because X is connected. We show that $X_1 \cap F$ is a dense set in F. To obtain a contradiction, assume that there exists $x_0 \in F$ and an open connected neighborhood U of x_0 in X such that $U \cap F \subseteq X_2$. Then $x_0 \in \overline{X}_1 \cap X_2$, consequently, $U \cap X_1 \neq \varnothing$. Take an arbitrary $a \in U \cap X_1$. It is easy to check that $a \notin \overline{X}_2$. Let G be a component of $X \setminus \overline{X}_2$ such that $a \in G$. Then G is open in X, since X is locally connected. Notice that $U \cap G \neq \varnothing \neq U \setminus G$. According to [5, p. 136], $U \cap \mathrm{fr}G \neq \varnothing$. Since G is closed in $X \setminus \overline{X}_2$, $\mathrm{fr}G \subseteq \overline{X}_2$. Moreover, $G \subseteq X_1$. Hence, $\mathrm{fr}G \subseteq F$. Take any $b \in U \cap \mathrm{fr}G$. Then $b \in X_2$. Since X_1 is $\mathcal{G}$-closed, $b \in \overline{G} \subseteq X_1$, a contradiction. Therefore, the set $X_1 \cap F$ is dense in F. In the same manner we can see that the set $X_2 \cap F$ is also dense in F. Then $X_1 \cap F$ and $X_2 \cap F$ are disjoint dense G_δ-subsets of F, which is impossible, because F is a Baire space. Hence, our assumption is false. Therefore, $X_1 = X$ or $X_2 = X$. □

Corollary 1. *Let X be a connected subset of $\mathbb{R}$ and Y be a topological space. Then every weakly Gibson F_σ-measurable mapping $f\colon X \to Y$ is a connectivity mapping.*

Proof. Remark that it is sufficient to show that the graph of f is connected. Assume it is not true. Then $\gamma_f(X) = W_1 \cup W_2$, where $W_1 \cap W_2 = \varnothing$ and W_i is a non-empty clopen set in $\gamma_f(X)$ for every $i \in \{1,2\}$. Let $X_i = \gamma_f^{-1}(W_i)$ for $i \in \{1,2\}$. Then $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \varnothing$.

We prove that the set X_i is $\mathcal{G}$-closed in X for $i \in \{1,2\}$. Consider an open connected subset U of X such that $U \subseteq X_1$. Then $\gamma_f(\overline{U}) \subseteq \overline{\gamma_f(U)}$. Indeed, let $x_0 \in U$ and W be a neighborhood of $\gamma_f(x_0)$ in $X \times Y$. Choose an open connected neighborhood U_0 of x_0 in

X and a neighborhood V of $f(x_0)$ in Y such that $U_0 \times V \subseteq W$. Taking into account that $U_0 \cap U$ is an open connected subset of X, $x_0 \in \overline{U_0 \cap U}$ and f is weakly Gibson, we obtain that there exists $x' \in U_0 \cap U$ such that $f(x') \in V$. Then $\gamma_f(x') \in U_0 \times V \subseteq W$. Hence,

$$\gamma_f(\overline{U}) \subseteq \overline{\gamma_f(U)} \subseteq \overline{W_1} \cap \gamma_f(X) = W_1.$$

Therefor, $\overline{U} \subseteq X_1$. Analogously we can prove that X_2 is $\mathcal{G}$-closed in X.

Observe that the mapping γ_f is F_σ-measurable. Indeed, let G be an open subset of $X \times Y$ and $\{B_k \colon k \in \mathbb{N}\}$ be a base of open sets in X. Put $V_k = \bigcup\{V \colon V$ is open in Y and $B_k \times V \subseteq G\}$. Then $W = \bigcup_{k=1}^\infty B_k \times V_k$. Since $\gamma_f^{-1}(G) = \bigcup_{k=1}^\infty (B_k \cap f^{-1}(V_k))$, the set $\gamma_f^{-1}(G)$ is F_σ in X.

