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ON LOGARITHM OF MAXIMAL TERM OF DIRICHLET SERIES CONVERGING IN A HALF-PLANE: THREE-TERM POWER ASYMPTOTICS

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We have found conditions on coefficients and exponents of Dirichlet series with null abscissa of absolute convergence, under which the maximal term satisfies the asymptotic equality $\ln\mu(\sigma,F)=T_1|\sigma|^{-\rho_1}+T_2|\sigma|^{-\rho_2}+(\tau+o(1))|\sigma|^{-\rho}(\sigma\uparrow 0)$, where $T_1>0,\ T_2\in\mathbb{R}\setminus\{0\},\ \tau\in\mathbb{R}\setminus\{0\}$, $0<\rho<\rho_2<\rho_1<+\infty$.

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Найдены условия на коэффициенты и показатели ряда Дирихле с нулевой абсциссой абсолютной сходимости, при выполнении которых максимальный член удовлетворяет асимптотическому равенству $\ln\mu(\sigma,F)=T_1|\sigma|^{-\rho_1}+T_2|\sigma|^{-\rho_2}+(\tau+o(1))|\sigma|^{-\rho}(\sigma\uparrow 0)$, где $T_1>0,\ T_2\in\mathbb{R}\setminus\{0\},\ \tau\in\mathbb{R}\setminus\{0\},\ 0<\rho<\rho_2<\rho_1<+\infty$.

1. Introduction. For an entire function f of order $\rho\in(0,+\infty)$ and type $\tau\in(0,+\infty)$ Lindelöf ([1]) has found conditions on the Taylor coefficients, under which $\ln\max\{|f(z)|\colon |z|=r\}=(1+o(1))\tau r^{\rho}$ as $r\to\infty$. For the functions of exponential type this result was obtained independently by N. V. Govorov and N. I. Chernykh ([2]). A Dirichlet series

$$F(s)=\sum_{n=0}^{\infty}a_n\exp\{s\lambda_n\},\quad s=\sigma+it, \tag{1}$$

where (λ_n) is a sequence of nonnegative numbers, increasing to $+\infty$ ($\lambda_0=0$), is a direct generalization of power expansion of an analytic function. If the abscissa of absolute convergence of the series (1) is equal to $\sigma_a\in(-\infty,+\infty]$, then the growth of the series is identified with the growth of the function $M(\sigma,F)=\sup\{|F(\sigma+it)|\colon t\in\mathbb{R}\}$ as $\sigma\uparrow\sigma_a$. The maximal term $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}\colon n\ge 0\}$ plays an important role in the investigation of connection between the growth of $M(\sigma,F)$ and the behavior of the coefficients.

For the entire ($\sigma_a=+\infty$) Dirichlet series (1) M. V. Zabolotskyi and M. M. Sheremeta ([3]), generalizing Lindelöf's Theorem, indicated necessary and sufficient conditions on a_n and λ_n, under which $\ln\mu(\sigma,F)=(1+o(1))\Phi(\sigma)$ as $\sigma\to+\infty$, where Φ is a positive unbounded

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on $(-\infty, +\infty)$ function such that its derivative Φ' is nonnegative continuous and increasing to $+\infty$ as $\sigma \to +\infty$. Ya. Ya. Prytula ([4]) solved a similar problem for Dirichlet series with an arbitrary abscissa of absolute convergence.

The study of connection between the growth of $\ln \mu(\sigma, F)$ and the behavior of coefficients in terms of two-term asymptotics originates from [5]. It was established there necessary and sufficient conditions on a_n, under which for the entire Dirichlet series F $\ln \mu(\sigma, F)$ has two-term exponential asymptotics of the form $\ln \mu(\sigma, F) = T\exp\{\rho_1\sigma\} + (1+o(1))\tau\exp\{\rho\sigma\}$, $\sigma \to +\infty$, where $0 < \rho < \rho_1 < +\infty$, $T > 0$ and $\tau \in \mathbb{R}\setminus\{0\}$. O. M. Sumyk solved a similar problem for two-term power asymptotics of $\ln \mu(\sigma, F)$ both for entire Dirichlet series and Dirichlet series with null abscissa of absolute convergence. In particular, in [6] she proved, that for the Dirichlet series F with null abscissa of absolute convergence the asymptotic equality $\ln \mu(\sigma, F) = T_1/|\sigma|^{\rho_1} + \tau(1+o(1))/|\sigma|^{\rho}$ as $\sigma \uparrow 0$ was correct, where $0 < \rho < \rho_1 < +\infty$, $T > 0$ and $\tau \in \mathbb{R} \setminus \{0\}$, if and only if for every $\varepsilon > 0$

1) there exists a number $n_0(\varepsilon)$ such that $\ln |a_n| \le T_1(\rho_1+1)(\frac{\lambda_n}{T_1\rho_1})^{\frac{\rho_1}{\rho_1+1}} + (\tau+\varepsilon)(\frac{\lambda_n}{T_1\rho_1})^{\frac{\rho}{\rho_1+1}}$ for all $n \ge n_0(\varepsilon)$;
2) there exists an increasing sequence (n_k) of positive integers such that $\ln |a_{n_k}| \ge T_1(\rho_1 + 1)(\frac{\lambda_{n_k}}{T_1\rho_1})^{\frac{\rho_1}{\rho_1+1}} + (\tau-\varepsilon)(\frac{\lambda_{n_k}}{T_1\rho_1})^{\frac{\rho}{\rho_1+1}}$ for all $k \ge k_0$ and $\lambda_{n_{k+1}} - \lambda_{n_k} = o\Big(\lambda_{n_k}^{\frac{\rho+\rho_1+2}{2(\rho_1+1)}}\Big)$ as $k \to +\infty$.

The general problem of exponential asymptotics for the logarithm of the maximal term of a Dirichlet series was considered by O. M. Sumyk ([7]). The result of this investigation in the case of three-term exponential asymptotics is specified in [8]. Finally, in [9] for entire Dirichlet series connection between the growth of $\ln \mu(\sigma, F)$ and the decrease of a_n in terms of three-term power asymptotics is established.

Here we are going to find conditions on coefficients a_n and exponents λ_n of the Dirichlet series (1) with null abscissa of absolute convergence, under which its maximal term has the following asymptotics

$$\ln \mu(\sigma, F) = \frac{T_1}{|\sigma|^{\rho_1}} + \frac{T_2}{|\sigma|^{\rho_2}} + \frac{\tau + o(1)}{|\sigma|^{\rho}}, \quad \sigma \uparrow 0, \tag{2}$$

where $T_1 > 0,\ T_2 \in \mathbb{R} \setminus \{0\},\ \tau \in \mathbb{R} \setminus \{0\},\ 0 < \rho < \rho_2 < \rho_1 < +\infty$.

We put

$$\tau^* = \tau I_{\{\rho:\, \rho \ge 2\rho_2-\rho_1\}}(\rho) - \frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)} I_{\{\rho:\, \rho \le 2\rho_2-\rho_1\}}(\rho),$$

where $I_E(\rho)$ is the characteristic function of a set E, i.e., $I_E(\rho) = 1$ for $\rho \in E$ and $I_E(\rho) = 0$ for $\rho \notin E$.

Theorem 1. *In order that $\ln \mu(\sigma, F)$ has three-term power asymptotics (2) it is necessary and in the case $\rho \ge 2\rho_2 - \rho_1$ sufficient that for every $\varepsilon > 0$:*

1) there exists a number $n_0(\varepsilon)$ such that for all $n \ge n_0(\varepsilon)$

$$\ln |a_n| \le T_1(\rho_1+1)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} + (\tau^*+\varepsilon)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\max\{\rho, 2\rho_2-\rho_1\}}{\rho_1+1}};$$

2) there exists an increasing sequence (n_k) of positive integers such that for all $k \ge k_0$

$$\ln |a_{n_k}| \ge T_1(\rho_1+1)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} + (\tau^*-\varepsilon)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\max\{\rho, 2\rho_2-\rho_1\}}{\rho_1+1}};$$

$$\lambda_{n_{k+1}} - \lambda_{n_k} = o\Big(\lambda_{n_k}^{\frac{\rho_1+\max\{\rho,2\rho_2-\rho_1\}+2}{2(\rho_1+1)}}\Big), \quad k \to +\infty.$$

To prove Theorem 1 and other statements we need results of the papers [3], [10], [11].

By $\Omega(0)$ we denote the class of functions Φ defined on $(-\infty, 0)$, positive unbounded and such that Φ' is positive, continuously differentiable on $(-\infty, 0)$ and increasing to $+\infty$ as $\sigma \uparrow 0$. For $\Phi \in \Omega(0)$ let φ be the function inverse to Φ' and $\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}$ be a function associated with Φ in the sense of Newton.

