УДК 517.5

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PROPERTIES OF THE SOLUTIONS OF THE GAUSS EQUATION

M. M. Sheremeta, Yu. S. Trukhan. *Properties of the solutions of the Gauss equation*, Mat. Stud. **41** (2014), 157–167.

It is proved that the Gauss equation $z(z-1)w'' + ((\alpha+\beta+1)z-\gamma)w' + \alpha\beta w = 0$ apart from the hypergeometric function has a power solution with negative exponents. Its possible growth, close-to-convexity and l-index boundedness are investigated.

М. Н. Шеремета, Ю. С. Трухан. *Свойства решений уравнения Гаусса* // Мат. Студії. – 2014. – Т.41, №2. – C.157–167.

Доказано, что уравнение Гаусса $z(z-1)w'' + ((\alpha+\beta+1)z-\gamma)w' + \alpha\beta w = 0$ кроме гипергеометрической функции имеет степенное решение с отрицательными показателями, исследованы его возможный рост, близость к выпуклости и ограниченность l-индекса.

1. Introduction. An analytic univalent function in $\mathbb{D} = \{z\colon |z| < 1\}$ of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. The condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for convexity of f ([1, p. 203]). A function f is said to be close-to-convex in $\mathbb{D}$ ([1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$ $(z \in \mathbb{D})$. For a close-to-convex function f exterior G of the domain $f(\mathbb{D})$ can be filled by rays L starting from ∂G and lying in G. Every close-to-convex function is univalent in $\mathbb{D}$, and therefore $a_1 \neq 0$.

For a positive continuous on $[0, 1)$ function l such that $l(r) > \beta/(1-r)$ for all $r \in [0, 1)$ and for some $\beta > 0$ an analytic in $\mathbb{D}$ function f is said to be of bounded l-index ([2, p. 7]), if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}$

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k! l^k(|z|)}\colon 0 \le k \le N\right\}. \tag{2}$$

The least such integer N is called the l-index and is denoted by $N(f, l)$. If $G \subset \mathbb{D}$ and there exists $N \in \mathbb{Z}_+$ such that inequality (2) holds for all $n \in \mathbb{Z}_+$ and $z \in G$ then the function f is said to be of bounded l-index on (or in) G, and the l-index is denoted by $N(f, l; G)$.

The function

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{k=1}^{\infty} A_k z^k = 1 + \sum_{k=1}^{\infty}\left(\prod_{j=0}^{k-1}\frac{(j+\alpha)(j+\beta)}{(j+1)(j+\gamma)}\right) z^k, \tag{3}$$

2010 *Mathematics Subject Classification:* 30C45.
Keywords: Gauss equation; hypergeometric function; close-to-convexity; l-index boundedness.

where $\gamma \notin \{0, -1, -2, \ldots\}$, is said to be hypergeometric ([3, p. 62–64]). The radius of convergence of the series in the right-hand side of (3) is 1, thus $F(\alpha, \beta, \gamma; z)$ is analytic in $\mathbb{D}$ and is a solution of Gauss equation ([3, p. 62])

$$z(z-1)w'' + ((\alpha+\beta+1)z - \gamma)w' + \alpha\beta w = 0. \tag{4}$$

We remark that $F(1, \beta, \beta; z) = 1/(1-z)$ is the sum of a geometric progression.

For positive parameters α, β, γ the following theorems are proved in [4].

Theorem A. *The hypergeometric function $F(\alpha, \beta, \gamma; z)$ has the following properties: 1) if $\gamma \ge \alpha + \beta + \alpha\beta$ then F is bounded and close-to-convex in $\mathbb{D}$; 2) if $\gamma = \alpha + \beta \ge \alpha\beta$ then $M_F(r) \asymp \ln \frac{1}{1-r}$ $(r \uparrow 1)$, where $M_f(r) = \max\{|f(z)|\colon |z| = r\}$; 3) if $\gamma < \alpha + \beta \le \alpha\beta$ then $M_F(r) \asymp \frac{1}{(1-r)^{\alpha+\beta-\gamma}}$ as $r \uparrow 1$.*

Theorem B. *If $\max\{\alpha\beta, \alpha + \beta - 1\} \le \gamma$ then $N(F, l) \le 1$ with $l(r) = \frac{9(\gamma+1)}{1-r}$, and if $\max\{\gamma, \alpha + \beta - 1\} \le \alpha\beta$ then $N(F, l) \le 1$ with $l(r) = \frac{18\alpha\beta(\alpha\beta+1)}{1-r}$.*

Investigation of the existence of other solutions of equation (4) and their properties is the purpose of this paper.

2. Meromorphic starlike and meromorphic convex solutions. Let Σ denote the class of functions of the form

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} g_n z^n, \tag{5}$$

analytic in $\mathbb{D}_0 = \{z\colon 0 < |z| < 1\}$. A function $g \in \Sigma$ is said to be meromorphic starlike if $\mathrm{Re}\{-zg'(z)/g(z)\} > 0$ for all $z \in \mathbb{D}_0$, and is said to be meromorphic convex if $\mathrm{Re}\{-(1 + zg''(z)/g'(z))\} > 0$ for all $z \in \mathbb{D}_0$. O. Juneja and T. Reddy ([5]) and M. Mogra ([6]) have proved that if $g_n \ge 0$ for all $n \ge 1$ then the condition $\sum_{n=1}^{\infty} n g_n \le 1$ is necessary and sufficient for the function defined by (5) to be meromorphic starlike, and the condition $\sum_{n=1}^{\infty} n^2 g_n \le 1$ is necessary and sufficient for the same function (5) to be meromorphic convex.

