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MULTIPLICATIVELY PERIODIC SUBHARMONIC FUNCTIONS IN THE PUNCTURED EUCLIDEAN SPACE

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It is proved that each multiplicatively periodic subharmonic function in $\mathbb{R}^m\setminus\{0\}$, $m \geq 3$, is constant. Examples of non-constant multiplicatively periodic differences of subharmonic in $\mathbb{R}^m\setminus\{0\}$ functions are constructed.

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Доказано, что мультипликативно периодическая функция в $\mathbb{R}^m\setminus\{0\}$, $m \geq 3$, является постоянной. Построен пример отличной от тождественной постоянной мультипликативно периодической в $\mathbb{R}^m\setminus\{0\}$ функции, являющейся разностью субгармонических функций.

1. Introduction. The theory of multiplicatively periodic (loxodromic) meromorphic functions in $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ was elaborated by O. Rausenberger ([1]). It is tightly connected with the theory of elliptic functions ([2], [3]).

We consider multiplicatively periodic subharmonic functions in the punctured Euclidean space $\mathring{\mathbb{R}}^m = \mathbb{R}^m\setminus\{0\}$, $m \geq 3$, that is, the functions satisfying the condition $u(qx) = u(x)$ for some q, $0 < q < 1$, and all $x \in \mathring{\mathbb{R}}^m$.

2. Multiplicatively periodic subharmonic functions in the punctured Euclidean space. Our main result is the following theorem.

Theorem. *Each multiplicatively periodic subharmonic function in $\mathring{\mathbb{R}}^m$ is constant. There are non-constant multiplicatively periodic differences of subharmonic functions in $\mathring{\mathbb{R}}^m$.*

Denote
$$BL(s,r) = \{x \in \mathring{\mathbb{R}}^m : s < |x| < r\}, \quad 0 < s < r.$$

Let μ be a measure in $\mathring{\mathbb{R}}^m$. Fix $t_0 > 0$ and a value $\nu(t_0)$. The function

$$\nu(t) = \begin{cases} \nu(t_0) + \mu\{z : t_0 < |z| \leq t\}, & t_0 < t; \\ \nu(t_0) - \mu\{z : 0 < t < |z| \leq t_0\}, & t < t_0, \end{cases}$$

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is said to be the *distribution function of the measure* μ.

Such a function is right hand continuous, nondecreasing and determined up to a constant. The difference $\nu(t_2) - \nu(t_1)$ gives a measure of the ball layer $\{z : t_1 < |z| \le t_2\}$.

Theorem A ([4]). *Let $u(x)$ be a subharmonic in $\overline{BL}(s,r)$ function, non-identical to $-\infty$, and let μ be its Riesz measure. Then*

$$N(s,r;u) := \frac{m-2}{1-r^{2-m}} \int_1^r \frac{\nu(t)}{t^{m-1}}dt - \frac{m-2}{s^{2-m}-1}\int_s^1 \frac{\nu(t)}{t^{m-1}}dt =$$
$$= \frac{1}{1-r^{2-m}}\left[\frac{1}{c_m r^{m-1}}\int_{S(0,r)} u(x)d\sigma(x) - \frac{1}{c_m}\int_{S(0,1)} u(x)d\sigma(x)\right] -$$
$$-\frac{1}{s^{2-m}-1}\left[\frac{1}{c_m}\int_{S(0,1)} u(x)d\sigma(x) - \frac{1}{c_m s^{m-1}}\int_{S(0,s)} u(x)d\sigma(x)\right] := F(s,r;u),\ \ s<1<r,$$

where c_m is the area of the unit sphere in $\mathbb{R}^m$, $\nu(t)$ is the distribution function of μ, $d\sigma(x)$ is the element of surface area, and $S(0,r)$ is the sphere of radius r centered at the origin.

The proof of Theorem needs an auxiliary lemma.

