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INTERASSOCIATES OF A FREE SEMIGROUP WITH TWO GENERATORS

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For any semigroup $(S;\cdot)$ let $(S;\circ)$ be a semigroup defined on the same set. Semigroup $(S;\circ)$ is called an interassociate of $(S;\cdot)$ if the following identities hold $x\cdot(y\circ z)=(x\cdot y)\circ z$ and $x\circ(y\cdot z)=(x\circ y)\cdot z$. All interassociates of the free semigroup over the two-element alphabet are described.

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Для произвольной полугруппы $(S;\cdot)$ пусть $(S;\circ)$ — полугруппа, определенная на том же множестве. Полугруппа $(S;\circ)$ называется интерассоциативной к полугруппе $(S;\cdot)$, если выполнены следующие тождества: $x\cdot(y\circ z)=(x\cdot y)\circ z$ и $x\circ(y\cdot z)=(x\circ y)\cdot z$. Описаны все полугруппы, интерассоциативные к свободной полугруппе над двухэлементным алфавитом.

1. Introduction. Let $S=(S;\cdot)$ be a semigroup. A semigroup $(S;\circ)$ is called an *interassociate* of S if for all $x,y,z\in S$ the following equalities are held:

$$x\cdot(y\circ z)=(x\cdot y)\circ z, \tag{1}$$
$$x\circ(y\cdot z)=(x\circ y)\cdot z. \tag{2}$$

We denote the set of all interassociates of S by $\mathrm{Int}(S)$.

The term *interassociativity* was introduced by D. Zupnik ([11]) in 1971. In 2004, interassociates of monogenic semigroups were fully described by M. Gould, K. A. Linton and A. W. Nelson ([3]). In the paper of B. N. Givens, K. A. Linton, A. Rosin and L. Dishman ([1]), for the free commutative semigroup with n generators it was shown when two interassociates of it are isomorphic and their general form was obtained. For the case of the infinitely generated free commutative semigroup the same problem was solved in [2].

Given any semigroup S, fix some $a\in S$ and define a *sandwich operation* $*_a$ on S by $x*_a y=xay$ $(x,y\in S)$. Clearly, $(S;*_a)$ is a semigroup, it is called a *variant* of S.

In 1967, K. D. Magill ([8]) considered variants of semigroups of relations. The general case was first studied by J. B. Hickey ([4]) in 1983. Variants of regular semigroups were considered in [5] and [6].

There is a natural connection between interassociates and variants of a semigroup. Obviously, every variant $(S;*_a)$ is an interassociate of S. One may easily check, that for any

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monoid M we have $\mathrm{Int}(M) = \{(M; *_a) : a \in M\}$. For the free commutative semigroup F it was shown that $\mathrm{Int}(F)$ consists of all variants of F and F itself (see [1] and [2]). Another reason to study interassociates of a semigroup is their connection with commutative dimonoids. Indeed, one can find that semigroups of any commutative dimonoid are interassociate to each other [9, 10].

In the present paper we study interassociates of the free semigroup with two generators using computer modeling and methods of the semigroup theory. This case is noncommutative, so it is different from the cases of free monogenic and free commutative semigroups. We have discovered that there are 18 types of such interassociates (not counting 12 dual cases) and identified all of them.

2. Notation and preliminaries. For any nonempty set X, by X^+ we denote the free semigroup over the alphabet X, and X^* denotes the free monoid over X with an empty word θ.

Let $x, y, w_1, w_2, \ldots, w_n \in X$ and $w = w_1 \ldots w_n$, we shall use the following notation:
$|w| = n$ — *length* of w;
$w_{(0)} = w_1$ and $w_{(1)} = w_n$ — the first and the last letters of w, respectively;
$w_l \in X^*$ and $w_r \in X^*$ — words obtained from w by deleting the last and the first letters respectively;
$X_{xy} = xX^* \cap X^*y$ — the set of all words $v \in X^+$ such that $v_{(0)} = x$ and $v_{(1)} = y$.