Since W_i is a clopen set, X_i is an ambiguous subset of X for $i \in \{1,2\}$. By Theorem 1 we have that either $X_1 = X$ or $X_2 = X$. Then either $W_1 = \varnothing$ or $W_2 = \varnothing$, which implies a contradiction. □

3. Uniform limit of l-Darboux functions. We call a mapping $f \colon X \to Y$ *an l-Darboux mapping* if $f(C)$ is connected for any arcwise connected set $C \subseteq X$. The collection of all l-Darboux mappings between X and Y we denote by $D^l(X,Y)$. Evidently, each Darboux mapping is l-Darboux. It is not hard to verify that $D(\mathbb{R},Y) = D^l(\mathbb{R},Y)$. But if the domain space is more complicated the inclusion $D^l(X,Y) \subset D(X,Y)$ might to be strict (see Example 1).

Proposition 1. *Let X and Y be topological spaces. Then $f \in D^l(X,Y)$ if and only if for any set $K \subseteq X$ with $K \simeq [0,1]$ the set $f(K)$ is connected.*

Proof. Clearly, we only need to prove the sufficiency. Let A be an arcwise connected subset of X. Fix an arbitrary $a \in A$. For all $x \in A$ we consider a homeomorphic embedding $\varphi_x \colon [0,1] \to A$ such that $\varphi_x(0) = a$ and $\varphi_x(1) = x$. Write $K_x = \varphi_x([0,1])$. Then $f(A) = \bigcup_{x \in A} f(K_x)$. Since $K_x \simeq [0,1]$, the set $f(K_x)$ is connected. Therefore, $f(A)$ is connected, provided $f(a) \in \bigcap_{x \in A} f(K_x)$. □

Example 1. For all $(x,y) \in \mathbb{R}^2$ define

$$f(x,y) = \begin{cases} \sin\frac{1}{x}, & x > 0, \\ 1, & x \le 0. \end{cases}$$

Then the mapping γ_f is l-Darboux and is not Darboux.

Proof. Let $K \subseteq \mathbb{R}^2$ be such a set that $K \simeq [0,1]$. It is easy to check that $f \in D(\mathbb{R}^2, \mathbb{R})$. Then $g = f|_K \in D(K, \mathbb{R})$. By Corollary 1, g is a connectivity function. In particular, g has a connected graph. Since $\gamma_f(K)$ is equal to the graph of g, $\gamma_f(K)$ is connected. Therefore, γ_f is l-Darboux by Proposition 1.

Consider the set $C = \{(x,y) \in \mathbb{R}^2 \colon y = \sin\frac{1}{x}, x > 0\} \cup \{0,0\}$. It is not hard to verify that the set $\gamma_f(C)$ is disconnected. Hence, γ_f does not satisfy the Darboux property. □

Let (X,d) be a metric space, $x_0 \in X$ and $\varepsilon > 0$. By $B(x_0,\varepsilon)$ we denote the set $\{x \in X \colon d(x,x_0) < \varepsilon\}$.

Theorem 2. *Let X be a topological space, (Y,d) be a metric space, $(f_n)_{n=1}^\infty$ be a uniformly convergent sequence of (weakly) Gibson mappings $f_n \colon X \to Y$ and let $f = \lim\limits_{n\to\infty} f_n$. Then f is (weakly) Gibson.*

Proof. Take an arbitrary open (connected) set $U \subseteq X$, a point $x_0 \in \overline{U}$ and $\varepsilon > 0$. Choose a number N so that $d(f_N(x), f(x)) < \varepsilon/2$ for all $x \in X$. In particular, $f_N(x_0) \in B(f(x_0), \varepsilon/2)$. Since f_N is (weakly) Gibson, there exists $x_1 \in U$ such that

$$f_N(x_1) \in B(f(x_0), \varepsilon/2).$$

Then

$$d(f(x_1), f(x_0)) \leq d(f(x_1), f_N(x_1)) + d(f_N(x_1), f(x_0)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $f(x_1) \in B(f(x_0), \varepsilon)$. Therefore, $f(x_0) \in \overline{f(U)}$. □