Gauss equation (4) does not have nontrivial meromorphic starlike and meromorphic convex solutions. It follows from the next statement.

Proposition 1. *If a function $g \in \Sigma$ is a solution of the Gauss equation then $\gamma = 2$, $1 + \alpha\beta = \alpha + \beta$ and $g_k = 0$ for all $k \ge 1$ and the solution $g(z) = 1/z$ is meromorphic starlike and meromorphic convex.*

Proof. For the function defined by (5) one has

$$g'(z) = -\frac{1}{z^2} + \sum_{n=1}^{\infty} n g_n z^{n-1}, \quad g''(z) = \frac{2}{z^3} + \sum_{n=2}^{\infty} n(n-1) g_n z^{n-2},$$

so (4) implies

$$\frac{2}{z} + \sum_{n=2}^{\infty} n(n-1) g_n z^n - \frac{2}{z^2} - \sum_{n=2}^{\infty} n(n-1) g_n z^{n-1} - \frac{\alpha+\beta+1}{z} + (\alpha+\beta+1)\sum_{n=1}^{\infty} n g_n z^n +$$
$$+ \frac{\gamma}{z^2} - \gamma \sum_{n=1}^{\infty} n g_n z^{n-1} + \frac{\alpha\beta}{z} + \alpha\beta \sum_{n=1}^{\infty} g_n z^n \equiv 0,$$

that is

$$\frac{\gamma-2}{z^2}+\frac{1+\alpha\beta-\alpha-\beta}{z}+\sum_{n=2}^{\infty}n(n-1)g_nz^n+\sum_{n=1}^{\infty}(\alpha+\beta+1)ng_nz^n+\sum_{n=1}^{\infty}\alpha\beta g_nz^n-$$
$$-\sum_{n=1}^{\infty}n(n+1)g_{n+1}z^n-\gamma\sum_{n=0}^{\infty}(n+1)g_{n+1}z^n\equiv 0.$$

Equating here the coefficients at identical powers of z, we get $\gamma=2$, $1+\alpha\beta=\alpha+\beta$, $\gamma g_1=0$, $(\alpha+\beta+1+\alpha\beta)g_1=(2+2\gamma)g_2$ and for $n\geq 2$ $(n(n+\alpha+\beta)+\alpha\beta)g_n=(n+1)(n+\gamma)g_{n+1}$, and it is easy to see that $0=g_1=g_2=g_3=\ldots=g_n=\ldots$. Since the equalities $\operatorname{Re}\{-zg'(z)/g(z)\}=\operatorname{Re}\{-(1+zg''(z)/g'(z))\}=1$ hold for the function $g(z)=1/z$. □

We remark that for the function $g(z)=1/z^n$ $(n\in\mathbb{N})$ equalities $\operatorname{Re}\{-zg'(z)/g(z)\}=$ $=\operatorname{Re}\{-(1+zg''(z)/g'(z))\}=n$ hold. It is easy to see that this function is a solution of Gauss equation if and only if $\gamma=n+1$, $\alpha\beta=n(\alpha+\beta-n)$. For $n\geq 2$ it does not belong to the class Σ, but satisfies conditions $\operatorname{Re}\{-zg'(z)/g(z)\}>0$ and $\operatorname{Re}\{-(1+zg''(z)/g'(z))\}>0$ for all $z\in\mathbb{D}_0$.

3. Series with negative powers. In view of the latter remark it is natural to ask under what assumptions the function

$$\varphi(z)=\frac{1}{z}+\sum_{k=2}^{\infty}\frac{b_k}{z^k}=\sum_{k=1}^{\infty}\frac{b_k}{z^k},\quad b_1=1, \tag{6}$$

is a solution of the Gauss equation and what are its properties. Since

$$\varphi'(z)=-\sum_{k=1}^{\infty}\frac{kb_k}{z^{k+1}},\quad \varphi''(z)=\sum_{k=1}^{\infty}\frac{k(k+1)b_k}{z^{k+2}},$$

putting φ in (4), we get

$$\sum_{k=1}^{\infty}\frac{k(k+1)b_k}{z^k}-\sum_{k=1}^{\infty}\frac{k(k+1)b_k}{z^{k+1}}-(\alpha+\beta+1)\sum_{k=1}^{\infty}\frac{kb_k}{z^k}+\gamma\sum_{k=1}^{\infty}\frac{kb_k}{z^{k+1}}+\alpha\beta\sum_{k=1}^{\infty}\frac{b_k}{z^k}\equiv 0,$$

that is

$$\sum_{k=1}^{\infty}\frac{(k(k-\alpha-\beta)+\alpha\beta)b_k}{z^k}-\sum_{k=2}^{\infty}\frac{(k-1)(k-\gamma)b_{k-1}}{z^k}\equiv 0,$$

so

$$1+\alpha\beta=\alpha+\beta, \tag{7}$$

and for $k\geq 2$ the equality $(k(k-\alpha-\beta)+\alpha\beta)b_k=(k-1)(k-\gamma)b_{k-1}$ holds. But $k(k-\alpha-$ $-\beta)+\alpha\beta=k^2-k(\alpha+\beta)+\alpha+\beta-1=(k-1)(k+1-\alpha-\beta)=(k-1)(k-\alpha\beta)$. Therefore, for $\alpha\beta\neq k$ and $\gamma\neq k$ and for all $k\geq 2$ we have the recurrent formula

$$b_k=\frac{k-\gamma}{k-\alpha\beta}b_{k-1},\quad k\geq 2. \tag{8}$$

We will consider a function of the form

$$F_1(z)=F_1(\alpha,\beta,\gamma;z)=\varphi(1/z)=z+\sum_{k=2}^{\infty}b_kz^k, \tag{9}$$

where coefficients b_k are defined by recurrent formula (8) and condition (7) holds. It is easy to see that the radius of convergence of the series in (9) equals 1. Assuming like in [4] that α, β, γ are positive, we shall prove the next statement.