Later we shall need the following lemma (see [7, p. 338]).

Lemma 1. *Let $u, v, w \in X^+$, then $vu = uw$ if and only if $v = ab$, $w = ba$ and $u = a(ba)^k$ for some $a \in X^+$, $b \in X^*$ and $k \in \mathbb{N}^0$.*

The next proposition is the starting point of our research, it gives us a key to construct interassociates of the free semigroup.

Proposition 1. *For any binary operation $\circ \colon X^+ \times X^+ \to X^+$, groupoid $(X^+; \circ)$ is an interassociate of X^+ if and only if the following conditions are satisfied:*

(i) $\forall x, y, z \in X \colon (x \circ y) \circ z = x \circ (y \circ z)$;

(ii) $\forall v, w \in X^+ \colon v \circ w = v_l(v_{(1)} \circ w_{(0)})w_r$.

Proof. Necessity. Let $(X^+; \circ) \in \mathrm{Int}(X^+)$, then (i) holds since $(X^+; \circ)$ is a semigroup. For all $v, w \in X^+$, by (1) and (2), we have $v \circ w = v_l(v_{(1)} \circ w) = v_l((v_{(1)} \circ w_{(0)})w_r) = v_l(v_{(1)} \circ w_{(0)})w_r$.
Sufficiency. Assume that (i) and (ii) are held. For all $u, v, w \in X^+$,

$$(uv) \circ w = (uv)_l((uv)_{(1)} \circ w_{(0)})w_r = uv_l(v_{(1)} \circ w_{(0)})w_r = u(v \circ w)$$

and we may dually check that $u \circ (vw) = (u \circ v)w$. Thus identities of an interassociativity are satisfied. Now let us prove an associativity. Consider the case $v \in X$

$$u \circ (v \circ w) = u \circ (v \circ w_{(0)})w_r = u_l(u_{(1)} \circ (v \circ w_{(0)}))w_r = u_l((u_{(1)} \circ v) \circ w_{(0)})w_r = \\ = u_l(u_{(1)} \circ v) \circ w = (u \circ v) \circ w.$$

Now let $v \in X^+ \backslash X$, notice that the following equalities are obvious $(v_l)_r = (v_r)_l$, $(v_l)_{(0)} = v_{(0)}$, $(v_r)_{(1)} = v_{(1)}$.

Then, by (i) and (ii), we have

$$u \circ (v \circ w) = u \circ v_l(v_{(1)} \circ w_{(0)})w_r = u_l(u_{(1)} \circ (v_l)_{(0)})(v_l)_r(v_{(1)} \circ w_{(0)})w_r = \\ = u_l(u_{(1)} \circ v_{(0)})(v_r)_l((v_r)_{(1)} \circ w_{(0)})w_r = u_l(u_{(1)} \circ v_{(0)})v_r \circ w = (u \circ v) \circ w.$$

Therefore, (i) and (ii) imply that $(X^+; \circ)$ is a semigroup and $(X^+; \circ) \in \mathrm{Int}(X^+)$. $\square$

Now we fix the set $B = \{0, 1\}$ and provide the following definitions.

Definition 1. Let $x, w_1, w_2 \dots w_n \in B$ and $w = w_1 \dots w_n$. Define $\bar{x}$ by $\bar{0} = 1$ and $\bar{1} = 0$. The word $\bar{w}$ is defined by $\bar{w} = \bar{w}_1 \dots \bar{w}_n$.

Definition 2. Given $I = (B^+; \circ) \in \mathrm{Int}(B^+)$, define a *dual interassociate* $\bar{I} = (B^+; *)$ of I by putting $x * y = \overline{(\bar{x} \circ \bar{y})}$ for all $x, y \in B$. It is a simple permutation of 0 and 1, hence it is clear that the assumptions of Proposition 1 hold. Thus $\bar{I} \in \mathrm{Int}(B^+)$.