Lemma. *Let $u(x)$ be a multiplicatively periodic subharmonic functions in $\mathring{\mathbb{R}}^m$, $\nu(t)$ is a distribution function of its Riesz measure. Then $\nu(t) \equiv \mathrm{const}$.*

Proof. Putting $s = q^n$, $r = q^{-n}$, $n \in \mathbb{N}$, in the function $F(s,r;u)$ from Theorem A and using the equalities $u(q^n x) = u(x)$, $u(q^{-n}x) = u(x)$, $x \in S(0,1)$, we have $F(q^n, q^{-n}; u) = 0$. Consequently, $N(q^n, q^{-n}; u) = 0$, i.e.

$$\frac{m-2}{(q^n)^{2-m}-1}\left[(q^n)^{2-m}\int_1^{q^{-n}} \frac{\nu(t)}{t^{m-1}}dt - \int_{q^n}^1 \frac{\nu(t)}{t^{m-1}}dt\right] = 0. \tag{1}$$

Denote

$$I_1 = \int_1^{q^{-n}} \frac{\nu(t)}{t^{m-1}}dt, \quad I_2 = \int_{q^n}^1 \frac{\nu(t)}{t^{m-1}}dt.$$

Then we can rewrite the equality (1), up to a factor, as follows

$$I_1 - (q^n)^{m-2} I_2 = 0. \tag{2}$$

The function $\nu(t)$ is monotone, then, applying the Bonnet theorem, we can represent the integrals I_1, I_2 as follows

$$I_1 = \nu(1)\int_1^{\xi} \frac{dt}{t^{m-1}} + \nu(q^{-n})\int_{\xi}^{q^{-n}} \frac{dt}{t^{m-1}}, \tag{3}$$
$$I_2 = \nu(q^n)\int_{q^n}^{\eta} \frac{dt}{t^{m-1}} + \nu(1)\int_{\eta}^1 \frac{dt}{t^{m-1}}, \tag{4}$$

where $\xi \in [1, q^{-n}]$, $\eta \in [q^n, 1]$.

If $\xi = q^{-n}$, then (3) implies

$$I_1 = \int_1^{q^{-n}} \frac{\nu(1)}{t^{m-1}}dt, \qquad \text{i.e.} \qquad \int_1^{q^{-n}} \frac{\nu(t)-\nu(1)}{t^{m-1}}dt = 0.$$

It means that $\nu(t) = \nu(1)$, $1 \le t < q^{-n}$. If $\eta = q^n$, then (4) implies similarly $\nu(t) = \nu(1)$, $q^n < t \le 1$. Thus we obtain

$$\nu(t) = \nu(1), \quad t \in (q^n, q^{-n}).$$

Consider now the case, when $1 \le \xi < q^{-n}$ and $q^n < \eta \le 1$. Calculating the integrals, we obtain

$$I_1 = \frac{1}{2-m}\left[\nu(1)\left(\frac{1}{\xi^{m-2}} - 1\right) + \nu(q^{-n})\left((q^n)^{m-2} - \frac{1}{\xi^{m-2}}\right)\right], \tag{5}$$

$$I_2 = \frac{1}{2-m}\left[\nu(q^n)\left(\frac{1}{\eta^{m-2}} - \frac{1}{(q^n)^{m-2}}\right) + \nu(1)\left(1 - \frac{1}{\eta^{m-2}}\right)\right]. \tag{6}$$

Using the equalities (5) and (6), we can rewrite (2) as follows

$$\frac{\big(\nu(1) - \nu(q^{-n})\big)}{2-m}\Big(\frac{1}{\xi^{m-2}} - (q^n)^{m-2}\Big) + \frac{\big(\nu(q^n) - \nu(1)\big)}{2-m}\Big(1 - \frac{(q^n)^{m-2}}{\eta^{m-2}}\Big) = 0. \tag{7}$$

As $1 \le \xi < q^{-n}$, $q^n < \eta \le 1$, we have

$$\frac{1}{\xi^{m-2}} - (q^n)^{m-2} > 0, \quad 1 - \left(\frac{q^n}{\eta}\right)^{m-2} > 0.$$

