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SOME RESULTS ON UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING SETS

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Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$. In 2011 A. Banerjee ([4]), in an attempt to answer on the question of F. Gross ([9]), by W. C. Lin and H. X. Yi ([18]), improved a result of I. Lahiri ([14]) by reducing the cardinality of the set shared by f and g from 7 to 6 under weaker condition on ramification index. In this paper we show that the cardinality of the shared set can further be reduced to 4 as well as the condition on ramification index can be replaced by weaker one to obtain the same conclusion as A. Banerjee ([4]).

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Пусть f и g — две мероморфные в открытой плоскости функции, не равные тождественно постоянной. В 2011 г. А. Банерджи ([4]), отвечая на вопрос Ф. Гросса ([9]), В. К. Лин и Г. Х. Йи ([18]), уточнили результат И. Лахира ([14]) уменьшив мощность множества разделённых значений f и g с 7 до 6 при более слабых условиях на индекс ветления. В этой статье показано, что мощность множества разделённых значений может быть уменьшена до 4, а условия на индекс ветления могут быть ослаблены, при этом сохраняется то же заключения, как и у А. Банерджи ([4]).

1. Introduction, definitions and results. Let f and g be two non constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (Counting Multiplicities) and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [11]. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity then we replace the above set by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In 1976 F. Gross([9]) raised the following question.

Question A. *Can one find finite sets S_j, $j \in \{1, 2\}$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j \in \{1, 2\}$ must be identical?*

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As a natural outcome of the above question W. C. Lin and H. X. Yi([18]) raised the following question in 2003.

Question B. *Can one find finite sets S_j, $j \in \{1,2\}$ such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j \in \{1,2\}$ must be identical?*

During last couple of years a great deal of works has been directed by researchers to answer the above questions ([1]–[8], [10], [14], [16]–[18], [20]–[26]).

In 2003, M. Fang and I. Lahiri exhibited a unique range set with smaller cardinalities than that obtained some previously imposing restrictions on the poles of f and g in the following result.

Theorem A([6]). *Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n(\geq 7)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two nonconstant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

In 2001 I. Lahiri introduced an idea of gradation of sharing of values and sets known as weighted sharing as follows.

Definition 1 ([12, 13])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ of multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2 ([13])**.** Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

With the notion of weighted sharing of sets improving Theorem A, I. Lahiri ([14]) proved the following theorem.

Theorem B([14]). *Let S be defined as in Theorem A. If f and g are two nonconstant meromorphic functions such that $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 1$ then $f \equiv g$.*

Suppose that the polynomial $P(w)$ is defined by

$$P(w) = a\omega^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 \qquad (1)$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. We also define

$$R(w) = \frac{aw^n}{n(n-1)(w - \alpha_1)(w - \alpha_2)}, \qquad (2)$$

where α_1, α_2 are two distinct roots of $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. It can be shown that $P(w)$ has only simple roots([25]).

In 2011 A. Banerjee improved Theorem B in the following result by lowering the cardinality of the shared set replacing it by a new one in the following theorem.

Theorem C([4]). *Let $S=\{w \mid P(w)=0\}$, where $P(w)$ is given by (1) and $n(\geq 6)$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ and $\Theta_f+\Theta_g+\min\{\Theta(b,f),\Theta(b,g)\}>8-n$, where $\Theta_f=2\Theta(0;f)+\Theta(b;f)+\Theta(\infty;f)$ and Θ_g is defined similarly. Then $f\equiv g$.*

Before proceeding further we need the following definitions.

Definition 3. For $a\in\mathbb{C}\cup\{\infty\}$ and a positive integer m we denote by $\overline{N}(r,a;f\mid\geq m)$ the reduced counting function of those a-points of f whose multiplicity is greater than or equal to m.

Definition 4. We put $N_2(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)$ and $\delta_2(a;f)=1-\overline{\lim\limits_{r\to\infty}}\frac{N_2(r,a;f)}{T(r,f)}$.

The aim of this paper is to improve Theorem C in the following way.

1. By replacing $n\geq 7$ in Theorem C with $n\geq 4$.
2. By replacing the condition on ramification index by weaker one.