Lemma 1 ([10, 11])**.** *Let $\Phi \in \Omega(0)$ and $\sigma_a = 0$. In order that $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$ it is necessary and sufficient that $\ln |a_n| \le -\lambda_n \Psi(\varphi(\lambda_n))$ for all $n \ge n_0$.*

Lemma 2 ([3, 11])**.** *For $\Phi \in \Omega(0)$ and positive numbers a, b, $a < b$, the following inequality $G_1(a, b, \Phi) < G_2(a, b, \Phi)$ holds, where*

$$G_1(a, b, \Phi) = \frac{ab}{b-a} \int_a^b \frac{\Phi(\varphi(t))}{t^2} dt,$$

$$G_2(a, b, \Phi) = \Phi\Big(\varkappa(a, b, \Phi)\Big), \quad \varkappa(a, b, \Phi) = \frac{1}{b-a} \int_a^b \varphi(t) dt.$$

Lemma 3 ([11])**.** *Let $\Phi \in \Omega(0)$, $\sigma_a = 0$, and $\ln |a_{n_k}| \ge -\lambda_{n_k} \Psi(\varphi(\lambda_{n_k}))$ for some sequence (n_k) of positive integers increasing to $+\infty$. Then for $k \ge k_0$ and $\sigma \in [\varphi(\lambda_{n_k}), \varphi(\lambda_{n_{k+1}})]$ the inequality $\ln \mu(\sigma, F) \ge \Phi(\sigma) - (G_2(\lambda_{n_k}, \lambda_{n_{k+1}}, \Phi) - G_1(\lambda_{n_k}, \lambda_{n_{k+1}}, \Phi))$ is valid.*

Lemma 4 ([11])**.** *Let $\Phi_j \in \Omega(0)$ $(j = 1, 2)$, $\sigma_a = 0$, and $\Phi_1(\sigma) \le \ln \mu(\sigma, F) \le \Phi_2(\sigma)$ for all $\sigma \in [\sigma_0, 0)$. Then $\ln |a_n| \le -\lambda_n \Psi_2(\varphi_2(\lambda_n))$ for all $n \ge n_0$ and there exists a sequence (n_k) of positive integers increasing to $+\infty$ such that $\ln |a_{n_k}| \ge -\lambda_{n_k} \Psi_1(\varphi_1(\lambda_{n_k}))$ and $G_1(\lambda_{n_k}, \lambda_{n_{k+1}}, \Phi_2) \ge \Phi_1\big(\frac{1}{\lambda_{n_{k+1}} - \lambda_{n_k}} \int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}} \varphi_2(t) dt\big)$.*

2. Main lemma. Suppose $\Phi \in \Omega(0)$ and Φ is a function of the form

$$\Phi(\sigma) = \frac{T_1}{|\sigma|^{\rho_1}} + \frac{T_2}{|\sigma|^{\rho_2}} + \frac{\tau}{|\sigma|^{\rho}} + \frac{\delta}{|\sigma|^{s}}, \quad \sigma_0 \le \sigma < 0, \tag{3}$$

where $T_1, T_2, \tau, \rho, \rho_2, \rho_1$ are the same as in the relation (2), $s \le \rho$ and $\delta \in \mathbb{R} \setminus \{0\}$. Set

$$U(x) = \Big(\frac{x}{T_1\rho_1}\Big)^{-\frac{1}{\rho_1+1}} + \frac{\rho_2 T_2}{\rho_1 T_1(\rho_1+1)} \Big(\frac{x}{T_1\rho_1}\Big)^{\frac{\rho_2-\rho_1-1}{\rho_1+1}}.$$

The following lemma is the main auxiliary statement in the paper.

Lemma 5. *Suppose that $\Phi \in \Omega(0)$ is of the form (3). Then the function φ admits the following asymptotics as $x \to +\infty$:*

(i) if $s = \rho > 2\rho_2 - \rho_1$ and $\tau + \delta \ne 0$, then $\varphi(x) = -U(x) - \frac{(\tau+\delta+o(1))\rho}{T_1\rho_1(\rho_1+1)} \big(\frac{x}{T_1\rho_1}\big)^{\frac{\rho-\rho_1-1}{\rho_1+1}}$;

(ii) if $s = \rho < 2\rho_2 - \rho_1$, then

$$\varphi(x) = -U(x) + \frac{2\rho_2 - \rho_1 + o(1)}{2} \Big(\frac{\rho_2 T_2}{\rho_1 T_1(\rho_1+1)}\Big)^2 \Big(\frac{x}{T_1\rho_1}\Big)^{\frac{2\rho_2-2\rho_1-1}{\rho_1+1}};$$

(iii) if $s=\rho=2\rho_2-\rho_1$, $\tau\neq\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$ *and* $\tau+\delta\neq\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, *then*

$$\varphi(x)=-U(x)-\frac{2\rho_2-\rho_1}{\rho_1T_1(\rho_1+1)}\left(\tau+\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-2\rho_1-1}{\rho_1+1}};$$

(iv) if $\rho=2\rho_2-\rho_1$, $s=3\rho_2-2\rho_1$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $(3\rho_2-2\rho_1)(3\rho_2-2\rho_1-1)\neq 0$ *and* $\delta\neq\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}$, *then*

$$\varphi(x)=-U(x)+\frac{3\rho_2-2\rho_1}{\rho_1T_1(\rho_1+1)}\left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}-\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{3\rho_2-3\rho_1-1}{\rho_1+1}};$$

(v) if $\rho=2\rho_2-\rho_1$, $s=4\rho_2-3\rho_1$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $3\rho_2-2\rho_1=0$ *and* $\delta\neq\frac{(\rho_1+3)(\rho_1+6)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}$, *then*

$$\varphi(x)=-U(x)-\frac{1}{3T_1(\rho_1+1)}\left(\frac{2\rho_1(\rho_1+3)(\rho_1+6)T_2^4}{2187(T_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{-\frac{4\rho_1+3}{3(\rho_1+1)}};$$

(vi) if $\rho=2\rho_2-\rho_1$, $s=4\rho_2-3\rho_1$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $3\rho_2-2\rho_1-1=0$, $\delta\neq\frac{(\rho_1-1)(\rho_1+2)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}$ *and* $\rho_1\neq 4$, *then*

$$\varphi(x)=-U(x)-\frac{\rho_1-4}{3\rho_1T_1(\rho_1+1)}\left(\frac{(\rho_1+2)(\rho_1-1)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{-\frac{4\rho_1-1}{\rho_1+1}};$$

(vii) if $\rho=2\rho_2-\rho_1$, $s=5\rho_2-4\rho_1$, $\rho_1=4$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $3\rho_2-2\rho_1-1=0$ *and* $\delta\neq-\frac{(T_2\rho_2)^5}{5(T_1\rho_1(\rho_1+1))^4}$, *then*

$$\varphi(x)=-U(x)+\frac{1}{20T_1}\left(\frac{(3T_2)^5}{5(20T_1)^4}+\delta+o(1)\right)\left(\frac{x}{4T_1}\right)^{-\frac{6}{5}}.$$

Proof. Since $\Phi'(\sigma)=\frac{T_1\rho_1}{|\sigma|^{\rho_1+1}}+\frac{T_2\rho_2}{|\sigma|^{\rho_2+1}}+\frac{\tau\rho}{|\sigma|^{\rho+1}}+\frac{\delta s}{|\sigma|^{s+1}}$, we need to solve the following equation

$$\frac{T_1\rho_1}{|\sigma|^{\rho_1+1}}+\frac{T_2\rho_2}{|\sigma|^{\rho_2+1}}+\frac{\tau\rho}{|\sigma|^{\rho+1}}+\frac{\delta s}{|\sigma|^{s+1}}=x \tag{4}$$

in order to find asymptotical behavior of φ. It is easy to see that the solution $\sigma=\sigma(x)$ of this equation satisfies the condition $\frac{T_1\rho_1}{|\sigma|^{\rho_1+1}}\times(1+o(1))=x$ $(x\to+\infty)$. Thus, we will find a solution of the form

$$|\sigma|=\left(\frac{T_1\rho_1}{x}\right)^{\frac{1}{\rho_1+1}}+\alpha(x), \tag{5}$$

where $\alpha=\alpha(x)=o(x^{-\frac{1}{\rho_1+1}})$ $(x\to+\infty)$. Substituting (5) into (4), we obtain

$$\left(1+\left(\frac{x}{T_1\rho_1}\right)^{\frac{1}{\rho_1+1}}\alpha\right)^{-(\rho_1+1)}+\frac{T_2\rho_2}{T_1\rho_1}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1}{\rho_1+1}}\left(1+\left(\frac{x}{T_1\rho_1}\right)^{\frac{1}{\rho_1+1}}\alpha\right)^{-(\rho_2+1)}+\frac{\tau\rho}{T_1\rho_1}\times$$
$$\times\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1}{\rho_1+1}}\left(1+\left(\frac{x}{T_1\rho_1}\right)^{\frac{1}{\rho_1+1}}\alpha\right)^{-(\rho+1)}+\frac{\delta\ s}{T_1\rho_1}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1}{\rho_1+1}}\left(1+\left(\frac{x}{T_1\rho_1}\right)^{\frac{1}{\rho_1+1}}\alpha\right)^{-(s+1)}=1.$$