Since the distribution function ν is nondecreasing, the equality (7) implies $\nu(1) - \nu(q^{-n}) = 0$, $\nu(q^n) - \nu(1) = 0$, i.e. $\nu(1) = \nu(q^n) = \nu(q^{-n})$, $n \in \mathbb{N}$. It means that $\nu(t) \equiv \text{const}$, because n is arbitrary. □

Proof of the theorem. Denote

$$BL = \{x \in \mathring{\mathbb{R}}^m : q < |x| \le 1\}.$$

Since u is subharmonic in $\overline{BL}$, it is bounded above on $\overline{BL}$. As $u(q^n x) = u(x)$, $n \in \mathbb{Z}$, then u is bounded above on $\mathring{\mathbb{R}}^m$. Hence [5] the point $0 = (0, 0, ..., 0)$ is removable, that is the function admits subharmonic continuation in whole $\mathbb{R}^m$, which we denote again by u. Since u is bounded above on $\mathbb{R}^m$, then ([6]) it is represented as follows

$$u(x) = C - \int_{\mathbb{R}^m} \frac{d\mu_u(\xi)}{|x - \xi|^{m-2}}, \tag{8}$$

where μ_u is the Riesz measure of u.

In virtue of Lemma, μ equals zero on $\mathring{\mathbb{R}}^m$. Thus it can be concentrating in $(0, 0, ..., 0)$ only. Then (8) implies

$$u(x) = C - \frac{C_1}{|x|^{m-2}}, \ x \in \mathbb{R}^m. \tag{9}$$

As $u(qx) = u(x)$, (9) gives $C_1 = 0$ and $u(x) = C$, $x \in \mathring{\mathbb{R}}^m$, which finishes the proof of the first part of the theorem.

Let now consider the case $m = 3$.

Fix $q \in (0, 1)$. Consider the following series

$$\sum_{n=1}^{+\infty}\Big(\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|} - \Big|\frac{x}{q^n} - \xi\Big|^{-1}\Big), \tag{10}$$

where $\xi \in BL$, $x \in \overset{\circ}{\mathbb{R}}{}^3$. Denote

$$BL_R = BL\Big(\frac{1}{R}, R\Big), \quad R > 1.$$

Fix $\xi \in BL$ and $R > 1$. Choose the minimal n, $n \in \mathbb{N}$, such that $q^{-n}|\xi| > R$, i.e. $|\xi| > q^n R$, and denote it by n_R. Put $|\xi| - q^{n_R} R = \delta_1$. Then we have $\delta_1 > 0$ and for each x from $\overline{BL}_R$ and any $n \geq n_R$

$$|q^n x - \xi| \geq |\xi| - q^n R \geq \delta_1.$$

Since $|\xi| \leq 1$, we find also for $n \geq n_R$ that $R < \frac{q^{-n}}{|\xi|}$, that is $q^n|\xi| < \frac{1}{R}$, and then

$$|x - q^n \xi| \geq |x| - q^n|\xi| \geq \frac{1}{R} - q^{n_R}|\xi| =: \delta_2$$

for any $x \in \overline{BL}_R$.

Denote the coordinates of x, ξ by x_1, x_2, x_3 and ξ_1, ξ_2, ξ_3, respectively. Hence for such a ξ fix the remainder

$$\mathcal{R}_R(x, \xi) = \sum_{n=n_R+1}^{+\infty} \Big(\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|} - \Big|\frac{x}{q^n} - \xi\Big|^{-1}\Big).$$

converges uniformly on $\overline{BL}_R$, because for all $x \in \overline{BL}_R$ we have

$$\left|\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|}\right| = \left|\frac{|q^n x - \xi| - |\xi|}{|\xi||q^n x - \xi|}\right| = \left|\frac{|q^n x - \xi|^2 - |\xi|^2}{|\xi||q^n x - \xi|(|\xi| + |q^n x - \xi|)}\right| =$$
$$= \left|\frac{q^n\big(q^n|x|^2 - 2(x_1\xi_1 + x_2\xi_2 + x_3\xi_3)\big)}{|\xi||q^n x - \xi|(|\xi| + |q^n x - \xi|)}\right| \leq \frac{q^n(R^2 + 2R)}{q^2|q^n x - \xi|} \leq q^{n-2}\frac{R^2 + 2R}{\delta_1}.$$