Note that in the definition of the polynomial $P(w)$, we require $ab^{n-2}\neq 2$. For our purpose, in addition to it we assume $ab^{n-2}\neq 1$, by which the polynomial $P(w)$ will not lose any of its properties mentioned above. Thus from now on our set S is given by $S=\{w\mid P(w)=0\}$ where $P(w)$ is given by (1) with $ab^{n-2}\neq 2,1$. We state below our theorem.

Theorem 1. *Let $S=\{w\mid P(w)=0\}$, where $P(w)$ is given by (1) and n (≥ 4) and $ab^{n-2}\neq 2$, 1. If f and g be two nonconstant meromorphic functions such that $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ and*

$$\Theta_f+\Theta_g+\min\{\Theta(b,f),\Theta(b,g)\}+\min\Big\{\sum_{a\notin S\cup\{0,b,\infty\}}\delta_2(a,f),\sum_{a\notin S\cup\{0,b,\infty\}}\delta_2(a,g)\Big\}>8-n \quad (3)$$

then $f\equiv g$, where $\Theta_f=2\Theta(0;f)+\Theta(b;f)+\Theta(\infty;f)$ and Θ_g is defined similarly.

Following Corollaries are the easy consequences of the above theorem.

Corollary 1. *Let $S=\{w\mid P(w)=0\}$, where $P(w)$ is given by (1) and n (≥ 7) and $ab^{n-2}\neq 2,1$. If f and g are two nonconstant entire functions such that $E_f(S,2)=E_g(S,2)$, then $f\equiv g$.*

Corollary 2. *Let $S=\{w\mid P(w)=0\}$, where $P(w)$ is given by (1) and n (≥ 7) and $ab^{n-2}\neq 2,1$. If f and g are two nonconstant meromorphic functions having no simple zeros and satisfying $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$, then $f\equiv g$.*

We conclude this section with the definition of a few more notations.

Definition 5 ([4, 13])**.** Let f and g be two nonconstant meromorphic functions such that f and g share $(a,0)$ for $a\in\mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity p, and an a-point of g of multiplicity q. We denote by $\overline{N}_L(r,a;f)(\overline{N}_L(r,a;g))$ the reduced counting function of those a-points of $f$and g where $p>q(q>p)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$. We also denote by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$.

2. Lemmas. In this section we present some lemmas which will be required to establish our results. In the lemmas several times we use the function H defined by $H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$ where F and G are two non-constant meromorphic functions.

Let f and g be two nonconstant meromorphic functions and

$$F = R(f),\ G = R(g), \tag{4}$$

where $R(w)$ is given by (2). From (2) and (5) it is clear that

$$T(r,f) = \frac{1}{n}T(r,F) + S(r,f), T(r,g) = \frac{1}{n}T(r,G) + S(r,g). \tag{5}$$

Lemma 1 ([2]). *Let F,G be given by (4) and $H \not\equiv 0$. If F,G share $(1,m)$ and f,g share (∞,k), then*

$$N_E^{1)}(r,1;F) \le \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,0;g) + \overline{N}(r,b;g) + \\ +\overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f' which are not the zeros of $f(f-b)$ and $F-1$. $\overline{N}_0(r,0;g')$ is defined similarly.

Lemma 2 ([19]). *Let f be a non-constant meromorphic function and let $R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$ be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \ne 0$, $b_m \ne 0$. Then $T(r,R(f)) = dT(r,f) + S(r,f)$, where $d = \max\{m,n\}$.*

Lemma 3 ([2]). *Let F and G be given by (4) and $H \not\equiv 0$. If F and G share $(1,m)$ and f,g share (∞,k), where $0 \le m < \infty$, $0 \le k < \infty$, then*

$$[(n-2)k+n-3]\overline{N}(r,\infty;f \mid\ge k+1) = [(n-2)k+n-3]\overline{N}(r,\infty;g \mid\ge k+1) \le \\ \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).$$

Lemma 4 ([4]). *Let f,g be two non-constant meromorphic functions sharing $(\infty,0)$ and suppose that α_1 and α_2 are two distinct roots of the equation $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$. Then $\frac{f^n}{(f-\alpha_1)(f-\alpha_2)} \cdot \frac{g^n}{(g-\alpha_1)(g-\alpha_2)} \not\equiv \frac{n^2(n-1)^2}{a^2}$, where $n \ge 3$ is an integer.*