Proposition 2. *If $\alpha+\beta=1+\alpha\beta\le\gamma<2$, then function $F_1(\alpha,\beta,\gamma;z)$ is close-to-convex.*

Indeed, in view of the condition $\gamma\ge 1+\alpha\beta$ the inequalities $2b_2=2\frac{2-\gamma}{2-\alpha\beta}\le 1$ and $kb_k = \frac{k(k-\gamma)}{(k-1)(k-\alpha\beta)}(k-1)b_{k-1}\le(k-1)b_{k-1}$ hold for $k\ge 3$, that is $1\ge 2b_2\ge 3b_3\ge\cdots\ge (k-1)b_{k-1}\ge kb_k\ge\cdots>0$ and using Alexander's criterion ([7, p. 10]) we see that the function $F_1(\alpha,\beta,\gamma;z)$ is close-to-convex.

Next we will study the possible growth of a function of form (9).

Proposition 3. *For the function $F_1(z)=F_1(\alpha,\beta,\gamma;z)$ with positive α, β, γ and $\gamma\notin\mathbb{N}\setminus\{1\}$, $\alpha\beta\notin\mathbb{N}\setminus\{1\}$ the following asymptotic equalities hold*

$$M_{F_1}(r)\asymp\begin{cases}1, & \alpha\beta-\gamma<-1;\\ \ln\frac{1}{1-r}, & \alpha\beta-\gamma=-1;\\ \frac{1}{(1-r)^{\alpha\beta-\gamma+1}}, & \alpha\beta-\gamma>-1,\end{cases}\quad r\uparrow 1.$$

Proof. Since $\frac{k-\gamma}{k-\alpha\beta}=1+\frac{\alpha\beta-\gamma}{k}+\frac{(\alpha\beta-\gamma)\alpha\beta}{k(k-\alpha\beta)}$, in view of (8)

$$\ln|b_k|=\sum_{j=2}^{k}\ln\left|1+\frac{\alpha\beta-\gamma}{j}+\frac{(\alpha\beta-\gamma)\alpha\beta}{j(j-\alpha\beta)}\right|=(\alpha\beta-\gamma)\ln k+O(1),\quad k\to+\infty,$$

so there exist constants $0<h\le H<+\infty$ such that

$$hk^{\alpha\beta-\gamma}\le|b_k|\le Hk^{\alpha\beta-\gamma},\quad k\ge 1. \tag{10}$$

Hence, if $\alpha\beta-\gamma<-1$ then $M_{F_1}(r)=O(1)$, and if $\alpha\beta-\gamma=-1$ then $M_{F_1}(r)\asymp\ln\frac{1}{1-r}$ as $r\uparrow 1$.

Finally, let $\alpha\beta-\gamma>-1$. We will consider the power series $F_1^*(r)=\sum_{k=1}^{\infty}k^{\alpha\beta-\gamma}r^k$. It is easy to show that

$$F_1^*(r)=\int_1^{\infty}x^{\alpha\beta-\gamma}e^{-x|\ln r|}dx+O(\exp\{\max\{(\alpha\beta-\gamma)\ln x-x|\ln r|\colon x\ge 1\}\}),\ r\uparrow 1.$$

But

$$\int_1^{\infty}x^{\alpha\beta-\gamma}e^{-x|\ln r|}dx=\left(\frac{1}{|\ln r|}\right)^{\alpha\beta-\gamma+1}\int_{|\ln r|}^{\infty}t^{\alpha\beta-\gamma}e^{-t}dt\sim\frac{\Gamma(\alpha\beta-\gamma+1)}{(1-r)^{\alpha\beta-\gamma+1}},\quad r\uparrow 1,$$

and

$$\max\{(\alpha\beta-\gamma)\ln x-x|\ln r|\colon x\ge 1\}=\begin{cases}-|\ln r|, & -1<\alpha\beta-\gamma\le 0;\\ (\alpha\beta-\gamma)\ln\frac{\alpha\beta-\gamma}{e|\ln r|}, & \alpha\beta-\gamma>0,\end{cases}$$

that is $\exp\{\max\{(\alpha\beta-\gamma)\ln x-x|\ln r|\colon x\ge 1\}\}=o(\frac{1}{(1-r)^{\alpha\beta-\gamma+1}})$ and $F_1^*(r)\sim\frac{\Gamma(\alpha\beta-\gamma+1)}{(1-r)^{\alpha\beta-\gamma+1}}$ as $r\uparrow 1$. Therefore, in view of (10), $M_{F_1}(r)\asymp\frac{1}{(1-r)^{\alpha\beta-\gamma+1}}$, $r\uparrow 1$. □

Uniting Propositions 2 and 3, we get the next analogue of Theorem A.

Theorem 1. *If $\gamma \notin \mathbb{N}\setminus\{1\}$, $\alpha\beta \notin \mathbb{N}\setminus\{1\}$ and $\alpha+\beta = 1+\alpha\beta$ then apart from the hypergeometric function $F(\alpha,\beta,\gamma;z)$ Gauss equation has a solution $w = F_1(\alpha,\beta,\gamma;1/z)$ with the following properties: 1) if $1+\alpha\beta < \gamma < 2$ then $F_1(z)$ is bounded and close-to-convex function; 2) if $\alpha\beta-\gamma = -1$ then $M_{F_1}(r) \asymp \ln\frac{1}{1-r}$ $(r\uparrow 1)$ and if $\gamma < 2$ then $F_1(z)$ is close-to-convex; 3) if $\alpha\beta-\gamma > -1$ then $M_{F_1}(r) \asymp \frac{1}{(1-r)^{\alpha\beta-\gamma+1}}$ as $r\uparrow 1$.*

4. l-index boundedness. We start with the following lemma.