Thus

$$\left|\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|}\right| \leq q^{n-2}\frac{R^2 + 2R}{\delta_1}.$$

Similarly

$$\left|\frac{x}{q^n} - \xi\right|^{-1} = \frac{q^n}{|x - q^n \xi|} \leq \frac{q^n}{\delta_2}.$$

Now the series (10) can be rewritten as follows

$$\sum_{n=1}^{+\infty} \Big(\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|} - \Big|\frac{x}{q^n} - \xi\Big|^{-1}\Big) = \mathcal{S}_{n_R}(x, \xi) + \mathcal{R}_R(x, \xi),$$

where $\mathcal{S}_{n_R}(x, \xi)$ is n_R-partial sum of (10).

$\mathcal{S}_{n_R}(x, \xi)$ is a subharmonic function in $\overset{\circ}{\mathbb{R}}{}^3$ as a sum of the Riesz kernels. The remainder $\mathcal{R}_R(x, \xi)$ as a series with harmonic in $\overline{BL}_R$ members converges uniformly, as we showed above, on $\overline{BL}_R$ to a harmonic function, which we denote by $h_R(x, \xi)$.

Then the function

$$v_R(x, \xi) = \mathcal{S}_{n_R}(x, \xi) + h_R(x, \xi)$$

is subharmonic in $\overline{BL}_R$ as the sum of the subharmonic and harmonic functions.

Now we are going to define a subharmonic function v in $\overset{\circ}{\mathbb{R}}{}^3$ as follows.

Fix $\xi \in BL$. For arbitrary x from $\overset{\circ}{\mathbb{R}}{}^3$ there is ρ, $\rho > 1$, such that $x \in BL_\rho$, and we put $v(x,\xi) = v_\rho(x,\xi)$. The function $v(x,\xi)$ does not depend on ρ. Indeed, if $\rho < R$ we have

$$v_R(x,\xi) - v_\rho(x,\xi) = \mathcal{S}_{n_R}(x,\xi) - \mathcal{S}_{n_\rho}(x,\xi) + h_R(x,\xi) - h_\rho(x,\xi) =$$

$$= \sum_{j=1}^{n_R}\left(\frac{1}{|\xi|} - \frac{1}{|q^j x - \xi|} - \frac{1}{\left|\frac{x}{q^j} - \xi\right|}\right) - \sum_{j=1}^{n_\rho}\left(\frac{1}{|\xi|} - \frac{1}{|q^j x - \xi|} - \frac{1}{\left|\frac{x}{q^j} - \xi\right|}\right) +$$

$$+ \sum_{j=n_R+1}^{+\infty}\left(\frac{1}{|\xi|} - \frac{1}{|q^j x - \xi|} - \frac{1}{\left|\frac{x}{q^j} - \xi\right|}\right) - \sum_{j=n_\rho+1}^{+\infty}\left(\frac{1}{|\xi|} - \frac{1}{|q^j x - \xi|} - \frac{1}{\left|\frac{x}{q^j} - \xi\right|}\right) =$$

$$= \sum_{j=n_\rho+1}^{n_R}\left(\frac{1}{|\xi|} - \frac{1}{|q^j x - \xi|} - \frac{1}{\left|\frac{x}{q^j} - \xi\right|}\right) - \sum_{j=n_\rho+1}^{n_R}\left(\frac{1}{|\xi|} - \frac{1}{|q^j x - \xi|} - \frac{1}{\left|\frac{x}{q^j} - \xi\right|}\right) = 0.$$