Proposition 1 (ii) implies that any interassociate $I = (B^+; \circ)$ of B^+ is defined by the following collection of words (we shall always use such a notation): $\omega = 0 \circ 0, \alpha = 0 \circ 1, \beta = 1 \circ 0, \varepsilon = 1 \circ 1$ that is why we may represent semigroups I and $\bar{I}$ by their *reduced Cayley tables* (see Table 1).

Table 1.

$\boldsymbol{I}$	**0**	**1**	$\boldsymbol{\bar{I}}$	**0**	**1**
0	ω	α	**0**	$\overline{\varepsilon}$	$\overline{\beta}$
1	β	ε	**1**	$\overline{\alpha}$	$\overline{\omega}$

3. The main result. To find all $I \in \mathrm{Int}\, B^+$ we shall consider the cases where $\varepsilon \in B_{a_1a_2}$, $\beta \in B_{a_3a_4}$, $\alpha \in B_{a_5a_6}, \omega \in B_{a_7a_8}$ for all combinations of elements $a_1, a_2, \dots, a_8 \in B$.

For each case we should check equations $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in B$.

There are $2^8 = 256$ such cases and each one contains 8 equations to solve, so we used computer modeling in Lazarus v.1.0.12 to reduce their number. We removed 120 cases which correspond to the dual interassociates of remaining 136. Then we removed 121 cases which contain at least one of the following contradictory equations $u1v = u0w, v1u = w0u$, $1v = 0w, v1 = w0$, for some variables $u, v, w \in B^*$.

We have also deleted trivial equations $v = v$. And, finally, we have deleted equations $vu = wu$ if there were equivalent ones of the form $vu' = wu'$ or $u'v = u'w$.

Example 1. For the case $\varepsilon \in B_{00}, \beta \in B_{11}, \alpha \in B_{01}, \omega \in B_{11}$, consider the following equation $(0 \circ 1) \circ 0 = 0 \circ (1 \circ 0) \Rightarrow \alpha_l(1 \circ 0) = (0 \circ 1)\beta_r \Rightarrow \alpha_l 1 \beta_r = \alpha_l 1 \beta_r$ hence it is trivial. We also have $(0 \circ 0) \circ 1 = 0 \circ (0 \circ 1) \Rightarrow \omega_l(1 \circ 1) = (0 \circ 0)\alpha_r \Rightarrow \omega_l 0 \varepsilon_r = \omega_l 1 \alpha_r$ a contradiction. Therefore, there are no interassociates of B^+ such that $\varepsilon \in B_{00}$, $\beta \in B_{11}$, $\alpha \in B_{01}$, $\omega \in B_{11}$.

There are 15 remaining cases and we shall consider all of them.

Case 1. $\varepsilon \in B_{00},\ \beta \in B_{00},\ \alpha \in B_{00},\ \omega \in B_{00}$:

1) $(0 \circ 1) \circ 0 = \alpha_l 0 \omega_r = \omega_l 0 \beta_r$, 3) $(1 \circ 1) \circ 0 = \varepsilon_l 0 \omega_r = \beta_l 0 \beta_r$,
2) $(0 \circ 1) \circ 1 = \alpha_l 0 \alpha_r = \omega_l 0 \varepsilon_r$, 4) $(1 \circ 1) \circ 1 = \varepsilon_l 0 \alpha_r = \beta_l 0 \varepsilon_r$.

Equality 1) implies that $\alpha_l \omega = \omega \beta_r$. Suppose that $\alpha_l = \theta$, then we have $\alpha = \beta = 0$. And from 2) it follows that $\omega = \varepsilon = 0$. Therefore, $(B; \circ)$ is the 2-element semigroup with zero multiplication. Denote $(B^+; \circ)$ by IO.