Lemma. *Let a function defined by (1) be analytic in the closed disc $\overline{\mathbb{D}}_R = \{z\colon |z|\le R\}$, $j = \min\{n\ge 0\colon a_n\ne 0\}$ and*

$$\sum_{n=1}^{\infty}\frac{(n+j)!}{n!j!}\frac{|a_{n+j}|}{|a_j|}R^n \le a_j(R) < 1. \tag{11}$$

Then $N(f,l;\mathbb{D}_R) = j$ with $l(|z|) = \frac{K_j(R)}{R-|z|}$, where $K_j(R) = \max\left\{1, \frac{1+a_j(R)}{(1+j)(1-a_j(R))}\right\}$.

Proof. Since

$$f^{(j)}(z) = j!a_j + \sum_{n=j+1}^{\infty} n(n-1)\dots(n-j+1)a_n z^{n-j} = j!a_j\left(1+\sum_{n=1}^{\infty}\frac{(n+j)!}{n!j!}\frac{a_{n+j}}{a_j}z^n\right),$$

we obtain $N(f^{(j)},l;\mathbb{D}_R) = N(\psi,l;\mathbb{D}_R)$, where $\psi(z) = 1+\sum_{n=1}^{\infty}\frac{(n+j)!}{n!j!}\frac{a_{n+j}}{a_j}z^n$. Inequality (11) implies for all $z\in\overline{\mathbb{D}}_R$ $1-a_j(R)\le|\psi(z)|\le 1+a_j(R)$.

Therefore, for all $z\in\mathbb{D}_R$ and $k\in\mathbb{N}$

$$\frac{|\psi^{(k)}(z)|}{k!} = \left|\frac{1}{2\pi i}\int_{|\tau-z|=R-|z|}\frac{\psi(\tau)d\tau}{(\tau-z)^{k+1}}\right| \le \frac{\max\{|\psi(z)|\colon z\in\overline{\mathbb{D}}_R\}}{(R-|z|)^k} \le$$
$$\le \frac{1+a_j(R)}{(R-|z|)^k} \le \frac{1+a_j(R)}{1-a_j(R)}\frac{|\psi(z)|}{(R-|z|)^k}.$$

But $\psi^{(k)}(z) = f^{(k+j)}(z)/(j!a_j)$ for all $k\ge 0$. Therefore, we get

$$\frac{|f^{(k+j)}(z)|}{(k+j)!} = \frac{|\psi^{(k)}(z)|j!|a_j|}{(k+j)!} \le \frac{k!}{(k+j)!}\frac{1+a_j(R)}{1-a_j(R)}\frac{|\psi(z)|j!|a_j|}{(R-|z|)^k} = \frac{k!}{(k+j)!}\frac{1+a_j(R)}{1-a_j(R)}\frac{|f^{(j)}(z)|}{(R-|z|)^k},$$

that is for all $k\in\mathbb{N}$

$$\frac{|f^{(k+j)}(z)|}{(k+j)!}(R-|z|)^{k+j} \le \frac{1+a_j(R)}{(1+j)(1-a_j(R))}\frac{|f^{(j)}(z)|}{j!}(R-|z|)^j. \tag{12}$$

If $\frac{1+a_j(R)}{(1+j)(1-a_j(R))}\le 1$ then (12) implies for all $z\in\mathbb{D}_R$ and $n\ge 0$

$$\frac{|f^{(n)}(z)|}{n!}(R-|z|)^n \le \max\left\{\frac{|f^{(m)}(z)|}{m!}(R-|z|)^m\colon 0\le m\le j\right\}.$$

If $\frac{(1+j)(1-a_j(R))}{1+a_j(R)} \le 1$ then (12) implies for all $k \in \mathbb{N}$

$$\frac{|f^{(k+j)}(z)|}{(k+j)!}\left(\frac{(1-a_j(R))(1+j)}{1+a_j(R)}(R-|z|)\right)^{k+j} \le$$
$$\le \left(\frac{(1-a_j(R))(1+j)}{1+a_j(R)}\right)^{k-1}\frac{|f^{(j)}(z)|}{j!}\left(\frac{(1-a_j(R))(1+j)}{1+a_j(R)}(R-|z|)\right)^{j} \le$$
$$\le \frac{|f^{(j)}(z)|}{j!}\left(\frac{(1-a_j(R))(1+j)}{1+a_j(R)}(R-|z|)\right)^{j}.$$

Thus, for all $z \in \mathbb{D}_R$ and $n \ge 0$

$$\frac{|f^{(n)}(z)|}{n!}\left(\frac{(1-a_j(R))(1+j)}{1+a_j(R)}(R-|z|)\right)^{n} \le$$
$$\le \max\left\{\frac{|f^{(m)}(z)|}{m!}\left(\frac{(1-a_j(R))(1+j)}{1+a_j(R)}(R-|z|)\right)^{m} : 0 \le m \le j\right\}.$$

Since $\frac{|f^{(j)}(0)|}{j!l^j(0)} = \frac{|a_j|}{l^j(0)} > 0 = |f(0)| = \frac{|f'(0)|}{1!l(0)} = \cdots = \frac{|f^{(j-1)}(0)|}{(j-1)!l^{j-1}(0)}$ for any positive function l, so (2) implies that $N(f, l; \mathbb{D}_R) \ge j$. Thus $N(f, l; \mathbb{D}_R) = j$. □

The lemma implies the following corollary.