Recalling the properties of $v_\rho(x,\xi)$, we see that the function $v(x,\xi)$ is subharmonic in $\overset{\circ}{\mathbb{R}}{}^3$. In other words, the function $v(x,\xi)$ is the subharmonic continuation of the subharmonic function $v_\rho(x,\xi)$ from BL_ρ in $\overset{\circ}{\mathbb{R}}{}^3$. Then we set

$$K(x,\xi) = \frac{1}{|\xi|} - \frac{1}{|x-\xi|} + v(x,\xi)$$

or

$$K(x,\xi) = \sum_{n=0}^{+\infty}\left(\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|}\right) - \sum_{n=1}^{+\infty}\frac{1}{\left|\frac{x}{q^n} - \xi\right|}.$$

The function $K(x,\xi)$ is a counterpart of Schottky-Klein prime function logarithm ([7]–[9]). It is subharmonic and equals to $-\infty$, when $x = \xi q^n$, $n \in \mathbb{Z}$. Then $K(qx,\xi) = -\infty$ for such x.

Let now $K(x,\xi) > -\infty$. Then $K(x,\xi)$ equals the sum of series (10) plus $\frac{1}{|\xi|} - \frac{1}{|x-\xi|}$. We have

$$K(qx,\xi) = \sum_{n=0}^{+\infty}\left(\frac{1}{|\xi|} - \frac{1}{|q^{n+1}x - \xi|}\right) - \sum_{n=1}^{+\infty}\frac{1}{\left|\frac{x}{q^{n-1}} - \xi\right|} = \sum_{n=1}^{+\infty}\left(\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|}\right) -$$

$$- \sum_{n=0}^{+\infty}\frac{1}{\left|\frac{x}{q^n} - \xi\right|} = \sum_{n=0}^{+\infty}\left(\frac{1}{|\xi|} - \frac{1}{|q^n x - \xi|}\right) - \frac{1}{|\xi|} + \frac{1}{|x-\xi|} - \sum_{n=1}^{+\infty}\frac{1}{\left|\frac{x}{q^n} - \xi\right|} - \frac{1}{|x-\xi|} =$$

$$= K(x,\xi) - \frac{1}{|\xi|}.$$

So the function $K(x,\xi)$ has the following property, when $x \neq \xi q^n$, $n \in \mathbb{Z}$,

$$K(qx,\xi) = K(x,\xi) - \frac{1}{|\xi|}. \tag{11}$$

Let $a \in BL$, $b \in BL$, $a \neq b$ and $|a| = |b|$. Consider the difference

$$u(x) = K(x,b) - K(x,a). \tag{12}$$

If $x = aq^n, bq^n$, $n \in \mathbb{Z}$, then we obtain

$$u(aq^n) = +\infty \quad \text{and} \quad u(bq^n) = -\infty. \tag{13}$$

Then, using the equalities (13), we obtain $u(qx) = u(x)$. Let now consider the case, when $x \neq aq^n, bq^n, n \in \mathbb{Z}$. Then, using (11), we have

$$u(qx) = K(qx, b) - K(qx, a) = K(x, b) - \frac{1}{|b|} - K(x, a) + \frac{1}{|a|}.$$

Since $|a| = |b|$, we have $u(qx) = u(x)$. Hence the function $u(x)$ is multiplicatively periodic.

In the case $m > 3$ we put

$$K(x, \xi) = \sum_{n=0}^{+\infty} \left(\frac{1}{|\xi|^{m-2}} - \frac{1}{|q^n x - \xi|^{m-2}} \right) - \sum_{n=1}^{+\infty} \left| \frac{x}{q^n} - \xi \right|^{2-m}$$

and define $u(x)$ by relation (12). The verification of the property $u(qx) = u(x)$, $x \in \mathring{\mathbb{R}}^m$, is similar to that presented above. $\square$

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