Now suppose that $\alpha_l \neq \theta$. By Lemma 1, equalities 1) and 4) imply that

$$\omega = a(ba)^k, \varepsilon = c(dc)^n, \alpha_l = ab, \alpha_r = dc, \beta_r = ba, \beta_l = cd \tag{3}$$

for some $a, c \in B^+$, $b, d \in B^*$ and $k, n \in \mathbb{N}^0$. Hence

$$\alpha = ab0 = 0dc,\ \beta = 0ba = cd0. \tag{4}$$

From 2) and equalities (3),(4) it follows that $abab0 = \omega\varepsilon_r = a(ba)^k\varepsilon_r \Rightarrow |abab0| \geq |a(ba)^k|$ and hence $k \leq 2$.

1. For $k = 0$ we have $\omega = a \in B_{00}$. By equality 2) we have $abab0 = a\varepsilon_r \Rightarrow \varepsilon_r = bab0 \Rightarrow \varepsilon = 0bab0$. From (3),(4) and equality 3) it follows that $\varepsilon_l\omega = \beta\beta_r \Rightarrow \varepsilon_l a = cd0ba \Rightarrow \varepsilon = (cd)^n c = cd0b0$.

   Hence $n \neq 0$. Also, 3) implies that $(cd)^n c\omega_r = cdcd0$, thus $n \leq 2$.

   If $n = 1$, then $c = 0b0$. By (4) we have $ab0 = 0d0b0$ and $\omega = a = 0d0$. Hence

$$\varepsilon = 0b0d0b0,\ \beta = 0b0d0,\ \alpha = 0d0b0,\ \omega = 0d0. \tag{5}$$

   Now, if $n = 2$, then the equality $cdcdc = cd0b0$ implies that $cdc = 0b0$. Since $|b| < |ba| = |dc| < |cdc| = |b| + 2 \Rightarrow |dc| = |b| + 1$ we have $dc = b0$ and so $c = 0$. From equality 3) we obtain $\omega_r = \theta$. Finally, we have $\varepsilon = 0d0d0, \beta = 0d0, \alpha = 0d0, \omega = 0$.

   We may combine this result with (5) as follows: $\varepsilon = 0yx_{00}y0, \beta = 0yx_{00},\ \alpha = x_{00}y0, \omega = x_{00}$, where $x_{00} \in B_{00}, y \in B^*$.

   Denote the corresponding interassociate by IO_1.

2. The case where $k = 2$ is dual to $k = 0$:

$$\varepsilon = x_{00},\ \beta = x_{00}y0,\ \alpha = 0yx_{00},\ \omega = 0yx_{00}y0, \tag{6}$$

   where $x_{00} \in B_{00}, y \in B^*$. Denote $(B^+; \circ)$ by IO_3.

3. If $k = 1$, then $\omega = aba$. Equality 2) implies that $abab0 = aba\varepsilon_r \ \Rightarrow\ \varepsilon_r = b0 \ \Rightarrow\ \varepsilon = 0b0$.

   If $n = 0$, then, dually to the case where $k = 0, n = 1$, we obtain $\varepsilon = 0b0,\ \beta = 0b0d0$, $\alpha = 0d0b0,\ \omega = 0d0b0d0$, and this result satisfies (6).

   If $n > 0$, then by (3) we have $c(dc)^n = \varepsilon = 0b0$ and hence $|b| < |ba| = |dc| < |c(dc)^n| = |b| + 2 \Rightarrow |dc| = |b| + 1$.

   Further, $c(dc)^n = 0b0$ implies that $dc = b0$ and $n = 1$. Thus $c = 0,\ d = b$ and from 2) and (4) it follows that $cdc\omega_r = cdcd0 \ \Rightarrow\ 0b0\omega_r = 0b0b0 \ \Rightarrow\ \omega = 0b0$.

   We may conclude that $\varepsilon = 0b0, \beta = 0b0, \alpha = 0b0, \omega = 0b0$. Denote $(B^+; \circ)$ by IO_2.

