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THIRD HANKEL DETERMINANT FOR THE INVERSE OF A FUNCTION WHOSE DERIVATIVE HAS A POSITIVE REAL PART

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Let RT be the class of functions f univalent in the unit disk $E = \{z\colon |z| < 1\}$ such that $\operatorname{Re} f'(z) > 0$, $(z \in E)$ and $H_3(1)$ the third Hankel determinant for inverse function to $f \in RT$. In the paper obtained the upper bound for $H_3(1)$ in the terms of Toeplitz determinants.

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Пусть RT — класс однолистых в единичном круге $E = \{z\colon |z| < 1\}$ функций такых, что $\operatorname{Re} f'(z) > 0$, $(z \in E)$, а $H_3(1)$ — определитель Ганкеля третьего порядка для функции, обратной к функции $f \in RT$. В статье получены оценки сверху определителя $H_3(1)$ в терминах определителя Тёплиця.

1. Introduction. Let A denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient of normalized univalent function readily yields the growth and distortion properties for univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Ch. Pommerenke ([20, 21]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by several authors in the literature. For example, J. W. Noonan and D. K. Thomas ([17]) studied about the second Hankel determinant of

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areally mean p-valent functions. K. I. Noor ([18]) determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. R. Ehrenborg ([7]) studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by J. W. Layman in [13]. One can easily observe that the Fekete-Szegő functional is $H_2(1)$. Fekete-Szegő then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. R. M. Ali ([2]) found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szegő functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$, when it belongs to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. Further sharp bounds for the functional $|a_2 a_4 - a_3^2|$, the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

were obtained by the authors ([1], [5], [10–12], [22], [24–27]) for various subclasses of univalent and multivalent analytic functions. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in A$, $a_1 = 1$, so that, we have $H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$ and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{2}$$

Incidentally, all of the functionals on the right hand side of the inequality (2) have known (and sharp) upper bounds except $|a_2 a_3 - a_4|$. The sharp upper bound to the second Hankel functional $H_2(2)$ for the subclass RT of S, consisting of functions whose derivative has a positive real part, studied by T. H. MacGregor ([16]) was obtained by A. Janteng ([12]). It was known that if $f \in RT$ then $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, \ldots\}$. Also the sharp upper bound for the functional $|a_3 - a_2^2|$ was $\frac{2}{3}$, stated in [4], for the class RT. Further, the best possible sharp upper bound for the functional $|a_2 a_3 - a_4|$ was obtained by K. O. Babalola ([3]) and hence the sharp inequality for $|H_3(1)|$, for the class RT.

Motivated by the result obtained by K. O. Babalola ([3]) in finding the sharp upper bound to the third Hankel determinant $|H_3(1)|$ for the class RT, in the present paper, we obtain an upper bound to the functional $|t_2 t_3 - t_4|$ and hence for $|H_3(1)|$, for the inverse of the function of f given in (1), when it belongs to the class RT, defined as follows.

Definition 1. A function $f(z) \in A$ is said to be *in the class RT*, consisting of functions whose derivative have a positive real part, if it satisfies the condition

$$(\forall\ z \in E):\ \operatorname{Re}\{f'(z)\} > 0.$$

Some preliminary lemmas required for proving our result are as follows.

2. Preliminary Results. Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{3}$$

which are regular in the open unit disc E and satisfy $\mathrm{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called the Caratheòdory function ([6]).

Lemma 1 ([19, 23]). *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\left(\frac{1+z}{1-z}\right)$.*

Lemma 2 ([9]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (3) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n \in \mathbb{N}, \quad c_{-k} = \overline{c}_k,$$

are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k P_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $P_0(z) = \left(\frac{1+z}{1-z}\right)$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [9] is due to Caratheòdory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2\,\mathrm{Re}\,\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\} \tag{4}$$

for some x, $|x| \leq 1$. For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{5}$$

Simplifying the relations (4) and (5), we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\} \tag{6}$$

with $|z| \leq 1$. To obtain our results, we refer to the classical method initiated by R. J. Libera and E. J. Zlotkiewicz ([14], [15]) and used by several authors in the literature.

3. Main Result.

Theorem 1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in RT$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f, then*

$$|t_2 t_3 - t_4| \leq \frac{1}{3}\left(\frac{13}{6}\right)^{\frac{3}{2}}.$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in RT$, from Definition 1, there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $\mathrm{Re}[p(z)] > 0$ such that

$$f'(z) = p(z). \tag{7}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (7), we have

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Upon simplification, we obtain

$$2a_2 + 3a_3 z + 4a_4 z^2 + 5a_5 z^3 + \ldots = c_1 + c_2 z + c_3 z^2 + c_4 z^3 + \ldots. \tag{8}$$

Equating the coefficients of like powers of z^0, z, z^2 and z^3 respectively on both sides of (8), we have

$$a_2 = \frac{c_1}{2}; \quad a_3 = \frac{c_2}{3}; \quad a_4 = \frac{c_3}{4}; \quad a_5 = \frac{c_4}{5}. \tag{9}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in RT$, from the definition of inverse function of f, we have

$$w = f\left(f^{-1}(w)\right) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n \left(f^{-1}(w)\right)^n \Leftrightarrow w = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \Big(w + \sum_{n=2}^{\infty} t_n w^n\Big)^n.$$

After simplifying, we get

$$\begin{aligned}(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 + \\ + (t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \ldots = 0.\end{aligned} \tag{10}$$