Corollary 1. *If a function of the form*

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n \tag{13}$$

is analytic in $\overline{\mathbb{D}}_R = \{z\colon |z| \le R\}$ *and*

$$\sum_{n=1}^{\infty}(n+1)|f_{n+1}|R^n \le a(R) < 1, \tag{14}$$

then $N(f, l; \mathbb{D}_R) = 1$ *with* $l(|z|) = \frac{K(R)}{R-|z|}$, *where* $K(R) = \max\left\{1, \frac{1+a(R)}{2(1-a(R))}\right\}$.

If $z \in \mathbb{D}_{R/2}$, then $R-|z| \ge R/2$ and the conclusion of Corollary 1 implies $N(f, l; \mathbb{D}_{R/2}) \le 1$ with $l(|z|) = \frac{2K(R)}{R}$, because if $l_*(r) \le l^*(r)$, it is easy to show ([2, p.23]), that $N(f, l^*; G) \le$ $\le N(f, l_*; G)$. Since for the function (13) we have $\frac{|f'(0)|}{1!l(0)} = \frac{1}{l(0)} > 0 = |f(0)|$, therefore, the next corollary is true.

Corollary 2. *If a function defined by (13) is analytic in* $\overline{\mathbb{D}}_R = \{z\colon |z| \le R\}$ *and satisfies (14), then* $N(f, l; \mathbb{D}_{R/2}) = 1$ *with* $l(|z|) \equiv \frac{2K(R)}{R}$, *where* $K(R) = \max\left\{1, \frac{1+a(R)}{2(1-a(R))}\right\}$.

We will apply Corollary 2 to investigate the l-index boundedness of the function (9). Assume, as earlier, that $0 < \gamma < 2$ and $0 < \alpha\beta < 2$. Since

$$\sup\left\{\frac{k-\gamma}{k-\alpha\beta}\colon k \ge 2\right\} = \begin{cases} 1, & \gamma \ge \alpha\beta; \\ \frac{2-\gamma}{2-\alpha\beta}, & \gamma < \alpha\beta, \end{cases}$$

so (8) implies that $b_k \le 1$ for all $k \ge 2$ if $\gamma \ge \alpha\beta$ and $b_k \le \left(\frac{2-\gamma}{2-\alpha\beta}\right)^{k-1}$ for all $k \ge 2$ if $\gamma < \alpha\beta$.

In the first case we choose, for example, $R = 1/4$. Then $\sum_{k=1}^{\infty}(k+1)b_{k+1}R^k \le \sum_{k=1}^{\infty}(k+1)/4^k = 7/9$. Thus, choosing $a(R) = 7/9$, we have $K(R) = 4$, and by Corollary 2 $N(F_1, 32; \mathbb{D}_{1/8}) = 1$.

In the second case we choose $R = \frac{2-\alpha\beta}{4(2-\gamma)}$. Then $a(R) = 7/9$ again, $K(R) = 4$ and by Corollary 2 $N(F_1, \frac{32(2-\gamma)}{2-\alpha\beta}; \mathbb{D}_{(2-\alpha\beta)/(8(2-\gamma))}) = 1$.

Therefore, the next proposition is true.

Proposition 4. *Let $0 < \gamma < 2$ and $0 < \alpha\beta < 2$. If $\gamma \ge \alpha\beta$ then $N(F_1, 32; \mathbb{D}_{1/8}) = 1$, and if $\gamma < \alpha\beta$ then $N(F_1, \frac{32(2-\gamma)}{2-\alpha\beta}; \mathbb{D}_{(2-\alpha\beta)/(8(2-\gamma))}) = 1$.*

Since the function $F_1(\alpha, \beta, \gamma; 1/z)$ is a solution of equation (4) and condition (7) holds, it is easy to show that the function $F_1(z) = F_1(\alpha, \beta, \gamma; z)$ satisfies the differential equation

$$(z^3 - z^2)w'' + ((2-\gamma)z^2 + \alpha\beta z)w' - \alpha\beta w = 0. \tag{15}$$

Using this fact we can investigate the l-index boundedness in $\mathbb{D} \setminus \mathbb{D}_{1/8}$. Indeed, from (15) we can get for $z \in \mathbb{D}$

$$|z|^2(1-|z|)|F_1''(z)| \le |z^3 - z^2||F_1''(z)| \le |(2-\gamma)z^2 + \alpha\beta z||F_1'(z)| + |\alpha\beta||F_1(z)|.$$

Suppose that $0 < \alpha\beta \le \gamma < 2$. Then the last inequality implies for $|z| \ge 1/8$

$$\frac{|F_1''(z)|}{2!}(1-|z|)^2 \le \frac{1}{2}\left(2-\gamma+\frac{\alpha\beta}{|z|}\right)\frac{|F_1'(z)|}{1!}(1-|z|) + \frac{\alpha\beta|F_1(z)|}{2|z|^2} \le$$
$$\le 9\frac{|F_1'(z)|}{1!}(1-|z|) + 64|F_1(z)|,$$

whence it follows that

$$\frac{|F_1''(z)|}{2!}\left(\frac{1-|z|}{32}\right)^2 \le \frac{9}{32}\frac{|F_1'(z)|}{1!}\left(\frac{1-|z|}{32}\right) + \frac{2}{32}|F_1(z)| \le$$
$$\le \max\left\{\frac{|F_1'(z)|}{1!}\left(\frac{1-|z|}{32}\right), |F_1(z)|\right\}. \tag{16}$$