Case 2. $\varepsilon \in B_{00},\ \beta \in B_{01},\ \alpha \in B_{10},\ \omega \in B_{11}$:

1) $(0 \circ 0) \circ 0 = \omega_l 0\beta_r = \alpha_l 0\omega_r$,	5) $(1 \circ 0) \circ 0 = \beta_l 0\beta_r = \varepsilon_l 0\omega_r$,
2) $(0 \circ 0) \circ 1 = \omega_l 0\varepsilon_r = \alpha_l 0\alpha_r$,	6) $(1 \circ 0) \circ 1 = \beta_l 0\varepsilon_r = \varepsilon_l 0\alpha_r$,
3) $(0 \circ 1) \circ 0 = \alpha_l 1\omega_r = \omega_l 1\beta_r$,	7) $(1 \circ 1) \circ 0 = \varepsilon_l 1\omega_r = \beta_l 1\beta_r$,
4) $(0 \circ 1) \circ 1 = \alpha_l 1\alpha_r = \omega_l 1\varepsilon_r$,	8) $(1 \circ 1) \circ 1 = \varepsilon_l 1\alpha_r = \beta_l 1\varepsilon_r$.

Equalities 1) and 3) imply that $\alpha_l = \omega_l$ and $\beta_r = \omega_r$, because we have $0 = 1$ otherwise. Similarly, from 2) and 4) it follows that $\alpha_r = \varepsilon_r$. Therefore, $\varepsilon = 0x0, \beta = 0x1, \alpha = 1x0, \omega = 1x1$, where $x \in B^*$. From equalities 5)–8) the same assertions follow. Denote $(B^+; \circ)$ by I_1. Notice that the interassociate I_1 is self-dual, i.e. $\bar{I}_1 = I_1$.

Case 3. $\varepsilon \in B_{00},\ \beta \in B_{11},\ \alpha \in B_{11},\ \omega \in B_{00}$:

$$1)\ (0 \circ 0) \circ 1 = \omega_l 1 \alpha_r = \alpha_l 1 \alpha_r \ \Rightarrow\ \omega_l = \alpha_l,\ 2)\ (1 \circ 0) \circ 0 = \beta_l 1 \beta_r = \beta_l 1 \omega_r \ \Rightarrow\ \beta_r = \omega_r,$$
$$3)\ (1 \circ 1) \circ 1 = \varepsilon_l 1 \alpha_r = \beta_l 1 \varepsilon_r.$$

We have $\omega_l = \alpha_l = \theta$, because $\omega_{(0)} \neq \alpha_{(0)}$. Hence $\omega = 0$ and $\alpha = 1$. And $\beta_r = \omega_r$ implies that $\beta = 1$. Thus 3) implies that $\varepsilon_l 1 = 1\varepsilon_r$ and hence $\varepsilon = 0$. Notice, that $(B;\circ)$ is isomorphic to the symmetric group S_2. So we denote $(B^+;\circ)$ by IS.

Case 4. $\varepsilon \in B_{01},\ \beta \in B_{00},\ \alpha \in B_{01},\ \omega \in B_{00}$: $(0 \circ 1) \circ 0 = \alpha_l 0 \beta_r = \omega_l 0 \beta_r \ \Rightarrow\ \alpha_l = \omega_l$, $(1 \circ 1) \circ 0 = \varepsilon_l 0 \beta_r = \beta_l 0 \beta_r \ \Rightarrow\ \varepsilon_l = \beta_l$. Hence we have $\varepsilon = 0x1, \beta = 0x0, \alpha = 0y1, \omega = 0y0$, where $x, y \in B^*$. Denote $(B^+;\circ)$ by I_2.