Using the asymptotic expansion for the function $(1+t)^\gamma$ as $t\to 0$, we can show that the following asymptotic equality holds

$$\alpha=\frac{(\rho_1+2)}{2}\alpha^2\left(\frac{x}{T_1\rho_1}\right)^{\frac{1}{\rho_1+1}}-\frac{(\rho_1+2)(\rho_1+3)}{6}\alpha^3\left(\frac{x}{T_1\rho_1}\right)^{\frac{2}{\rho_1+1}}+$$
$$+\frac{(\rho_1+2)(\rho_1+3)(\rho_1+4)}{24}\alpha^4\left(\frac{x}{T_1\rho_1}\right)^{\frac{3}{\rho_1+1}}-\frac{(\rho_1+2)(\rho_1+3)(\rho_1+4)(\rho_1+5)}{120}\alpha^5\left(\frac{x}{T_1\rho_1}\right)^{\frac{4}{\rho_1+1}}+$$
$$+O\left(\alpha^6x^{\frac{5}{\rho_1+1}}\right)+\frac{\rho_2T_2}{\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1-1}{\rho_1+1}}-\frac{\rho_2T_2(\rho_2+1)}{\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1}{\rho_1+1}}\alpha+$$
$$+\frac{\rho_2T_2(\rho_2+1)(\rho_2+2)}{2\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1+1}{\rho_1+1}}\alpha^2-\frac{\rho_2T_2(\rho_2+1)(\rho_2+2)(\rho_2+3)}{6\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1+2}{\rho_1+1}}\alpha^3+$$
$$+\frac{\rho_2T_2(\rho_2+1)(\rho_2+2)(\rho_2+3)(\rho_2+4)}{24\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1+3}{\rho_1+1}}\alpha^4+O\left(\alpha^5x^{\frac{\rho_2-\rho_1+4}{\rho_1+1}}\right)+$$
$$+\frac{\tau\rho}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1-1}{\rho_1+1}}-\frac{\tau\rho(\rho+1)}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1}{\rho_1+1}}\alpha+$$
$$+\frac{\tau\rho(\rho+1)(\rho+2)}{2T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1+1}{\rho_1+1}}\alpha^2-\frac{\tau\rho(\rho+1)(\rho+2)(\rho+3)}{6T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1+2}{\rho_1+1}}\alpha^3+$$
$$+O\left(\alpha^4x^{\frac{\rho-\rho_1+3}{\rho_1+1}}\right)+\frac{(1+o(1))\delta s}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1-1}{\rho_1+1}},\quad x\to+\infty. \tag{6}$$

Since $\alpha(x)x^{\frac{1}{\rho_1+1}}\to 0$ $(x\to+\infty)$, (6) gives

$$\alpha=\frac{\rho_2T_2(1+o(1))}{\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1-1}{\rho_1+1}}=\frac{\rho_2T_2}{\rho_1T_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-\rho_1-1}{\rho_1+1}}+\beta, \tag{7}$$

where $\beta=\beta(x)=o(x^{\frac{\rho_2-\rho_1-1}{\rho_1+1}}),\ x\to+\infty$. Substituting (7) into (6), we obtain the asymptotic equality

$$(1+o(1))\beta(x)=\frac{\rho_1-2\rho_2}{2}\left(\frac{\rho_2T_2}{\rho_1T_1(\rho_1+1)}\right)^2\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-2\rho_1-1}{\rho_1+1}}-$$
$$-\frac{(\rho_1+2)(\rho_1+3)-3(\rho_2+1)(\rho_2+2)}{6}\left(\frac{\rho_2T_2}{\rho_1T_1(\rho_1+1)}\right)^3\left(\frac{x}{T_1\rho_1}\right)^{\frac{3\rho_2-3\rho_1-1}{\rho_1+1}}+$$
$$+\frac{(\rho_1+2)(\rho_1+3)(\rho_1+4)-4(\rho_2+1)(\rho_2+2)(\rho_2+3)}{24}\left(\frac{\rho_2T_2}{\rho_1T_1(\rho_1+1)}\right)^4\left(\frac{x}{T_1\rho_1}\right)^{\frac{4\rho_2-4\rho_1-1}{\rho_1+1}}-$$
$$-\frac{(\rho_1+2)(\rho_1+3)(\rho_1+4)(\rho_1+5)-5(\rho_2+1)(\rho_2+2)(\rho_2+3)(\rho_2+4)}{120}\left(\frac{\rho_2T_2}{\rho_1T_1(\rho_1+1)}\right)^5\times$$
$$\times\left(\frac{x}{T_1\rho_1}\right)^{\frac{5\rho_2-5\rho_1-1}{\rho_1+1}}+\frac{\tau\rho}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1-1}{\rho_1+1}}-$$
$$-\frac{\tau\rho(\rho+1)\rho_2T_2}{(\rho_1T_1(\rho_1+1))^2}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2-2\rho_1+\rho-1}{\rho_1+1}}+\frac{\tau\rho(\rho+1)(\rho+2)(\rho_2T_2)^2}{2(\rho_1T_1(\rho_1+1))^3}\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-3\rho_1+\rho-1}{\rho_1+1}}-$$

$$-\frac{\tau\rho(\rho+1)(\rho+2)(\rho+3)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^4}\left(\frac{x}{T_1\rho_1}\right)^{\frac{3\rho_2-4\rho_1+\rho-1}{\rho_1+1}}+\frac{(1+o(1))\delta s}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1-1}{\rho_1+1}}+$$
$$+O\left(x^{\frac{6\rho_2-6\rho_1-1}{\rho_1+1}}\right)+O\left(x^{\frac{4\rho_2-5\rho_1+\rho-1}{\rho_1+1}}\right). \tag{8}$$

If $s=\rho>2\rho_2-\rho_1$ then $\rho-\rho_1-1>2(\rho_2-\rho_1)-1>3(\rho_2-\rho_1)-1>\ldots>6(\rho_2-\rho_1)-1$ and from (8) we get

$$\beta(x)=\frac{\tau\rho}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1-1}{\rho_1+1}}+\frac{(1+o(1))\delta\rho}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1-1}{\rho_1+1}},\quad x\to+\infty,$$

and under the condition $\tau+\delta\neq 0$ we have

$$\beta(x)=\frac{(\tau+\delta+o(1))\rho}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho-\rho_1-1}{\rho_1+1}},\quad x\to+\infty.$$

Therefore, from (5) and (7) we get statement (i) of Lemma 5.

If $0<s=\rho<2\rho_2-\rho_1$, then from (8) we obtain the asymptotic equality

$$\beta(x)=-(1+o(1))\frac{2\rho_2-\rho_1}{2}\left(\frac{T_2\rho_2}{T_1\rho_1(\rho_1+1)}\right)^2\left(\frac{x}{T_1\rho_1}\right)^{\frac{2(\rho_2-\rho_1)-1}{\rho_1+1}},$$

and in view of (5) and (7) statement (ii) of Lemma 5 is valid.

Suppose now that $s=\rho=2\rho_2-\rho_1$ and $\tau\neq\frac{(T_2\rho_2)^2}{2T_1\rho_1(\rho_1+1)}$. Then the equality (8) implies

$$(1+o(1))\beta(x)=\frac{2\rho_2-\rho_1}{T_1\rho_1(\rho_1+1)}\left(\tau-\frac{(T_2\rho_2)^2}{2T_1\rho_1(\rho_1+1)}\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2(\rho_2-\rho_1)-1}{\rho_1+1}}+$$
$$+\frac{(1+o(1))\delta s}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1-1}{\rho_1+1}},\quad x\to+\infty.$$

In the case $\tau+\delta\neq\frac{(T_2\rho_2)^2}{2T_1\rho_1(\rho_1+1)}$ the last relation is equivalent to the following one

$$\beta(x)=\frac{2\rho_2-\rho_1}{T_1\rho_1(\rho_1+1)}\left(\tau+\delta-\frac{(T_2\rho_2)^2}{2T_1\rho_1(\rho_1+1)}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2(\rho_2-\rho_1)-1}{\rho_1+1}},\quad x\to+\infty,$$

and in view of (5) and (7) assertion (iii) of Lemma 5 is proved.