Lemma 5 ([8]). *Let $Q(w) = (n-1)^2(w^n-1)(w^{n-2}-1) - n(n-2)(w^{n-1}-1)^2$, then $Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)\dots(w-\beta_{2n-6})$ where $\beta_j \in \mathbb{C}\setminus\{0,1\}$, $(j \in \{1,2,\dots,2n-6\})$ which are pairwise distinct.*

Lemma 6. *Let F,G be given by (5), where $n \ge 4$ is an integer. If f,g share $(\infty,0)$ then $F \equiv G \Rightarrow f \equiv g$.*

Proof. From the definitions of F,G we observe that $F \equiv G \Rightarrow \frac{f^n}{(f-\alpha_1)(f-\alpha_2)} \equiv \frac{g^n}{(g-\alpha_1)(g-\alpha_2)}$. Therefore f,g share $(0,\infty)$ and (∞,∞). Then from above and in view of the definitions of $R(w)$ we obtain

$$n(n-1)f^2g^2(f^{n-2}-g^{n-2}) - 2n(n-2)bfg(f^{n-1}-g^{n-1}) + (n-1)(n-2)b^2(f^n-g^n) = 0. \tag{6}$$

Let $h = \frac{f}{g}$, that is $f = gh$ which on substitution in (6) yields

$$n(n-1)h^2g^2(h^{n-2}-1) - 2n(n-2)bhg(h^{n-1}-1) + (n-1)(n-2)b^2(h^n-1) = 0. \tag{7}$$

Note that since f and g share $(0,\infty)$ and (∞,∞), $(0,\infty)$ are the exceptional values of Picard of h. If h is non-constant then from Lemma 6 and (7) we have

$$\{n(n-1)h(h^{n-2}-1)g - n(n-2)b(h^{n-1}-1)\}^2 = -n(n-2)b^2Q(h) \tag{8}$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\dots(h-\beta_{2n-6})$, $\beta_j \in \mathbb{C}\setminus\{0,1\}$, $j \in \{1,2,\dots,2n-6\}$ which are pairwise distinct. From (8)we observe that each zero of $h-\beta_j$, $j \in \{1,2,\dots,2n-6\}$ is of order at least two. Therefore by the second main theorem we obtain

$$(2n-6)T(r,h) \le \overline{N}(r,\infty;h)+\overline{N}(r,0;h)+\sum_{j=1}^{2n-6}\overline{N}(r,\beta_j;h)+S(r,h) \le \frac{1}{2}(2n-6)T(r,h)+S(r,h),$$

which is a contradiction for $n \ge 4$. Thus h must be a constant. From (8) it follows that $h^{n-2}-1=0$ and $h^{n-1}-1=0$ which implies that $h \equiv 1$. Therefore $f \equiv g$. □

Lemma 7 ([4])**.** *Let F, G be given by (4) and S be defined as in Theorem 1, where $n \ge 4$. If $E_f(S,0) = E_g(S,0)$ then $S(r,f) = S(r,g)$.*

Lemma 8 ([15])**.** *If $N(r,0;f^{(k)} \mid f \ne 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then $N(r,0;f^{(k)} \mid f \ne 0) \le k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\ge k) + S(r,f)$ where $N(r,0;f \mid< k)$ is the counting function of the zeros of f with multiplicity $< k$ each zero being counted according to its multiplicity.*

3. Proof of Theorem 1. Let F, G be given by (4). Suppose first that $H \not\equiv 0$. Let p be any positive integer and $a_j \notin S \cup \{0,b,\infty\}$, $j \in \{1,2,\dots,p\}$ be distinct complex numbers. We denote by $N_*(r,0;f')$ the counting function of the zeros of f' which are not the zeros of $f(f-b)\prod_{j=1}^{p}(f-a_j)$ and $F-1$. Similarly may we define $N_*(r,0;g')$ and the corresponding reduced counting functions $\overline{N}_*(r,0;f')$ and $\overline{N}_*(r,0;g')$. Note that $\overline{N}_0(r,0;f') = \overline{N}_*(r,0;f') + \sum_{j=1}^{p}\overline{N}(r,a_j,f \mid\ge 2)$. Since $E_f(S,2) = E_g(S,2)$, it follows that F, G share (1,2). Also since $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$ we see that $\overline{N}_*(r,\infty;f,g) \equiv 0$. We denote the elements of S by $w_1, w_2, \dots, w_n$. By Lemma 8 we note that $\overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\ge 2) + \overline{N}_*(r,1;F,G) \le \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\ge 2) + \overline{N}(r,1;G \mid\ge 3) \le \overline{N}_0(r,0;g') + \sum_{j=1}^{n}\{\overline{N}(r,w_j;g \mid= 2) + 2\overline{N}(r,w_j;g \mid\ge 3)\} \le N(r,0;g' \mid g \ne 0) + S(r,g) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g)$.