Case 5. $\varepsilon \in B_{01},\ \beta \in B_{00},\ \alpha \in B_{11},\ \omega \in B_{00}$: $(0 \circ 0) \circ 1 = \omega_l 1 \alpha_r = \alpha_l 1 \alpha_r \ \Rightarrow\ \omega_l = \alpha_l$, $(1 \circ 0) \circ 1 = \beta_l 1 \alpha_r = \varepsilon_l 1 \alpha_r \ \Rightarrow\ \beta_l = \varepsilon_l$. We have $\omega_l = \alpha_l = \theta$ because $\omega_{(0)} \neq \alpha_{(0)}$. Thus $\omega = 0$ and $\alpha = 1$. The equality $\varepsilon_r = \beta_l$ implies that $\varepsilon = 0x1$ and $\beta = 0x0$ for $x \in B^*$. Denote $(B^+;\circ)$ by I_3.

Case 6. $\varepsilon \in B_{01},\ \beta \in B_{00},\ \alpha \in B_{11},\ \omega \in B_{10}$: $(0 \circ 0) \circ 0 = \omega_l 1 \omega_r = \alpha_l 1 \omega_r \ \Rightarrow\ \omega_l = \alpha_l$, $(1 \circ 0) \circ 0 = \beta_l 1 \omega_r = \varepsilon_l 1 \omega_r \ \Rightarrow\ \beta_l = \varepsilon_l$. Therefore, $\varepsilon = 0x1, \beta = 0x0, \alpha = 1y1, \omega = 1y0$, where $x, y \in B^*$. Denote $(B^+;\circ)$ by I_4. Such an interassociate is self-dual.

Case 7. $\varepsilon \in B_{10},\ \beta \in B_{10},\ \alpha \in B_{00},\ \omega \in B_{00}$: $(0 \circ 1) \circ 0 = \alpha_l 0 \omega_r = \alpha_l 0 \beta_r \ \Rightarrow\ \omega_r = \beta_r$, $(0 \circ 1) \circ 1 = \alpha_l 0 \alpha_r = \alpha_l 0 \varepsilon_r \ \Rightarrow\ \alpha_r = \varepsilon_r$. Again, we have a similar situation as in Case 6: $\varepsilon = 1x0, \beta = 1y0, \alpha = 0x0, \omega = 0y0$, where $x, y \in B^*$. Denote $(B^+;\circ)$ by I_5.

Case 8. $\varepsilon \in B_{10},\ \beta \in B_{11},\ \alpha \in B_{00},\ \omega \in B_{00}$: $(0 \circ 1) \circ 0 = \alpha_l 0 \omega_r = \alpha_l 0 \beta_r \ \Rightarrow\ \omega_r = \beta_r$, $(0 \circ 1) \circ 1 = \alpha_l 0 \alpha_r = \alpha_l 0 \varepsilon_r \ \Rightarrow\ \alpha_r = \varepsilon_r$. We have $\omega_r = \beta_r = \theta$ since $\omega_{(1)} \neq \beta_{(1)}$. Hence $\omega = 0$ and $\beta = 1$. And by $\alpha_r = \varepsilon_r$ we obtain $\varepsilon = 1x0$ and $\alpha = 0x0$ for $x \in B^*$. Denote $(B^+;\circ)$ by I_6.

Case 9. $\varepsilon \in B_{10},\ \beta \in B_{11},\ \alpha \in B_{00},\ \omega \in B_{01}$: $(0 \circ 0) \circ 0 = \omega_l 1 \beta_r = \omega_l 1 \omega_r \ \Rightarrow\ \beta_r = \omega_r$, $(0 \circ 0) \circ 1 = \omega_l 1 \varepsilon_r = \omega_l 1 \alpha_r \ \Rightarrow\ \varepsilon_r = \alpha_r$. Previous equalities imply that $\varepsilon = 1x0, \beta = 1y1, \alpha = 0x0, \omega = 0y1$, where $x, y \in B^*$. Denote $(B^+;\circ)$ by I_7, it is self-dual.