If $\rho=2\rho_2-\rho_1$ and $\tau=\frac{(T_2\rho_2)^2}{2T_1\rho_1(\rho_1+1)}$, then (8) can be rewritten as

$$(1+o(1))\beta(x)=-\frac{(3\rho_2-2\rho_1)(3\rho_2-2\rho_1-1)}{6}\left(\frac{T_2\rho_2}{T_1\rho_1(\rho_1+1)}\right)^3\left(\frac{x}{T_1\rho_1}\right)^{\frac{3(\rho_2-\rho_1)-1}{\rho_1+1}}+$$
$$+\left(\frac{(\rho_1+2)(\rho_1+3)(\rho_1+4)-4(\rho_2+1)(\rho_2+2)(\rho_2+3)}{24}+\right.$$
$$\left.+\frac{(2\rho_2-\rho_1)(2\rho_2-\rho_1+1)(2\rho_2-\rho_1+2)}{4}\right)\left(\frac{T_2\rho_2}{T_1\rho_1(\rho_1+1)}\right)^4\left(\frac{x}{T_1\rho_1}\right)^{\frac{4(\rho_2-\rho_1)-1}{\rho_1+1}}-$$

$$-\left(\frac{(\rho_1+2)(\rho_1+3)(\rho_1+4)(\rho_1+5)-5(\rho_2+1)(\rho_2+2)(\rho_2+3)(\rho_2+4)}{120}+\right.$$

$$+\left.\frac{(2\rho_2-\rho_1)(2\rho_2-\rho_1+1)(2\rho_2-\rho_1+2)(2\rho_2-\rho_1+3)}{12}\right)\left(\frac{T_2\rho_2}{T_1\rho_1(\rho_1+1)}\right)^5\left(\frac{x}{T_1\rho_1}\right)^{\frac{5(\rho_2-\rho_1)-1}{\rho_1+1}}+$$

$$+O\left(x^{\frac{6(\rho_2-\rho_1)-1}{\rho_1+1}}\right)+\frac{(1+o(1))\delta s}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1-1}{\rho_1+1}},\quad x\to+\infty. \quad (9)$$

Under the conditions $(3\rho_2-2\rho_1)(3\rho_2-2\rho_1-1)\neq 0$ and $s=3\rho_2-2\rho_1$ the second, third and fourth addends are $o(x^{\frac{s-\rho_1-1}{\rho_1+1}})$ as $x\to+\infty$. Moreover, if $\delta\neq\frac{(3\rho_2-2\rho_1-1)(T_2\rho_2)^3}{6(T_1\rho_1(\rho_1+1))^2}$, then

$$\beta(x)=\frac{3\rho_2-2\rho_1}{T_1\rho_1(\rho_1+1)}\left(\delta-\frac{(3\rho_2-2\rho_1-1)(T_2\rho_2)^3}{6(T_1\rho_1(\rho_1+1))^2}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{3(\rho_2-\rho_1)-1}{\rho_1+1}},\quad x\to+\infty,$$

and in view of (5) and (7) we come to assertion (iv) of Lemma 5.

If $(3\rho_2-2\rho_1)(3\rho_2-2\rho_1-1)=0$, then the first addend in the right-hand side of (9) is equal to zero.

At first we consider the case $3\rho_2-2\rho_1=0$. Then $\rho_2=2\rho_1/3$ and $2\rho_2-\rho_1=\rho_1/3$. Therefore, (9) implies

$$(1+o(1))\beta(x)=\frac{\rho_1(\rho_1+3)(\rho_1+6)}{648}\left(\frac{T_2\rho_2}{T_1\rho_1(\rho_1+1)}\right)^4\left(\frac{x}{T_1\rho_1}\right)^{\frac{4(\rho_2-\rho_1)-1}{\rho_1+1}}+$$

$$+O\left(x^{\frac{5(\rho_2-\rho_1)-1}{\rho_1+1}}\right)+\frac{(1+o(1))\delta s}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1-1}{\rho_1+1}},\quad x\to+\infty,$$

and so under conditions $s=4\rho_2-3\rho_1$ and $\delta\neq\frac{(\rho_1+3)(\rho_1+6)(T_2\rho_2)^4}{216(T_1\rho_1(\rho_1+1))^3}$ we have

$$\beta(x)=\frac{1}{3T_1(\rho_1+1)}\left(\frac{(\rho_1+3)(\rho_1+6)(T_2\rho_2)^4}{216(T_1\rho_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{4(\rho_2-\rho_1)-1}{\rho_1+1}},\quad x\to+\infty,$$

and in view of (5) and (7) we obtain assertion (v) of Lemma 5.

Finally, let $3\rho_2-2\rho_1-1=0$. Then $\rho_2=(2\rho_1+1)/3$, $2\rho_2-\rho_1=(\rho_1+2)/3$ and (9) implies

$$(1+o(1))\beta(x)=\frac{(\rho_1+2)(\rho_1-1)(\rho_1-4)}{648}\left(\frac{T_2\rho_2}{T_1\rho_1(\rho_1+1)}\right)^4\left(\frac{x}{T_1\rho_1}\right)^{\frac{4(\rho_2-\rho_1)-1}{\rho_1+1}}+$$

$$+O\left(x^{\frac{5(\rho_2-\rho_1)-1}{\rho_1+1}}\right)+\frac{(1+o(1))\delta s}{T_1\rho_1(\rho_1+1)}\left(\frac{x}{T_1\rho_1}\right)^{\frac{s-\rho_1-1}{\rho_1+1}},\quad x\to+\infty. \quad (10)$$

If $(\rho_1+2)(\rho_1-1)(\rho_1-4)\neq 0$, $s=4\rho_2-3\rho_1$, and $\delta\neq\frac{(\rho_1+2)(\rho_1-1)(T_2\rho_2)^4}{216(T_1\rho_1(\rho_1+1))^3}$, then from (10) we get

$$\beta(x)=\frac{\rho_1-4}{3T_1\rho_1(\rho_1+1)}\left(\frac{(\rho_1+2)(\rho_1-1)}{216}\frac{(T_2\rho_2)^4}{(T_1\rho_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{4(\rho_2-\rho_1)-1}{\rho_1+1}},$$

as $x \to +\infty$. We note that $(\rho_1+2)(\rho_1-1)(\rho_1-4) \neq 0$ if and only if $\rho_1 \neq 4$, because if $\rho_1 = 1$, then $\rho_1 = \rho_2 = 1$ which is impossible. Thus, we get statement (vi) of Lemma 5.

What is left is to consider the case $\rho_1 = 4$. In this case from (10) under condition $s = 5\rho_2 - 4\rho_1 = -1$ we obtain

$$\beta(x) = -\frac{1}{T_1\rho_1(\rho_1+1)}\left(\frac{(T_2\rho_2)^5}{5(T_1\rho_1(\rho_1+1))^4}+\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{5(\rho_2-\rho_1)-1}{\rho_1+1}}, \quad x \to +\infty,$$

and statement (vii) follows. □

Using Lemma 5, we can find asymptotics of the functions $|\varphi(x)|^{-p}$, where $p = \rho_1, \rho_2, \rho, s$ and hence asymptotics of $\Phi(\varphi(x))$ and, since $x\Psi(\varphi(x)) = x\varphi(x) - \Phi(\varphi(x))$, asymptotics of $x\Psi(\varphi(x))$ also. Setting

$$V(x) = T_1\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(1-\frac{\rho_2}{\rho_1+1}\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}},$$

$$W(x) = T_1(\rho_1+1)\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}},$$

we come to the following lemma.