Theorem 3. *Let X be a topological space, (Y, d) be a metric space and $(f_n)_{n=1}^{\infty}$ be a uniformly convergent sequence of F_σ-measurable mappings $f_n \in D^l(X, Y)$. Then $f = \lim\limits_{n\to\infty} f_n$ is an F_σ-measurable l-Darboux mapping.*

Proof. Assume that $f \notin D^l(X, Y)$ and choose a subset $K \simeq [0, 1]$ of X so that $f(K) = W_1 \cup W_2$, where W_1 and W_2 are disjoint non-empty clopen subsets of $Z = f(K)$. Let $K_i = f^{-1}(W_i)$ for $i \in \{1, 2\}$. Set $g = f|_K$ and $g_n = f_n|_K$ for all $n \in \mathbb{N}$.

Show that $g_n \colon K \to Z$ is weakly Gibson for every n. Let $U \subseteq K$ be an open connected set, $x_0 \in \overline{U}$ and let V be an open neighborhood of $f(x_0)$ in Y. Suppose that $f(U) \cap V = \varnothing$. Denote $C = U \cup \{x_0\}$. Then

$$f(C) = f(U) \cup \{f(x_0)\} \subseteq (Y \setminus V) \cup \{f(x_0)\},$$

which contradicts the connectedness of C.

According to Theorem 2 the mapping $g \colon K \to Z$ is weakly Gibson. Then K_1 and K_2 are $\mathcal{G}$-closed subsets of K. Moreover, g is F_σ-measurable as a uniform limit of F_σ-measurable mappings ([5, p. 395]). Therefore, K_1 and K_2 are ambiguous subsets of K. Hence, $K_1 = K$ or $K_2 = K$ by Theorem 1. Consequently, $W_1 = \varnothing$ or $W_2 = \varnothing$, a contradiction. □

4. Pointwise limit of Gibson functions. Recall that a topological space X is *ω-resolvable* if there exists a sequence $(X_n)_{n=1}^{\infty}$ of dense subspaces of X such that $X_n \cap X_m = \varnothing$ for $n \neq m$ and $X = \bigcup_{n=1}^{\infty} X_n$.

Theorem 4. *Let X be an ω-resolvable space and Y be a separable space. Then any mapping $f \colon X \to Y$ is a pointwise limit of sequence of Gibson mappings.*

Proof. Let $(X_n)_{n=1}^{\infty}$ be a sequence of dense subspaces of X such that $X_n \cap X_m = \varnothing$ for $n \neq m$ and $X = \bigcup_{n=1}^{\infty} X_n$, and let $D = \{y_n \colon n \in \mathbb{N}\}$ be a dense subspace of Y. For all $n \in \mathbb{N}$ and $x \in X$ define

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in X_1 \cup \cdots \cup X_n, \\ y_k, & \text{if } x \in X_{n+k} \text{ for some } k. \end{cases}$$

Fix $n \in \mathbb{N}$ and show that $f_n \colon X \to Y$ is Gibson. Indeed, take an open set $U \subseteq X$, a point $x_0 \in \overline{U}$ and a neighborhood V of $f_n(x_0)$. Choose a number N so that $y_N \in V$. Since X_{N+n} is dense in X, there is a point $x \in U \cap X_{N+n}$. Then $y_N = f_n(x)$. Hence, $f_n(\overline{U}) \subseteq \overline{f_n(U)}$.

It can be easily seen that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$. □

Remark 1. Let us consider the function $f \colon \mathbb{R} \to \{0, 1\}$, $f = \chi_{\{0\}}$. According to the previous theorem f is a pointwise limit of a sequence of Gibson functions. But f is not a pointwise limit of Darboux functions, because every Darboux function $h \colon \mathbb{R} \to \{0, 1\}$ is constant.

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