Hence by above and the second main theorem we obtain for $\epsilon > 0$ using Lemma 1 and Lemma 2,

$$(n+p+1)T(r,f) \le \overline{N}(r,1;F) + \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + \sum_{j=1}^{p}\overline{N}(r,a_j;f) -$$

$$-N_*(r,0;f') + S(r,f) = \overline{N}(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\ge 2) + \overline{N}(r,0;f) + \overline{N}(r,b;f) +$$

$$+\overline{N}(r,\infty;f) + \sum_{j=1}^{p}\overline{N}(r,a_j;f) - N_*(r,0;f') + S(r,f) \le$$

$$\le 2\{\overline{N}(r,0;f) + \overline{N}(r,b;f)\} + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,b;g) + \overline{N}(r,1;G \mid\ge 2) +$$

$$+\overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;g') + \overline{N}_0(r,0;f') + \sum_{j=1}^{p}\overline{N}(r,a_j;f) - N_*(r,0;f') + S(r,g) +$$

$$+S(r,f)\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)\}+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+$$

$$+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_{*}(r,1;F,G)+\overline{N}_{0}(r,0;g')+\overline{N}_{*}(r,0;f')+\sum_{j=1}^{p}\overline{N}(r,a_j,f\mid\geq 2)+$$

$$+\sum_{j=1}^{p}\overline{N}(r,a_j;f)-N_{*}(r,0;f')+S(r,g)+S(r,f)\leq$$

$$\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)\}+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,b;g)+\overline{N}(r,1;G\mid\geq 2)+$$

$$+\overline{N}_{*}(r,1;F,G)+\overline{N}_{0}(r,0;g')+\overline{N}_{*}(r,0;f')+\sum_{j=1}^{p}\overline{N}(r,a_j,f\mid\geq 2)+\sum_{j=1}^{p}\overline{N}(r,a_j;f)-$$

$$-N_{*}(r,0;f')+S(r,g)+S(r,f)\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,0;g)\}+$$

$$+\overline{N}(r,\infty;f)+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)+\sum_{j=1}^{p}N_2(r,a_j;f)+S(r,g)+S(r,f),$$

and hence

$$(n+p+1)T(r,f)\leq(9+p-2\Theta(0;f)-2\Theta(0;g)-\Theta(\infty;f)-\Theta(\infty;g)-2\Theta(b;f)-$$
$$-\Theta(b;g)-\sum_{j=1}^{p}\delta_2(a_j,f)+\epsilon)T(r)+S(r), \tag{9}$$

where $T(r)=\max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$ as $r\to\infty$, $r\notin E$. Similarly we obtain

$$(n+p+1)T(r,g)\leq(9+p-2\Theta(0;f)-2\Theta(0;g)-\Theta(\infty;f)-\Theta(\infty;g)-2\Theta(b;g)-$$
$$-\Theta(b;f)-\sum_{j=1}^{p}\delta_2(a_j,g)+\epsilon)T(r)+S(r). \tag{10}$$

Combining (9) and (10) we obtain

$$(n+p+1)T(r)\leq\Big[9+p-2\Theta(0;f)-2\Theta(0;g)-\Theta(\infty;f)-\Theta(\infty;g)-\Theta(b;f)-\Theta(b;g)-$$

$$-\min\Big\{\Theta(b;f),\Theta(b;g)\Big\}-\min\Big\{\sum_{j=1}^{p}\delta_2(a_j,f),\sum_{j=1}^{p}\delta_2(a_j,g)\Big\}+\epsilon\Big]T(r)+S(r)\Rightarrow\Big[\Theta_f+\Theta_g+$$

$$+\min\Big\{\Theta(b;f),\Theta(b;g)\Big\}+\min\Big\{\sum_{j=1}^{p}\delta_2(a_j,f),\sum_{j=1}^{p}\delta_2(a_j,g)\Big\}-(8-n)-\epsilon\Big]T(r)\leq S(r).$$