Lemma 6. *Suppose that $\Phi \in \Omega(0)$ and Φ has a form (3). Then functions $\Phi(\varphi(x))$ and $x\Psi(\varphi(x))$ admit the following asymptotics, as $x \to +\infty$:*

(i) under conditions of Lemma 5 (i)

$$\Phi(\varphi(x)) = V(x) + \left(\frac{(\tau+\delta)(\rho_1-\rho+1)}{\rho_1+1}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}},$$

$$x\Psi(\varphi(x)) = -W(x) - (\tau+\delta+o(1))\left(\frac{x}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}};$$

(ii) under conditions of Lemma 5 (ii)

$$\Phi(\varphi(x)) = V(x) + \frac{2\rho_2-2\rho_1-1}{\rho_1+1}\left(\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}},$$

$$x\Psi(\varphi(x)) = -W(x) + \left(\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

(iii) under conditions of Lemma 5 (iii)

$$\Phi(\varphi(x)) = V(x) - \frac{2\rho_2-2\rho_1-1}{\rho_1+1}\left(\tau+\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}},$$

$$x\Psi(\varphi(x)) = -W(x) - \left(\tau+\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

(iv) under conditions of Lemma 5 (iv)

$$\Phi(\varphi(x)) = V(x) + \frac{3\rho_2-3\rho_1-1}{\rho_1+1}\left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}-\delta+o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{3\rho_2-2\rho_1}{\rho_1+1}},$$

$$x\Psi(\varphi(x)) = -W(x) + \left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2} - \delta + o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{3\rho_2-2\rho_1}{\rho_1+1}};$$

(v) under conditions of Lemma 5 (v)

$$\Phi(\varphi(x)) = V(x) + \frac{4\rho_1T_2^3}{81(T_1(\rho_1+1))^2} -$$
$$-\frac{4\rho_1+3}{3(\rho_1+1)}\left(\frac{2\rho_1(\rho_1+3)(\rho_1+6)T_2^4}{2187(T_1(\rho_1+1))^3} - \delta + o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{-\frac{\rho_1}{3(\rho_1+1)}},$$
$$x\Psi(\varphi(x)) = -W(x) - \frac{4\rho_1T_2^3}{81(T_1(\rho_1+1))^2} + \left(\frac{2\rho_1(\rho_1+3)(\rho_1+6)T_2^4}{2187(T_1(\rho_1+1))^3} - \delta + o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{-\frac{\rho_1}{3(\rho_1+1)}};$$

(vi) under conditions of Lemma 5 (vi)

$$\Phi(\varphi(x)) = V(x) - \frac{4\rho_1-1}{3(\rho_1+1)}\left(\frac{(\rho_1+2)(\rho_1-1)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3} - \delta + o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{-\rho_1+4}{3(\rho_1+1)}},$$
$$x\Psi(\varphi(x)) = -W(x) + \left(\frac{(\rho_1+2)(\rho_1-1)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3} - \delta + o(1)\right)\left(\frac{x}{T_1\rho_1}\right)^{\frac{-\rho_1+4}{3(\rho_1+1)}};$$

(vii) under conditions of Lemma 5 (vii)

$$\Phi(\varphi(x)) = V(x) - \frac{1}{12}\cdot\frac{(3T_2)^4}{(20T_1)^3} + \frac{6}{5}\left(\frac{(3T_2)^5}{5(20T_1)^4} + \delta + o(1)\right)\left(\frac{x}{4T_1}\right)^{-\frac{1}{5}},$$
$$x\Psi(\varphi(x)) = -W(x) + \frac{1}{12}\cdot\frac{(3T_2)^4}{(20T_1)^3} - \left(\frac{(3T_2)^5}{5(20T_1)^4} + \delta + o(1)\right)\left(\frac{x}{4T_1}\right)^{-\frac{1}{5}}.$$

3. Asymptotic behavior of quantities $G_1(t_k, t_{k+1}, \Phi)$ and $G_2(t_k, t_{k+1}, \Phi)$. Let $0 < t_k \uparrow +\infty$ $(k \to +\infty)$ and $t_{k+1} = t_k(1+\theta_k)$. Using Lemma 6, it is easy to prove the following three lemmas.

Lemma 7. *If there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j} \to +\infty$ as $j \to +\infty$, then for the function* (3)

$$G_1(t_{k_j}, t_{k_j}(1+\theta_{k_j}), \Phi) = (1+o(1))T_1(\rho_1+1)\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} \quad (j \to +\infty)$$

and

$$G_2(t_{k_j}, t_{k_j}(1+\theta_{k_j}), \Phi) = (1+o(1))T_1\left(\frac{\rho_1}{\rho_1+1}\right)^{\rho_1}\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\theta_{k_j}^{\frac{\rho_1}{\rho_1+1}} \quad (j \to +\infty)$$

for this sequence (θ_{k_j}) under assumptions of any statement of Lemma 5.

Lemma 8. *If there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j} \to \theta \in (0, +\infty)$ as $j \to +\infty$, then for the function* (3)

$$G_1(t_{k_j}, t_{k_j}(1+\theta_{k_j}), \Phi) = (1+o(1))T_1(\rho_1+1)\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\frac{1+\theta}{\theta}(1-(1+\theta)^{-\frac{1}{\rho_1+1}}) \quad (j \to +\infty)$$

and

$$G_2(t_{k_j}, t_{k_j}(1+\theta_{k_j}), \Phi) = (1+o(1))T_1\left(\frac{\rho_1}{\rho_1+1}\right)^{\rho_1}\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\left(\frac{((1+\theta)^{\frac{\rho_1}{\rho_1+1}}-1)}{\theta}\right)^{-\rho_1},$$

for this sequence (θ_{k_j}) under assumptions of any statement of Lemma 5.

Lemma 9. *Suppose that $\Phi \in \Omega(0)$ and Φ has a form (3). Let (t_k) be such sequence as defined above and $\theta_k \to 0$ $(k \to +\infty)$. Then*

$$G_1(t_k, t_k(1+\theta_k), \Phi) = A^*(t_k, \theta_k) + O\Big(t_k^{\frac{\rho_1}{\rho_1+1}}\theta_k^3\Big) + O\Big(t_k^{\frac{\rho_2}{\rho_1+1}}\theta_k^2\Big) + g_1(t_k, \theta_k),\ k \to +\infty,$$

where

$$A^*(t_k,\theta_k) = T_1\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + \frac{T_1\rho_1\theta_k}{2(\rho_1+1)}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} - \frac{T_1\rho_1(\rho_1+2)}{6(\rho_1+1)^2}\theta_k^2\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} +$$
$$+\frac{T_2(\rho_1+1-\rho_2)}{\rho_1+1}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} + \frac{T_2\rho_2(\rho_1+1-\rho_2)}{2(\rho_1+1)^2}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}}\theta_k,$$

and $g_1(t_k,\theta_k)$ has the following asymptotics as $k \to +\infty$:

(i) under conditions of Lemma 5 (i)

$$g_1(t_k,\theta_k) = \frac{\rho_1-\rho+1}{\rho_1+1}(\tau+\delta+o(1))\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}};$$

(ii) under conditions of Lemma 5 (ii)

$$g_1(t_k,\theta_k) = \frac{2\rho_2-2\rho_1-1}{\rho_1+1}\left(\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

(iii) under conditions of Lemma 5 (iii)

$$g_1(t_k,\theta_k) = -\frac{2\rho_2-2\rho_1-1}{\rho_1+1}\left(\tau+\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

(iv) under conditions of Lemma 5 (iv)

$$g_1(t_k,\theta_k) = \frac{3\rho_2-3\rho_1-1}{\rho_1+1}\left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}-\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{3\rho_2-2\rho_1}{\rho_1+1}};$$

(v) under conditions of Lemma 5 (v)

$$g_1(t_k,\theta_k) = \frac{4\rho_1T_2^3}{81(T_1(\rho_1+1))^2} - \frac{4\rho_1+3}{3(\rho_1+1)}\left(\frac{2\rho_1(\rho_1+3)(\rho_1+6)T_2^4}{2187(T_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{-\frac{\rho_1}{3(\rho_1+1)}};$$

(vi) under conditions of Lemma 5 (vi)

$$g_1(t_k,\theta_k) = -\frac{4\rho_1-1}{3(\rho_1+1)}\left(\frac{(\rho_1+2)(\rho_1-1)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{-\rho_1+4}{3(\rho_1+1)}};$$

(vii) under conditions of Lemma 5 (vii)

$$g_1(t_k,\theta_k) = -\frac{1}{12}\cdot\frac{(3T_2)^4}{(20T_1)^3} + \frac{6}{5}\left(\frac{(3T_2)^5}{5(20T_1)^4}+\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{-\frac{1}{5}}.$$