Differentiating (15) with $w = F_1(z)$ we have

$$(z^3 - z^2)w''' + ((5-\gamma)z^2 + (\alpha\beta - 2)z)w'' + 2(2-\gamma)zw' = 0. \tag{17}$$

Since $|\alpha\beta - 2| \le 2$ from (17), as above, for $|z| \ge 1/8$ we get

$$\frac{|F_1'''(z)|}{3!}(1-|z|)^3 \le \frac{21}{3}\frac{|F_1''(z)|}{2!}(1-|z|)^2 + \frac{16}{3}\frac{|F_1'(z)|}{1!}(1-|z|).$$

Therefore, in view of (16)

$$\frac{|F_1'''(z)|}{3!}\left(\frac{1-|z|}{32}\right)^3 \le \frac{21}{96}\frac{|F_1''(z)|}{2!}\left(\frac{1-|z|}{32}\right)^2 + \frac{16}{96}\frac{|F_1'(z)|}{1!}\left(\frac{1-|z|}{32}\right) \le \tag{18}$$
$$\le \max\left\{\frac{|F_1''(z)|}{2!}\left(\frac{1-|z|}{32}\right)^2, \frac{|F_1'(z)|}{1!}\left(\frac{1-|z|}{32}\right)\right\} \le \max\left\{\frac{|F_1'(z)|}{1!}\left(\frac{1-|z|}{32}\right), |F_1(z)|\right\}.$$

Differentiating (15) $n \geq 2$ times we obtain

$$(z^3 - z^2)w^{(n+2)} + \{(3n + 2 - \gamma)z^2 + (\alpha\beta - 2n)z\}w^{(n+1)} + \\ + \{(3n(n-1) + 2n(2-\gamma))z - (n-1)(n-\alpha\beta)\}w^{(n)} + n(n-1)(n-\gamma)w^{(n-1)} = 0. \tag{19}$$

Thus for $|z| \geq 1/8$ we have

$$\frac{|F_1^{(n+2)}(z)|}{(n+2)!}(1-|z|)^{n+2} \leq \frac{3n + 2 - \gamma + 8(2n - \alpha\beta)}{n+2}\frac{|F_1^{(n+1)}(z)|}{(n+1)!}(1-|z|)^{n+1} + \\ + \frac{8(3n(n-1) + 2n(2-\gamma)) + 64(n-1)(n-\alpha\beta)}{(n+2)(n+1)}\frac{|F_1^{(n)}(z)|}{n!}(1-|z|)^{n} + \\ + \frac{64n(n-1)(n-\gamma)}{(n+2)(n+1)n}\frac{|F_1^{(n-1)}(z)|}{(n-1)!}(1-|z|)^{n-1} \leq \frac{19n+2}{n+2}\frac{|F_1^{(n+1)}(z)|}{(n+1)!}(1-|z|)^{n+1} + \\ + \frac{88n^2 - 56n}{(n+2)(n+1)}\frac{|F_1^{(n)}(z)|}{n!}(1-|z|)^{n} + \frac{64n(n-1)}{(n+2)(n+1)}\frac{|F_1^{(n-1)}(z)|}{(n-1)!}(1-|z|)^{n-1} \leq \\ \leq 19\frac{|F_1^{(n+1)}(z)|}{(n+1)!}(1-|z|)^{n+1} + 88\frac{|F_1^{(n)}(z)|}{n!}(1-|z|)^{n} + 64\frac{|F_1^{(n-1)}(z)|}{(n-1)!}(1-|z|)^{n-1}.$$

Hence the next inequality easy follow for $n \geq 2$

$$\frac{|F_1^{(n+2)}(z)|}{(n+2)!}\left(\frac{1-|z|}{32}\right)^{n+2} \leq \frac{19}{32}\frac{|F_1^{(n+1)}(z)|}{(n+1)!}\left(\frac{1-|z|}{32}\right)^{n+1} + \\ + \frac{88}{1024}\frac{|F_1^{(n)}(z)|}{n!}\left(\frac{1-|z|}{32}\right)^{n} + \frac{2}{1024}\frac{|F_1^{(n-1)}(z)|}{(n-1)!}\left(\frac{1-|z|}{32}\right)^{n-1} \leq \\ \leq \max\left\{\frac{|F_1^{(n+1)}(z)|}{(n+1)!}\left(\frac{1-|z|}{32}\right)^{n+1}, \frac{|F_1^{(n)}(z)|}{n!}\left(\frac{1-|z|}{32}\right)^{n}, \frac{|F_1^{(n-1)}(z)|}{(n-1)!}\left(\frac{1-|z|}{32}\right)^{n-1}\right\}. \tag{20}$$

Inequalities (16), (18) and (20) imply that for all $m \geq 2$ and $|z| \geq 1/8$

$$\frac{|F_1^{(m)}(z)|}{m!}\left(\frac{1-|z|}{32}\right)^{m} \leq \max\left\{\frac{|F_1'(z)|}{1!}\left(\frac{1-|z|}{32}\right), |F_1(z)|\right\},$$

that is $N(F_1, l, \mathbb{D} \setminus \mathbb{D}_{1/8}) \leq 1$ with $l(|z|) = 32/(1-|z|)$, and in view of Proposition 4 we obtain the next statement.

Proposition 5. *Let* $0 < \alpha\beta \leq \gamma < 2$. *Then the function* F_1 *is of bounded* l*-index* $N(F_1, l) = 1$ *with* $l(|z|) = 32/(1-|z|)$.