Case 10. $\varepsilon \in B_{11},\ \beta \in B_{00},\ \alpha \in B_{00},\ \omega \in B_{00}$:

$$1)\ (0 \circ 1) \circ 0 = \alpha_l 0 \omega_r = \omega_l 0 \beta_r,$$
$$2)\ (0 \circ 1) \circ 1 = \alpha_l 0 \alpha_r = \alpha_l 0 \varepsilon_r \ \Rightarrow\ \alpha_r = \varepsilon_r,$$
$$3)\ (1 \circ 1) \circ 0 = \varepsilon_l 0 \beta_r = \beta_l 0 \beta_r \ \Rightarrow\ \varepsilon_l = \beta_l.$$

Similarly to the Case 3, equalities 2) and 3) imply that $\varepsilon = 1,\ \beta = 0$ and $\alpha = 0$. Hence equality 1) is equivalent to $\omega = \omega$ and so ω is an arbitrary word from B_{00}. If $\omega = 0$, then $(B;\circ)$ becomes a semigroup with the usual multiplication on the set of integers $\{0, 1\}$. Denote $(B^+;\circ)$ by IM.

Case 11. $\varepsilon \in B_{11},\ \beta \in B_{00},\ \alpha \in B_{01},\ \omega \in B_{00}$: $(0 \circ 1) \circ 0 = \alpha_l 0 \beta_r = \omega_l 0 \beta_r \ \Rightarrow\ \alpha_l = \omega_l$, $(1 \circ 1) \circ 0 = \varepsilon_l 0 \beta_r = \beta_l 0 \beta_r \ \Rightarrow\ \varepsilon_l = \beta_l$.

Hence we obtain $\varepsilon = 1, \beta = 0, \alpha = 0x1, \omega = 0x0$, where $x \in B^*$. Denote $(B^+;\circ)$ by I_8.

Case 12. $\varepsilon \in B_{11},\ \beta \in B_{00},\ \alpha \in B_{11},\ \omega \in B_{00}$: $(0 \circ 0) \circ 1 = \omega_l 1 \alpha_r = \alpha_l 1 \alpha_r \ \Rightarrow\ \omega_l = \alpha_l$, $(1 \circ 0) \circ 1 = \beta_l 1 \alpha_r = \varepsilon_l 1 \alpha_r \ \Rightarrow\ \beta_l = \varepsilon_l$. Since $\omega_{(0)} \neq \alpha_{(0)}$ and $\beta_{(0)} \neq \varepsilon_{(0)}$ we have $\varepsilon = 1,\ \beta = 0,\ \alpha = 1,\ \omega = 0$, hence $(B; \circ)$ is the right zero semigroup and we denote $(B^+; \circ)$ by IR. Notice that the interassociate IR is self-dual.

Case 13. $\varepsilon \in B_{11},\ \beta \in B_{10},\ \alpha \in B_{00},\ \omega \in B_{00}$: $(0 \circ 1) \circ 0 = \alpha_l 0 \omega_r = \alpha_l 0 \beta_r \ \Rightarrow\ \omega_r = \beta_r$, $(0 \circ 1) \circ 1 = \alpha_l 0 \alpha_r = \alpha_l 0 \varepsilon_r \ \Rightarrow\ \alpha_r = \varepsilon_r$. As $\varepsilon_{(0)} \neq \alpha_{(0)}$ and $\omega_r = \beta_r$ we have $\varepsilon = 1$, $\beta = 1x0$, $\alpha = 0$, $\omega = 0x0$. We denote $(B^+; \circ)$ by I_9.

Case 14. $\varepsilon \in B_{11},\ \beta \in B_{10},\ \alpha \in B_{01},\ \omega \in B_{00}$. For this case all eight equations are trivial. Thus, ε, β, α and ω are arbitrary words from B_{11}, B_{10}, B_{01} and B_{00} respectively. Denote $(B^+; \circ)$ by I_{10}, this interassociate is self-dual.