Lemma 10. *Suppose that $\Phi\in\Omega(0)$ and Φ has a form (3). Let (t_k) be such sequence as defined above and $\theta_k\to 0$ $(k\to+\infty)$. Then*

$$G_2(t_k,t_k(1+\theta_k),\Phi)=B^*(t_k,\theta_k)+O(t_k^{\frac{\rho_1}{\rho_1+1}}\theta_k^3)+O(t_k^{\frac{\rho_2}{\rho_1+1}}\theta_k^2)+g_2(t_k,\theta_k),\quad k\to+\infty,\quad (11)$$

where

$$B^*(t_k,\theta_k)=T_1\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}+\frac{T_1\rho_1\theta_k}{2(\rho_1+1)}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}-\frac{T_1\rho_1(\rho_1+5)}{24(\rho_1+1)^2}\theta_k^2\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}+$$
$$+\frac{T_2(\rho_1+1-\rho_2)}{\rho_1+1}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}}+\frac{T_2\rho_2(\rho_1+1-\rho_2)}{2(\rho_1+1)^2}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}}\theta_k,$$

and $g_2(t_k,\theta_k)$ has the following asymptotics as $k\to+\infty$:

(i) under conditions of Lemma 5 (i)

$$g_2(t_k,\theta_k)=\frac{\rho_1-\rho+1}{\rho_1+1}(\tau+\delta+o(1))\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}};$$

(ii) under conditions of Lemma 5 (ii)

$$g_2(t_k,\theta_k)=\frac{2\rho_2-2\rho_1-1}{\rho_1+1}\left(\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

(iii) under conditions of Lemma 5 (iii)

$$g_2(t_k,\theta_k)=\frac{2\rho_1-2\rho_2+1}{\rho_1+1}\left(\tau+\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

(iv) under conditions of Lemma 5 (iv)

$$g_2(t_k,\theta_k)=\frac{3\rho_2-3\rho_1-1}{\rho_1+1}\left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}-\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{3\rho_2-2\rho_1}{\rho_1+1}};$$

(v) under conditions of Lemma 5 (v)

$$g_2(t_k,\theta_k)=\frac{4\rho_1T_2^3}{81(T_1(\rho_1+1))^2}-\frac{4\rho_1+3}{3(\rho_1+1)}\left(\frac{2\rho_1(\rho_1+3)(\rho_1+6)T_2^4}{2187(T_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{-\frac{\rho_1}{3(\rho_1+1)}};$$

(vi) under conditions of Lemma 5 (vi)

$$g_2(t_k,\theta_k)=-\frac{4\rho_1-1}{3(\rho_1+1)}\left(\frac{(\rho_1+2)(\rho_1-1)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}-\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{-\rho_1+4}{3(\rho_1+1)}};$$

(vii) under conditions of Lemma 5 (vii)

$$g_2(t_k,\theta_k)=-\frac{1}{12}\frac{(3T_2)^4}{(20T_1)^3}+\frac{6}{5}\left(\frac{(3T_2)^5}{5(20T_1)^4}+\delta+o(1)\right)\left(\frac{t_k}{T_1\rho_1}\right)^{-\frac{1}{5}}.$$

Since $B^*(t_k,\theta_k)-A^*(t_k,\theta_k)=\frac{T_1\rho_1}{8(\rho_1+1)}\big(\frac{t_k}{T_1\rho_1}\big)^{\frac{\rho_1}{\rho_1+1}}\theta_k^2$, Lemmas 9 and 12 imply the following statement.

Lemma 11. *Suppose that $\Phi\in\Omega(0)$ and Φ has a form (3). Let (t_k) be such sequence as defined above and $\theta_k\to 0$ $(k\to+\infty)$. Then*

$$G_2(t_k,t_k(1+\theta_k),\Phi)-G_1(t_k,t_k(1+\theta_k),\Phi)=\frac{T_1\rho_1}{8(\rho_1+1)}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\theta_k^2+$$
$$+O(\theta_k^2 t_k^{\frac{\rho_2}{\rho_1+1}})+O(\theta_k^3 t_k^{\frac{\rho_1}{\rho_1+1}})+g(t_k,\theta_k),\quad k\to+\infty,$$

and $g(t_k,\theta_k)$ has the following asymptotics as $k\to+\infty$:

(i) under conditions of Lemma 5 (i) $g(t_k,\theta_k)=o\Big(t_k^{\frac{\rho}{\rho_1+1}}\Big)$;

(ii) under conditions of Lemma 5 (ii) $g(t_k,\theta_k)=o\Big(t_k^{\frac{2\rho_2-\rho_1}{\rho_1+1}}\Big)$;

(iii) under conditions of Lemma 5 (iii) $g(t_k,\theta_k)=o\Big(t_k^{\frac{2\rho_2-\rho_1}{\rho_1+1}}\Big)$;

(iv) under conditions of Lemma 5 (iv) $g(t_k,\theta_k)=o\Big(t_k^{\frac{3\rho_2-2\rho_1}{\rho_1+1}}\Big)$;

(v) under conditions of Lemma 5 (v) $g(t_k,\theta_k)=o\Big(t_k^{-\frac{\rho_1}{3(\rho_1+1)}}\Big)$;

(vi) under conditions of Lemma 5 (vi) $g(t_k,\theta_k)=o\Big(t_k^{\frac{-\rho_1+4}{3(\rho_1+1)}}\Big)$;

(vii) under conditions of Lemma 5 (vii) $g(t_k,\theta_k)=o\Big(t_k^{-\frac{1}{5}}\Big)$.

We will also need the following lemma.

Lemma 12. *Let $\Phi_1(\sigma)\in\Omega(0)$ and $\Phi_2(\sigma)\in\Omega(0)$ be such functions that*

$$\Phi_1(\sigma)=\frac{T_1}{|\sigma|^{\rho_1}}+\frac{T_2}{|\sigma|^{\rho_2}}+\frac{\tau}{|\sigma|^{\rho}}-\frac{\delta}{|\sigma|^{s}},\quad \Phi_2(\sigma)=\frac{T_1}{|\sigma|^{\rho_1}}+\frac{T_2}{|\sigma|^{\rho_2}}+\frac{\tau}{|\sigma|^{\rho}}+\frac{\delta}{|\sigma|^{s}},\quad \sigma_0\le\sigma<0,$$

where $\delta>0$ and $s\le\rho$. We suppose that $t_{k+1}=(1+\theta_k)t_k$ and for all $k\ge k_0$

$$G_1(t_k,t_{k+1},\Phi_2)\ge\Phi_1(\varkappa(t_k,t_{k+1},\Phi_2)).\tag{12}$$

Then $\theta_k\to 0$ $(k\to+\infty)$ and

$$\theta_k^2\le\frac{16\delta(\rho_1+1)}{T_1\rho_1}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{s-\rho_1}{\rho_1+1}}+g^*(t_k,\theta_k),\tag{13}$$

where the following asymptotic equalities are valid as $k\to+\infty$:

(i) if $s=\rho>2\rho_2-\rho_1$ and $\tau\pm\delta\ne 0$, then $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{\rho-\rho_1}{\rho_1+1}}\Big)$;

(ii) if $s=\rho<2\rho_2-\rho_1$, then $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{2(\rho_2-\rho_1)}{\rho_1+1}}\Big)$;

(iii) if $s=\rho=2\rho_2-\rho_1$, $\tau\neq\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$ *and* $\tau\pm\delta\neq\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$*, then* $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{2(\rho_2-\rho_1)}{\rho_1+1}}\Big)$;

(iv) if $\rho=2\rho_2-\rho_1$, $s=3\rho_2-2\rho_1$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $(3\rho_2-2\rho_1)(3\rho_2-2\rho_1-1)\neq 0$ *and* $\delta\neq\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}$*, then* $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{3(\rho_2-\rho_1)}{\rho_1+1}}\Big)$;

(v) if $\rho=2\rho_2-\rho_1$, $s=4\rho_2-3\rho_1$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $3\rho_2-2\rho_1=0$ *and* $\delta\neq\frac{(\rho_1+3)(\rho_1+6)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}$*, then* $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{4(\rho_2-\rho_1)}{\rho_1+1}}\Big)$;

(vi) if $\rho=2\rho_2-\rho_1$, $s=4\rho_2-3\rho_1$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $3\rho_2-2\rho_1-1=0$, $\delta\neq\frac{(\rho_1-1)(\rho_1+2)(\rho_2T_2)^4}{216(\rho_1T_1(\rho_1+1))^3}$ *and* $\rho_1\neq 4$*, then* $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{4(\rho_2-\rho_1)}{\rho_1+1}}\Big)$;

(vii) if $\rho=2\rho_2-\rho_1$, $s=5\rho_2-4\rho_1$, $\rho_1=4$, $\tau=\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}$, $3\rho_2-2\rho_1-1=0$ *and* $\delta\neq\frac{(T_2\rho_2)^5}{5(T_1\rho_1(\rho_1+1))^4}$*, then* $g^*(t_k,\theta_k)=o\Big(t_k^{\frac{5(\rho_2-\rho_1)}{\rho_1+1}}\Big)$.