Suppose now that $0 < \gamma \leq \alpha\beta < 2$ and $z \in \mathbb{D} \setminus \mathbb{D}_{(2-\alpha\beta)/(8(2-\gamma))}$. Then $\frac{1}{|z|} \leq \frac{8(2-\gamma)}{(2-\alpha\beta)}$ and like in proof of (16) we get

$$\frac{|F_1''(z)|}{2!}(1-|z|)^2 \leq \frac{1}{2}\left(2 - \gamma + \frac{8\alpha\beta(2-\gamma)}{2-\alpha\beta}\right)\frac{|F_1'(z)|}{1!}(1-|z|) + \frac{64\alpha\beta(2-\gamma)^2}{2(2-\alpha\beta)^2}|F_1(z)| \leq \\ \leq \frac{2-\gamma}{2}\left(1 + \frac{32}{2-\alpha\beta}\right)\frac{|F_1'(z)|}{1!}(1-|z|) + \frac{64(2-\gamma)^2}{(2-\alpha\beta)^2}|F_1(z)|$$

and

$$\frac{|F_1''(z)|}{2!}\left(\frac{2-\alpha\beta}{32(2-\gamma)}(1-|z|)\right)^2 \le$$
$$\le \frac{2-\gamma}{2}\left(1+\frac{32}{2-\alpha\beta}\right)\frac{2-\alpha\beta}{32(2-\gamma)}\frac{|F_1'(z)|}{1!}\left(\frac{2-\alpha\beta}{32(2-\gamma)}(1-|z|)\right)+$$
$$+\frac{64(2-\gamma)^2}{(2-\alpha\beta)^2}\left(\frac{2-\alpha\beta}{32(2-\gamma)}\right)^2|F_1(z)| = \frac{2-\alpha\beta+32}{64}\frac{|F_1'(z)|}{1!}\left(\frac{2-\alpha\beta}{32(2-\gamma)}(1-|z|)\right)+$$
$$+\frac{1}{16}|F_1(z)| \le \max\left\{\frac{|F_1'(z)|}{1!}\left(\frac{2-\alpha\beta}{32(2-\gamma)}(1-|z|)\right),\, |F_1(z)|\right\}. \tag{21}$$

From (17) for $1/|z| \le 8(2-\gamma)/(2-\alpha\beta)$ we have

$$\frac{|F_1'''(z)|}{3!}(1-|z|)^3 \le$$
$$\le \frac{1}{3}\left(5-\gamma+\frac{8(2-\alpha\beta)(2-\gamma)}{2-\alpha\beta}\right)\frac{|F_1''(z)|}{2!}(1-|z|)^2+\frac{16(2-\gamma)^2}{6(2-\alpha\beta)}\frac{|F_1'(z)|}{1!}(1-|z|)^2,$$

and since $\frac{2-\alpha\beta}{2-\gamma} \le 1$ and $2-\alpha\beta \le 2$ in view of (21)

$$\frac{|F_1'''(z)|}{3!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^3 \le \frac{1}{3}\left(\frac{5}{32}+\frac{16}{32}\right)\frac{|F_1''(z)|}{2!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^2+$$
$$+\frac{1}{6}\,\frac{1}{32}\frac{|F_1'(z)|}{1!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right) \le \max\left\{\frac{|F_1''(z)|}{2!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^2,\right.$$
$$\left.\frac{|F_1'(z)|}{1!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)\right\} \le \max\left\{\frac{|F_1'(z)|}{1!}\left(\frac{2-\alpha\beta}{32(2-\gamma)}(1-|z|)\right),\, |F_1(z)|\right\}. \tag{22}$$

Finally, from (19) for $n \ge 2$ and $1/|z| \le 8(2-\gamma)/(2-\alpha\beta)$ we get

$$\frac{|F_1^{(n+2)}(z)|}{(n+2)!}(1-|z|)^{n+2} \le \frac{1}{n+2}\left\{3n+2-\gamma+\frac{8(2n-\alpha\beta)(2-\gamma)}{2-\alpha\beta}\right\}\frac{|F_1^{(n+1)}(z)|}{(n+1)!}(1-|z|)^{n+1}+$$
$$+\frac{1}{(n+1)(n+2)}\left\{\frac{8n(3n+1-2\gamma)(2-\gamma)}{2-\alpha\beta}+\frac{64(n-1)(n-\alpha\beta)(2-\gamma)^2}{(2-\alpha\beta)^2}\right\}\times$$
$$\times\frac{|F_1^{(n)}(z)|}{n!}(1-|z|)^n+\frac{n(n-1)(n-\gamma)}{(n+2)(n+1)n}\frac{64(2-\gamma)^2}{(2-\alpha\beta)^2}\frac{|F_1^{(n-1)}(z)|}{(n-1)!}(1-|z|)^{n-1},$$

and since $\frac{2-\alpha\beta}{2-\gamma} \le 1$ and $2n-\alpha\beta \le 2n$ we have

$$\frac{|F_1^{(n+2)}(z)|}{(n+2)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n+2} \le$$
$$\le \frac{1}{n+2}\left\{\frac{(3n+2)(2-\alpha\beta)}{32(2-\gamma)}+\frac{16n}{32}\right\}\frac{|F_1^{(n+1)}(z)|}{(n+1)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n+1}+$$
$$+\frac{1}{(n+1)(n+2)}\left\{\frac{8n(3n+1)(2-\alpha\beta)}{32^2(2-\gamma)}+\frac{2n(n-1)}{32}\right\}\frac{|F_1^{(n)}(z)|}{n!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^n+$$

$$+\frac{2n(n-1)}{32^2(n+1)(n+2)}\frac{2-\alpha\beta}{2-\gamma}\frac{|F_1^{(n-1)}(z)|}{(n-1)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n-1}\le$$
$$\le\left(\frac{3}{32}+\frac{16}{32}\right)\frac{|F_1^{(n+1)}(z)|}{(n+1)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n+1}+$$
$$+\left(\frac{3}{64}+\frac{1}{16}\right)\frac{|F_1^{(n)}(z)|}{n!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n}+\frac{1}{256}\frac{|F_1^{(n-1)}(z)|}{(n-1)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n-1}\le$$
$$\le\max\left\{\frac{|F_1^{(n+1)}(z)|}{(n+1)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n+1},\frac{|F_1^{(n)}(z)|}{n!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n},\right.$$
$$\left.\frac{|F_1^{(n-1)}(z)|}{(n-1)!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{n-1}\right\}. \tag{23}$$