Since p is arbitrary we have from above,

$$\Big[\Theta_f+\Theta_g+\min\{\Theta(b;f),\Theta(b;g)\}+$$
$$+\min\Big\{\sum_{a\notin S\cup\{0,b,\infty\}}\delta_2(a,f),\sum_{a\notin S\cup\{0,b,\infty\}}\delta_2(a,g)\Big\}-(8-n)-\epsilon\Big]T(r)\leq S(r),$$

but this contradicts (3). Hence $H\equiv 0$. Then

$$F\equiv\frac{AG+B}{CG+D} \tag{11}$$

where A, B, C, D are constants such that $AD - BC \neq 0$. Also $T(r, F) = T(r, G) + O(1)$, and hence from (5)

$$T(r, f) = T(r, g) + O(1). \tag{12}$$

Since $R(w) - c = \frac{a(w-b)^3 Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}$, where $c = \frac{ab^{n-2}}{2} \neq 1, \frac{1}{2}$ and $Q_{n-3}(w)$ is a polynomial in w of degree $n-3$, in view of the definitions of F and G we notice that

$$\begin{aligned} \overline{N}(r, c; F) \leq \overline{N}(r, b; f) + (n-3)T(r, f) \leq (n-2)T(r, f) + S(r, f), \\ \overline{N}(r, c; G) \leq \overline{N}(r, b; g) + (n-3)T(r, g) \leq (n-2)T(r, g) + S(r, g). \end{aligned} \tag{13}$$

Now we consider the following cases.

Case 1. $C \neq 0$.

Since f, g share (∞, ∞) it follows from (11) that ∞ is an exceptional value of Picard of f and g. Therefore from (2) and (4) it follows that

$$\overline{N}(r, \infty; F) = \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f), \quad \overline{N}(r, \infty; G) = \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g). \tag{14}$$

Subcase 1.1. $A \neq 0$.

Suppose $B \neq 0$. Then from (11) it follows that $\overline{N}(r, -\frac{B}{A}; G) = \overline{N}(r, 0; F)$. Thus from the second main theorem and (13) we have

$$\begin{aligned} nT(r, g) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, -\frac{B}{A}; G) + S(r, G) \leq \\ \leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, 0; f) + S(r, g). \end{aligned} \tag{15}$$

Clearly (14) leads to a contradiction if $n \geq 5$. Let $n = 4$. Note that if either $\overline{N}(r, 0; g) < T(r, g) + S(r, g)$ or $\overline{N}(r, 0; f) < T(r, f) + S(r, f)$ then also above leads to a contradiction.

So let $\overline{N}(r, 0; g) \sim T(r, g) + S(r, g)$ and $\overline{N}(r, 0; f) \sim T(r, f) + S(r, f)$ that is $\Theta(0, g) = 0$ and $\Theta(0, f) = 0$. Since $\Theta(\infty, g) = 1$, and $\Theta(\infty, f) = 1$, from (3) we obtain with $n = 4$, $\Theta(b, f) + \Theta(b, g) + \min\{\sum_{a \notin S \cup \{0, b, \infty\}} \delta_2(a, f), \sum_{a \notin S \cup \{0, b, \infty\}} \delta_2(a, g)\} > 2$, which is not possible. Therefore $B = 0$. Then $F \equiv \frac{\frac{A}{C}.G}{G + \frac{D}{C}}$ and therefore $\overline{N}(r, \frac{-D}{C}; G) = \overline{N}(r, \infty; F)$. We also note that $c = \frac{ab^{n-2}}{2} \neq 0$. If possible suppose $c = \frac{-D}{C}$. Also suppose that F has no 1-points. This amounts to saying that f has no w_i-points where $w_i \in S$ and $i \in \{1, 2, \ldots, n(\geq 4)\}$, which is not possible. Therefore F must have some 1-points. Since F, G share 1-points, we have $A = C + D = C - cC$ and hence $F = \frac{(C - cC)G}{CG - cC} = \frac{(1-c)G}{G - c}$, since $C \neq 0$ by our assumption. Then since $c \neq \frac{1}{2}$, from above $\overline{N}(r, c; F) = \overline{N}(r, \frac{c^2}{2c-1}; G)$ and since $c \neq 1$, $c \neq \frac{c^2}{2c-1}$. Thus by the second main theorem and (13) we have $2nT(r, g) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, c; G) + \overline{N}\left(r, \frac{c^2}{2c-1}; G\right) + S(r, g) \leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) + (n-2)T(r, f) + S(r, g) \leq (5 + n - 2)T(r, g) + S(r, g)$, which leads to a contradiction for $n \geq 4$. Next let $c \neq \frac{-D}{C}$. Hence as before by the second main theorem and (13) we have $2nT(r, g) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, \frac{-D}{C}; G) + \overline{N}(r, c; G) + S(r, G) \leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) + (n-2)T(r, g) + S(r, g) \leq (5 + n - 2)T(r, g) + S(r, g)$, which leads to a contradiction for $n \geq 4$.