Case 15. $\varepsilon \in B_{11},\ \beta \in B_{11},\ \alpha \in B_{00},\ \omega \in B_{00}$: $(0 \circ 1) \circ 0 = \alpha_l 0 \omega_r = \alpha_l 0 \beta_r \ \Rightarrow\ \omega_r = \beta_r$, $(0 \circ 1) \circ 1 = \alpha_l 0 \alpha_r = \alpha_l 0 \varepsilon_r \ \Rightarrow\ \alpha_r = \varepsilon_r$. Analogously to the Case 12 we have $\varepsilon = 1$, $\beta = 1$, $\alpha = 0$, $\omega = 0$, thus $(B; \circ)$ is the left zero semigroup and we denote $(B^+; \circ)$ by IL. The interassociate IL is self-dual.

Finally, we have reached the goal of our research.

Theorem 1. *If $I \in \mathrm{Int}(B^+)$, then the reduced Cayley table of I or $\bar{I}$ is contained in Table 2. Conversely, for any $x, y \in B^*$, $z \in B_{00}$ and $x_{ij} \in B_{ij}$ $(i, j \in B)$, each reduced Cayley table from Table 2 defines an interassociate of B^+.*

Table 2. All interassociates of B^+ ($x, y \in B^*$, $z \in B_{00}$, $x_{ij} \in B_{ij}$).

$\boldsymbol{IO}$	**0**	**1**	$\boldsymbol{IO}_1$	**0**	**1**	$\boldsymbol{IS}$	**0**	**1**	$\boldsymbol{IM}$	**0**	**1**
0	0	0	**0**	z	$zx0$	**0**	0	1	**0**	z	0
1	0	0	**1**	$0xz$	$0xzx0$	**1**	1	0	**1**	0	1
$\boldsymbol{IO}_2$	**0**	**1**	$\boldsymbol{IO}_3$	**0**	**1**	$\boldsymbol{IR}$	**0**	**1**	$\boldsymbol{IL}$	**0**	**1**
0	$0x0$	$0x0$	**0**	$0xzx0$	$0xz$	**0**	0	1	**0**	0	0
1	$0x0$	$0x0$	**1**	$zx0$	z	**1**	0	1	**1**	1	1
$\boldsymbol{I}_1$	**0**	**1**	$\boldsymbol{I}_2$	**0**	**1**	$\boldsymbol{I}_3$	**0**	**1**	$\boldsymbol{I}_4$	**0**	**1**
0	$1x1$	$1x0$	**0**	$0x0$	$0x1$	**0**	0	1	**0**	$1x0$	$1x1$
1	$0x1$	$0x0$	**1**	$0y0$	$0y1$	**1**	$0x0$	$0x1$	**1**	$0y0$	$0y1$
$\boldsymbol{I}_5$	**0**	**1**	$\boldsymbol{I}_6$	**0**	**1**	$\boldsymbol{I}_7$	**0**	**1**	$\boldsymbol{I}_8$	**0**	**1**
0	$0x0$	$0y0$	**0**	0	$0x0$	**0**	$0x1$	$0y0$	**0**	$0x0$	$0x1$
1	$1x0$	$1y0$	**1**	1	$1x0$	**1**	$1x1$	$1y0$	**1**	0	1
			$\boldsymbol{I}_9$	**0**	**1**	$\boldsymbol{I}_{10}$	**0**	**1**			
			0	$0x0$	0	**0**	x_{00}	x_{01}			
			1	$1x0$	1	**1**	x_{10}	x_{11}			

Examine Table 2, the connection between variants and interassociates of B^+ is entirely clear now. Indeed, the operations in Table 2 are very similar to the sandwich operation on B^+. Another task is to describe all interassociates of the free semigroup with $n > 2$ generators. As we have seen, every semigroup over X gives a variant which is an interassociate of X^+. It is well known that the number of semigroups over an n-element set grows extremely fast, whence it is impractical to build the set $\mathrm{Int}(X^+)$ by considering of all operations as we did for B^+. But it is possible to study some special interassociates of X^+ like the variants of X^+, interassociates $(X^+, \circ) \in \mathrm{Int}(X^+)$ such that $(X, \circ)$ is a semigroup and so forth.

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