Proof. Since $\Phi_1(\sigma)=\Phi_2(\sigma)-\frac{2\delta}{|\sigma|^s}$, $\Phi_1(\varkappa(t_k,t_{k+1},\Phi_2))=G_2(t_k,t_{k+1},\Phi_2)-\frac{2\delta}{|\varkappa(t_k,t_{k+1},\Phi_2)|^s}$, from (12) we have

$$G_1(t_k,t_{k+1},\Phi_2)\geq G_2(t_k,t_{k+1},\Phi_2)-\frac{2\delta}{|\varkappa(t_k,t_{k+1},\Phi_2)|^s}. \tag{14}$$

We suppose that $\overline{\lim}_{k\to+\infty}\theta_k=+\infty$. Then there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j}\to+\infty$ $(j\to+\infty)$ and for this sequence according to Lemma 7, and the asymptotics $|\varkappa(t_k,t_{k+1},\Phi_2)|^s=(1+o(1))(\frac{\rho_1+1}{\rho_1})^s\cdot(\frac{t_k}{T_1\rho_1})^{-\frac{s}{\rho_1+1}}\cdot\theta_k^{-\frac{s}{\rho_1+1}}$ $(k\to+\infty)$ we have

$$T_1(\rho_1+1)\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\geq(1+o(1))T_1\left(\frac{\rho_1}{\rho_1+1}\right)^{\rho_1}\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\theta_{k_j}^{\frac{\rho_1}{\rho_1+1}},\quad j\to+\infty,$$

which is impossible.

If $\overline{\lim}_{k\to+\infty}\theta_k=\theta\in(0,+\infty)$, then there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j}\to\theta$ $\ (j\to+\infty)$ and for this sequence according to Lemma 8, and the asymptotics $|\varkappa(t_k,t_{k+1},\Phi_2)|^s=(1+o(1))(\frac{\rho_1+1}{\rho_1})^s\cdot\big(\frac{(1+\theta)^{\frac{\rho_1}{\rho_1+1}}-1}{\theta}\big)^s\cdot(\frac{t_k}{T_1\rho_1})^{-\frac{s}{\rho_1+1}}$ $(k\to+\infty)$ we obtain

$$T_1(\rho_1+1)\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\frac{1+\theta}{\theta}\left(1-(1+\theta)^{-\frac{1}{\rho_1+1}}\right)\geq$$
$$\geq(1+o(1))T_1\left(\frac{\rho_1}{\rho_1+1}\right)^{\rho_1}\left(\frac{t_{k_j}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\left(\frac{\theta}{(1+\theta)^{\frac{\rho_1}{\rho_1+1}}-1}\right)^{\rho_1},\quad j\to+\infty,$$

whence it follows that

$$(\rho_1+1)\frac{1+\theta}{\theta}\left(1-(1+\theta)^{-\frac{1}{\rho_1+1}}\right)\geq\left(\frac{\rho_1}{\rho_1+1}\right)^{\rho_1}\left(\frac{\theta}{(1+\theta)^{\frac{\rho_1}{\rho_1+1}}-1}\right)^{\rho_1}. \tag{15}$$

Similarly, using the inequality $G_1(t_k, t_{k+1}, \Phi_2) < G_2(t_k, t_{k+1}, \Phi_2)$ and Lemma 8, we obtain the opposite inequality, i.e. we can put an equal sign in (15). In [6] it is shown that such equality is possible only if $\theta = 0$.

Thus, $\theta_k \to 0$ $(k \to +\infty)$, and we get the asymptotics $|\varkappa(t_k, t_{k+1}, \Phi_2)|^s = (1 + o(1)) (\frac{t_k}{T_1\rho_1})^{-\frac{s}{\rho_1+1}}$ as $k \to +\infty$. In this case, for example, according to Lemma 11 (i) in virtue of (14)

$$\frac{T_1\rho_1}{8(\rho_1+1)}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\theta_k^2 \le \frac{2\delta}{|\varkappa(t_k, t_{k+1}, \Phi_2)|^s} + O\left(\theta_k^2 t_k^{\frac{\rho_2}{\rho_1+1}}\right) + O\left(\theta_k^3 t_k^{\frac{\rho_1}{\rho_1+1}}\right) + o\left(t_k^{\frac{\rho}{\rho_1+1}}\right)$$

as $k \to +\infty$. So if $s = \rho$, then

$$\frac{T_1\rho_1(1+o(1))}{8(\rho_1+1)}\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\theta_k^2 \le 2\delta\left(\frac{t_k}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}} + o\left(t_k^{\frac{\rho}{\rho_1+1}}\right) \quad (k \to +\infty),$$

whence the assertion (i) of Lemma 12 follows. The rest of the assertions of this lemma can be proved similarly. □

4. Proof of Theorem 1 and other results. It is easy to verify that Theorem 1 follows from the following four theorems.

Theorem 2. *If $\rho > 2\rho_2 - \rho_1$, then for $\ln\mu(\sigma, F)$ to have three-term power asymptotics* (2) *it is necessary and sufficient that for every $\varepsilon > 0$:*

1) there exists a number $n_0(\varepsilon)$ such that for all $n \ge n_0(\varepsilon)$

$$\ln|a_n| \le T_1(\rho_1+1)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} + (\tau+\varepsilon)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}};$$

2) there exists an increasing sequence (n_k) of positive integers such that for all $k \ge k_0$

$$\ln|a_{n_k}| \ge T_1(\rho_1+1)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} + (\tau-\varepsilon)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho}{\rho_1+1}};$$

and $\lambda_{n_{k+1}} - \lambda_{n_k} = o\left(\lambda_{n_k}^{\frac{\rho+\rho_1+2}{2(\rho_1+1)}}\right)$, $k \to +\infty$.

Theorem 3. *If $\rho < 2\rho_2 - \rho_1$, then for $\ln\mu(\sigma, F)$ to have three-term power asymptotics* (2) *it is necessary that for every $\varepsilon > 0$:*

1) there exists a number $n_0(\varepsilon)$ such that for all $n \ge n_0(\varepsilon)$

$$\ln|a_n| \le T_1(\rho_1+1)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} - \left(\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)} - \varepsilon\right)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

2) there exists an increasing sequence (n_k) of positive integers such that for all $k \ge k_0$

$$\ln|a_{n_k}| \ge T_1(\rho_1+1)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} - \left(\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)} + \varepsilon\right)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

and $\lambda_{n_{k+1}} - \lambda_{n_k} = o\left(\lambda_{n_k}^{\frac{\rho_2+1}{\rho_1+1}}\right)$, $k \to +\infty$.

Theorem 4. *Let $\rho = 2\rho_2 - \rho_1$ and $\tau \neq \frac{(\rho_2 T_2)^2}{2\rho_1 T_1(\rho_1+1)}$. Then for $\ln\mu(\sigma, F)$ to have three-term power asymptotics (2) it is necessary and sufficient that for every $\varepsilon > 0$:*

1) there exists a number $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$

$$\ln|a_n| \leq T_1(\rho_1+1)\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{\rho_1}{\rho_1+1}} + T_2\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{\rho_2}{\rho_1+1}} + \Big(\tau+\varepsilon-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}\Big)\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{2\rho_2-\rho_1}{\rho_1+1}}; \tag{16}$$

2) there exists an increasing sequence (n_k) of positive integers such that for all $k \geq k_0$

$$\ln|a_{n_k}| \geq T_1(\rho_1+1)\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{\rho_1}{\rho_1+1}} + T_2\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{\rho_2}{\rho_1+1}} + \Big(\tau-\varepsilon-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}\Big)\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{2\rho_2-\rho_1}{\rho_1+1}}, \tag{17}$$

$$\lambda_{n_{k+1}} - \lambda_{n_k} = o\Big(\lambda_{n_k}^{\frac{\rho_2+1}{\rho_1+1}}\Big), \quad k \to +\infty. \tag{18}$$

Theorem 5. *Let $\rho = 2\rho_2 - \rho_1$ and $\tau = \frac{(\rho_2 T_2)^2}{2\rho_1 T_1(\rho_1+1)}$. Then for $\ln\mu(\sigma, F)$ to have three-term power asymptotics (2) it is necessary and sufficient that for every $\varepsilon > 0$:*

1) there exists a number $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$

$$\ln|a_n| \leq T_1(\rho_1+1)\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{\rho_1}{\rho_1+1}} + T_2\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{\rho_2}{\rho_1+1}} + \varepsilon\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