Subcase 1.2. $A = 0$.

Then clearly $B \neq 0$ and $F \equiv \frac{1}{\gamma G+\delta}$ where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$. Since F and G have some 1-points, then $\gamma + \delta = 1$ and so $F \equiv \frac{1}{\gamma G+1-\gamma}$. Suppose $\gamma \neq 1$. If $\frac{1}{1-\gamma} \neq c$ then by second main theorem and (13) we have

$$2nT(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,\frac{1}{1-\gamma};F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,F) \leq$$
$$\leq \overline{N}(r,0;f) + (n-2)T(r,f) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f)$$
$$\Rightarrow (n+2)T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f),$$

which is a contradiction for $n \geq 4$. If $c = \frac{1}{1-\gamma}$, then $F \equiv \frac{c}{(c-1)G+1}$. If $c \neq \frac{1}{1-c}$, then by the second main theorem and (13) we obtain

$$2nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}\Big(r,\frac{1}{1-c};G\Big) + \overline{N}(r,\infty;G) + S(r,g) \leq$$
$$\leq \overline{N}(r,0;g) + (n-2)T(r,g) + \overline{N}(r,\infty;F) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g) \leq$$
$$\leq \overline{N}(r,0;g) + (n-2)T(r,g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g).$$

Thus $(n+2)T(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + S(r,g)$, which leads to a contradiction for $n \geq 4$.

If $c = \frac{1}{1-c}$ then $G \equiv \frac{c(F-c)}{F}$ and by the second main theorem we obtain $nT(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + S(r,f)$. Above leads to a contradiction for $n \geq 5$. Let $n = 4$. If either $\overline{N}(r,0;f) < T(r,f) + S(r,f)$ or $\overline{N}(r,0;g) < T(r,g) + S(r,g)$ then also above leads to a contradiction. Therefore suppose $\overline{N}(r,0;f) \sim T(r,f)$ and $\overline{N}(r,0;g) \sim T(r,g)$ that is $\Theta(0,f) = 0$ and $\Theta(0,g) = 0$.

Since $\Theta(\infty,f) = \Theta(\infty,g) = 1$, from (3) we get for $n = 4$ and $\Theta(b,f) + \Theta(b,g) + \min\{\Theta(b,f),\Theta(b,g)\} + \min\{\sum_{a \notin S\cup\{0,b,\infty\}} \delta_2(a,f), \sum_{a \notin S\cup\{0,b,\infty\}} \delta_2(a,g)\} > 2$, which is not possible. Therefore we must have $\gamma = 1$ and hence $FG \equiv 1$, which is again impossible by Lemma 4.

Case 2. $C = 0$.

Clearly $A \neq 0$ and $F \equiv \alpha G + \beta$, where $\alpha = \frac{A}{D}, \beta = \frac{B}{D}$. Since F and G must have some 1-points, $\alpha + \beta = 1$ and so $F \equiv \alpha G + 1 - \alpha$. Suppose $\alpha \neq 1$. If $1 - \alpha \neq c$, then by the second main theorem and (13) we obtain

$$2nT(r,f) \leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1-\alpha;F) + S(r,f) \leq \overline{N}(r,0;f) +$$
$$+ \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + (n-2)T(r,f) + \overline{N}(r,0;G) + S(r,f).$$

Thus $(n+2)T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,0;g) + S(r,f)$ which leads to a contradiction for $n \geq 4$. If $1 - \alpha = c$, then $F \equiv (1-c)G + c$. Since $c \neq 1$ we obtain from the second main theorem and (13)

$$2nT(r,g) \leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}(r,\infty;G) + \overline{N}\Big(r,\frac{c}{c-1};G\Big) + S(r,g) \leq$$
$$\leq \overline{N}(r,0;g) + (n-2)T(r,g) + \overline{N}(r,\infty;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,0;F) + S(r,g).$$

Thus $(n+2)T(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;f) + S(r,f)$ which leads to a contradiction for $n \geq 4$.