Inequalities (21), (22) and (23) implies for all $m\geq 2$ and $|z|\geq 1/8$

$$\frac{|F_1^{(m)}(z)|}{m!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right)^{m}\le\max\left\{\frac{|F_1'(z)|}{1!}\left(\frac{(2-\alpha\beta)(1-|z|)}{32(2-\gamma)}\right),|F_1(z)|\right\},$$

thus $N(F_1,l,\mathbb{D}\setminus\mathbb{D}_{(2-\alpha\beta)/(8(2-\gamma))})\le 1$ with $l(|z|)=\frac{32(2-\gamma)}{(2-\alpha\beta)(1-|z|)}$, and in view of Proposition 4 we get the following proposition.

Proposition 6. *Let $0<\gamma\le\alpha\beta<2$. Then the function F_1 is of bounded l-index $N(F_1,l)=$ $=1$ with $l(|z|)=\frac{32(2-\gamma)}{(2-\alpha\beta)(1-|z|)}$.*

From Propositions 5 and 6 we get the following analogue of Theorem B.

Theorem 2. *Let $0<\gamma<2$ and $0<\alpha\beta<2$. If $\alpha\beta\le\gamma$ then $N(F_1,l)=1$ with $l(|z|)=$ $=\frac{32}{1-|z|}$, and if $\gamma\le\alpha\beta$ then $N(F_1,l)=1$ with $l(|z|)=\frac{32(2-\gamma)}{(2-\alpha\beta)(1-|z|)}$.*

5. Conclusions and remarks. Theorems A–B and 1–2 imply the following statement.

Corollary 3. *If $\gamma\notin\mathbb{Z}$, $\alpha\beta\notin\{2,3,4,\dots\}$ and $\alpha+\beta=1+\alpha\beta$ then the common solution of the Gauss equation is of the form $F(\alpha,\beta,\gamma;z)+cF_1(\alpha,\beta,\gamma;1/z)$, where, if α,β,γ are positive, the function $F(\alpha,\beta,\gamma;z)$ has the properties from Theorems A and B, and a function $F_1(\alpha,\beta,\gamma;z)$ possesses the assumptions of Theorems 1 and 2.*

For a function analytic in $\mathbb{C}\setminus\overline{\mathbb{D}}=\{z\colon 1<|z|<+\infty\}$ defined by (6) like for the functions from the class Σ one can give a definition of meromorphic starlikeness (or starlikeness with respect to ∞) and of meromorphic convexity (or convexity with respect to ∞) respectively by the conditions $\mathrm{Re}\{-z\varphi'(z)/\varphi(z)\}>0$ and $\mathrm{Re}\{-(1+z\varphi''(z)/\varphi'(z))\}>0$ for all $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$. It is easy to show that if a function φ satisfies one of the above conditions then $f(z)=\varphi(1/z)$ is starlike or convex in $\mathbb{D}$, respectively. By analogy, an analytic in $\mathbb{C}\setminus\overline{\mathbb{D}}$ function (6) is said to be close-to-convex with respect to ∞, if there exists convex with respect to ∞ function $\Phi(z)=1/z+\sum_{k=2}^{\infty}B_k/z^k$ such that $\mathrm{Re}\{\varphi'(z)/\Phi'(z)\}>0$ for all $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$. Then for a function defined by (6) to be close-to-convex with respect to ∞ is necessary and sufficient that the function $f(z)=\varphi(1/z)$ is close-to-convex in $\mathbb{D}$. Proposition 2 implies that if $1+\alpha\beta\le\gamma<2$ then the function $F_1(\alpha,\beta,\gamma;1/z)$ is close-to-convex with respect to ∞ in $\mathbb{C}\setminus\overline{\mathbb{D}}$.

REFERENCES

1. Golusin G.M., Geometric theory of functions of a complex variable. – American Math. Soc., 1966. – 676 p.
2. Sheremeta M.M., Analytic functions of bounded index. – Lviv: VNTL Publishers, 1999. – 141 p.
3. Kuznetsov D.S., Special functions. – M.: Vysshaya Shkola, 1965. – 423 p. (in Russian)
4. Sheremeta M.M. *Properties of the hypergeometric function with positive parameters*// Visnyk Lviv Univ., Ser. Mech. Math. – 2009. – V.70. – P. 183–190. (in Ukrainian)
5. Juneja O.P., Reddy T.R. *Meromorphic starlike and univalent functions with positive coefficients*// Ann. Univ. Mariae Curie-Sklodowska. – 1985. – V.39. – P. 65–76.
6. Mogra M.L. *Hadamard product certain meromorphic univalent functions*// J. Math. Anal. Appl. – 1991. – V.157. – P. 10–16.
7. Goodman A.W., Univalent function. – V.II, Mariner Publishing Co., 1983. – 158 p.

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Received 13.01.2014
Revised 11.03.2014