2) there exists an increasing sequence (n_k) of positive integers such that for all $k \geq k_0$

$$\ln|a_{n_k}| \geq T_1(\rho_1+1)\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{\rho_1}{\rho_1+1}} + T_2\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{\rho_2}{\rho_1+1}} - \varepsilon\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{2\rho_2-\rho_1}{\rho_1+1}};$$

and $\lambda_{n_{k+1}} - \lambda_{n_k} = o\big(\lambda_{n_k}^{\frac{\rho_2+1}{\rho_1+1}}\big), \quad k \to +\infty.$

On account of analogy we focus only on the proof of Theorem 4. We begin with the necessity. Asymptotics (2) implies for every $0 < \delta < \min\{|\tau|, |\tau - \frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}|\}$ and all $\sigma \in [\sigma_0(\delta), 0)$ the condition of Lemma 4 is true with $\Phi_1(\sigma) = \frac{T_1}{|\sigma|^{\rho_1}} + \frac{T_2}{|\sigma|^{\rho_2}} + \frac{\tau-\delta}{|\sigma|^{\rho}}$ and $\Phi_2(\sigma) = \frac{T_1}{|\sigma|^{\rho_1}} + \frac{T_2}{|\sigma|^{\rho_2}} + \frac{\tau+\delta}{|\sigma|^{\rho}}$. Therefore, according to this lemma the inequalities $\ln|a_n| \leq -\lambda_n\Psi_2(\varphi_2(\lambda_n))$ for all $n \geq n_0$ and $\ln|a_{n_k}| \geq -\lambda_{n_k}\Psi_1(\varphi_1(\lambda_{n_k}))$ for an increasing sequence (n_k) of positive integers such that $G_1(\lambda_{n_k}, \lambda_{n_{k+1}}, \Phi_2) \geq \Phi_1(\frac{1}{\lambda_{n_{k+1}}-\lambda_{n_k}}\int_{\lambda_{n_k}}^{\lambda_{n_{k+1}}}\varphi_2(t)dt)$ hold. But by assertion (iii) of Lemma 6 with $s = \rho = 2\rho_2 - \rho_1$ we obtain

$$\lambda_n\Psi_2(\varphi_2(\lambda_n)) = -T_1(\rho_1+1)\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{\rho_1}{\rho_1+1}} - T_2\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{\rho_2}{\rho_1+1}} - \\ -\Big(\tau+\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\Big)\Big(\frac{\lambda_n}{T_1\rho_1}\Big)^{\frac{2\rho_2-\rho_1}{\rho_1+1}}, \quad n \to +\infty,$$

$$\lambda_{n_k}\Psi_1(\varphi_1(\lambda_{n_k})) = -T_1(\rho_1+1)\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{\rho_1}{\rho_1+1}} - T_2\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{\rho_2}{\rho_1+1}} - \\ -\Big(\tau-\delta-\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}+o(1)\Big)\Big(\frac{\lambda_{n_k}}{T_1\rho_1}\Big)^{\frac{2\rho_2-\rho_1}{\rho_1+1}}, \quad k \to +\infty,$$

and by assertion (iii) of Lemma 12 we have

$$\left(\frac{\lambda_{n_{k+1}}-\lambda_{n_k}}{\lambda_{n_k}}\right)^2=\theta_k^2\le\frac{16(\rho_1+1)}{T_1\rho_1}(\delta+o(1))\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{2\rho_2-2\rho_1}{\rho_1+1}},\quad k\to+\infty,$$

i.e. $\lambda_{n_{k+1}}-\lambda_{n_k}\le 4\sqrt{\rho_1+1}(\sqrt{\delta}+o(1))(T_1\rho_1)^{\frac{\rho_1-2\rho_2-1}{2(\rho_1+1)}}\lambda_{n_k}^{\frac{\rho_2+1}{\rho_1+1}}$ as $k\to+\infty$.

Taking into account arbitrariness of δ these relations imply (16)–(18).

We will now prove the sufficiency of conditions (16)–(18). Using Lemma 1 and assertion (iii) of Lemma 6 with $s=\rho=2\rho_2-\rho_1$, it is easy to show that condition (16) implies the asymptotic inequality

$$\ln\mu(\sigma)\le\frac{T_1}{|\sigma|^{\rho_1}}+\frac{T_2}{|\sigma|^{\rho_2}}+\frac{\tau+\delta}{|\sigma|^{\rho}},\quad\sigma\in[\sigma_1,0)\tag{19}$$

for arbitrary positive δ. Further by Lemma 3 and assertion (iii) of Lemma 11 with $s=\rho=2\rho_2-\rho_1$ from the inequality (17) for $k\ge k_0$ and $\sigma\in[\varphi_1(\lambda_{n_k}),\varphi_1(\lambda_{n_{k+1}})]$ in view of condition (18) we obtain

$$\ln\mu(\sigma)\ge\Phi_1(\sigma)-(G_2(\lambda_{n_k},\lambda_{n_{k+1}},\Phi_1)-G_1(\lambda_{n_k},\lambda_{n_{k+1}},\Phi_1))=\Phi_1(\sigma)-$$
$$-\frac{T_1\rho_1}{8(\rho_1+1)}\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}\theta_k^2+O(\theta_k^2\lambda_{n_k}^{\frac{\rho_2}{\rho_1+1}})+O(\theta_k^3\lambda_{n_k}^{\frac{\rho_1}{\rho_1+1}})+o(\lambda_{n_k}^{\frac{2\rho_2-\rho_1}{\rho_1+1}})\ge\Phi_1(\sigma)-\delta\cdot\lambda_{n_k}^{\frac{2\rho_2-\rho_1}{\rho_1+1}}$$

for arbitrary positive δ, as $k\to+\infty$ and, thus,

$$\ln\mu(\sigma)\ge\Phi_1(\sigma)-\delta\cdot(\Phi_1'(\sigma))^{\frac{2\rho_2-\rho_1}{\rho_1+1}}\ge\Phi_1(\sigma)-\tilde{\delta}\cdot\left(\frac{1}{|\sigma|^{\rho_1+1}}\right)^{\frac{2\rho_2-\rho_1}{\rho_1+1}}=\Phi_1(\sigma)-\tilde{\delta}\cdot\frac{1}{|\sigma|^{2\rho_2-\rho_1}}.$$

Since $\rho=2\rho_2-\rho_1$, we have

$$\ln\mu(\sigma)\ge\frac{T_1}{|\sigma|^{\rho_1}}+\frac{T_2}{|\sigma|^{\rho_2}}+\frac{(\tau-\delta_1)}{|\sigma|^{\rho}},\quad\sigma\in[\sigma_2,0)\tag{20}$$

for arbitrary positive δ_1. On account of arbitrariness of δ and δ_1 (19) and (20) imply (2). Theorem 4 is proved.

Finally, we note that similar results can be obtained in cases which correspond to the statements (iv)–(vii) of Lemma 5. For example, the following theorem is correct.

Theorem 6. *If $2\rho_2-\rho_1>0$ and $(3\rho_2-2\rho_1)(3\rho_2-2\rho_1-1)\ne0$, then for*

$$\ln\mu(\sigma,F)=\frac{T_1}{|\sigma|^{\rho_1}}+\frac{T_2}{|\sigma|^{\rho_2}}+\frac{(\rho_2T_2)^2}{2\rho_1T_1(\rho_1+1)}\frac{1}{|\sigma|^{2\rho_2-\rho_1}}+o\left(\frac{1}{|\sigma|^{3\rho_2-2\rho_1}}\right),\quad\sigma\uparrow0,$$

it is necessary and sufficient that for every $\varepsilon>0$:

1) there exists a number $n_0(\varepsilon)$ such that for all $n\ge n_0(\varepsilon)$

$$\ln|a_n|\le T_1(\rho_1+1)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}}+T_2\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}}-$$
$$-\left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}-\varepsilon\right)\left(\frac{\lambda_n}{T_1\rho_1}\right)^{\frac{3\rho_2-2\rho_1}{\rho_1+1}};$$

2) there exists an increasing sequence (n_k) of positive integers such that for all $k \geq k_0$

$$\ln|a_{n_k}| \geq T_1(\rho_1+1)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_1}{\rho_1+1}} + T_2\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{\rho_2}{\rho_1+1}} -$$

$$-\left(\frac{(3\rho_2-2\rho_1-1)(\rho_2T_2)^3}{6(\rho_1T_1(\rho_1+1))^2}+\varepsilon\right)\left(\frac{\lambda_{n_k}}{T_1\rho_1}\right)^{\frac{3\rho_2-2\rho_1}{\rho_1+1}}; \ \lambda_{n_{k+1}}-\lambda_{n_k} = o\left(\lambda_{n_k}^{\frac{3\rho_2-\rho_1+2}{2(\rho_1+1)}}\right), \quad k\to+\infty.$$

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