So $\alpha = 1$. Hence $F \equiv G$ and therefore by Lemma 6, $f \equiv g$.

REFERENCES

1. A. Banerjee, *Uniqueness of meromorphic functions that share two sets*, Southeast Asian Bull. Math., **31** (2007), 7–17.
2. A. Banerjee, *On uniqueness of meromorphic functions that share two sets*, Georgian Math. J., **15** (2008), №1, 21–28.
3. A. Banerjee, S. Mukherjee, *Uniqueness of meromorphic functions sharing two or three sets*, Hokkaido Math. J., **37** (2008), №3, 507–530.
4. A. Banerjee, *A uniqueness result on meromorphic functions sharing two sets*, J. Adv. Math. Stud., **4** (2011), №1, 1–10.
5. M. Fang, H. Guo, *On meromorphic functions sharing two values*, Analysis, **17** (1997), №4, 355–366.
6. M. Fang, I. Lahiri, *Uniquene range set for certain meromorphic functions*, Indian J. Math., **45** (2003), №2, 141–150.
7. M. Fang, W. Xu, *A note on a problem of Gross*, Chinese Ann. Math. Ser. A, **18** (1997), №4, 355–366.
8. G. Frank, M.R. Reinders, *A unique range set for meromorphic functions with 11 elements*, Complex Var. Theory Appl., **37** (1998), 185–193.
9. F. Gross, *Factorization of meromorphic functions and some open problems*, Proc. Conf. Univ. Kentucky, Lexington, Lecture notes in Math., Springer, Berlin, **599** (1977), 51–69.
10. Q. Han, H.X. Yi, *Some further results on meromorphic functions that share two sets*, Ann. Polon. Math. **91** (2008), №1, 17–31.
11. W.K. Hayman, Meromorphic functions. – Clarendon Press, Oxford, 1964.
12. I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161** (2001), 193–206.
13. I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241–253.
14. I. Lahiri, *On a question of Hong Xun Yi*, Arch Math. (Brno), **38** (2002), 119–128.
15. I. Lahiri, S. Dewan, *Value distribution of the product of a meromorphic function and its derivative*, Kodai Math. J., **26** (2003), 95–100.
16. P. Li, C.C. Yang, *On the unique range sets for meromorphic functions*, Proc. Amer. Math. Soc., **124** (1996), 177–185.
17. P. Li, C.C. Yang, *Some further results on the unique range sets for meromorphic functions*, Kodai Math. J., **18** (1995), 437–450.
18. W.C. Lin, H.X. Yi, *Some further results on meromorphic functions that share two sets*, Kyungpook Math. J., **43** (2003), 73–85.
19. A.Z. Mohon'ko, *On the Nevanlinna characteristics of some meromorphic functions*, Theory of Funct. Funct. Anal. Appl., **14** (1971), 83–87.
20. H.X. Yi, *Uniqueness of meromorphic functions and a question of Gross*, Science in China(A), **37** (1994), №7, 802–813.
21. H.X. Yi, *Unicity theorem for meromorphic functions or entire functions*, Bull. Austral. Math. Soc. II, **52** (1995), 215–224.
22. H.X. Yi, *On a question of Gross concerning uniqueness of entire functions*, Bull. Austral. Math. Soc., **57** (1998), 343–349.
23. H.X. Yi, *Meromorphic functions that share two sets* (Chinese), Acta Math. Sinica (Chinese Ser.), **45** (2002), №1, 75–82.
24. H.X. Yi, *Uniqueness of meromorphic functions II*, Indian J. Pure Appl. Math., **28** (1997), 509–519.
25. H.X. Yi, W.C. Lin, *Uniqueness of meromorphic and a question of Gross*, Kyungpook Math. J., **46** (2006), 437–444.
26. H.X. Yi, W.R. Lu, *Meromorphic functions that share two sets II*, Acta Math. Sci. Ser. B, Eng Ed., **24** (2004), №1, 83–90.

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