Equating the coefficients of like powers of w^2, w^3, w^4 and w^5 on both sides of (10), respectively, further simplification gives

$$\begin{aligned}t_2 = -a_2;\ t_3 = \{-a_3 + 2a_2^2\};\ t_4 = \{-a_4 + 5a_2a_3 - 5a_2^3\}; \\ t_5 = \{-a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4\}.\end{aligned} \tag{11}$$

Using the values of a_2, a_3, a_4 and a_5 in (9) along with (11), upon simplification, we obtain

$$\begin{aligned}t_2 = \frac{-c_1}{2};\ t_3 = \frac{1}{6}\{-2c_2 + 3c_1^2\};\ t_4 = \frac{1}{24}\{-6c_3 + 20c_1c_2 - 15c_1^3\}; \\ t_5 = \frac{1}{120}\{-24c_5 + 90c_1c_3 - 210c_1^2c_2 + 40c_2^2 + 105c_1^4\}.\end{aligned} \tag{12}$$

Substituting the values of t_2, t_3 and t_4 from (12) in the functional $|t_2t_3 - t_4|$ for the inverse function of $f \in RT$, after simplifying, we get

$$|t_2t_3 - t_4| = \frac{1}{24} \times |6c_3 - 16c_1c_2 + 9c_1^3|. \tag{13}$$

Substituting the values of c_2 and c_3 from (4) and (6) respectively from Lemma 2 on the right-hand side of (13), we have

$$|6c_3 - 16c_1c_2 + 9c_1^3| = \Big|6 \times \frac{1}{4}\{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\} -$$

$$-16c_1 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + 9c_1^3\Big|.$$

Using the facts $|z| < 1$ and $|pa+qb| \le |p||a| + |q||b|$, where p, q, a and b are real numbers, on the right-hand side of the above expression, after simplifying, we get

$$2|6c_3 - 16c_1c_2 + 9c_1^3| \le |5c_1^3 + 6(4-c_1^2) - 10(4-c_1^2)|x| - 3(c_1+2)(4-c_1^2)|x|^2|.$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality, using the property $(c+a) \ge (c-a)$, where $a \ge 0$ and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned} 2|6c_3 - 16c_1c_2 + 9c_1^3| &\le [5c^3 + 6(4-c^2) + 10c(4-c^2)\mu + 3(c-2)(4-c^2)\mu^2] = \\ &= F(c,\mu), \text{ for } 0 \le \mu = |x| \le 1, \end{aligned} \tag{14}$$

where

$$F(c,\mu) = [5c^3 + 6(4-c^2) + 10c(4-c^2)\mu + 3(c-2)(4-c^2)\mu^2]. \tag{15}$$

We next maximize the function $F(c,\mu)$ given in the left hand side of (14) on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (15) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = [10c + 6(c-2)\mu] \times (4-c^2). \tag{16}$$

For $0 < \mu < 1$, fixed c with $0 < c < 2$, from (16), we observe that $\frac{\partial F}{\partial \mu} > 0$. This implies that $F(c,\mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. Further, for fixed $c \in [0,2]$, we have

$$\max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = G(c). \tag{17}$$

Therefore, simplifying the relations (15) and (17), we obtain

$$G(c) = -8c^3 + 52c, \tag{18}$$

$$G'(c) = -24c^2 + 52, \tag{19}$$

$$G''(c) = -48c. \tag{20}$$

For extreme values of G(c), consider $G'(c) = 0$. From (19), we have

$$-24c^2 + 52 = 0 \Leftrightarrow c^2 = \frac{13}{6} \in [0,2]. \tag{21}$$

Substituting the value of c^2 from (21) in (20), which simplifies to $G''(c) = -104 < 0$. By the second derivative test, $G(c)$ has maximum value at c, where c^2 given in (21). Using the obtained value of c^2 in (18), after simplifying, we get

$$\max_{0 \le c \le 2} G(c) = \frac{104}{3}\sqrt{\frac{13}{6}}. \tag{22}$$

Considering, the maximum value of $G(c)$ only at c^2, simplifying the relations (14) and (22), we obtain

$$|6c_3 - 16c_1c_2 + 9c_1^3| \le \frac{52}{3}\sqrt{\frac{13}{6}}. \tag{23}$$

From (13) and (23), after simplifying, we get $|t_2t_3 - t_4| \le \frac{1}{3}\left(\frac{13}{6}\right)^{\frac{3}{2}}$. □

The following theorems are straight forward verification on applying the same procedure as described in Theorem 1.

Theorem 2. *If $f(z) \in RT$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f, then $|t_2 t_3 - t_4| \leq \frac{137}{288}$.*

Theorem 3. *If $f(z) \in RT$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f, then $|t_3 - t_2^2| \leq \frac{2}{3}$ and is sharp for the values $c_1 = c = 0, c_2 = 2$ and $x = 1$.*

Theorem 4. *If $f(z) \in RT$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f, then we have the following inequalities: (i) $|t_2| \leq 1$; (ii) $|t_3| \leq \frac{4}{3}$; (iii) $|t_4| \leq \frac{13}{6}$; (iv) $|t_5| \leq \frac{59}{15}$.*

Proof. Using the fact that $|c_n| \leq 2, n \in \mathbb{N}$, with the help of c_2 and c_3 values given in (4) and (6) respectively together with the values in (12), we at once obtain all the above inequalities. □

Using the results of Theorems 1, 2, 3 and 4 in (2), we obtain the following corollary.

Corollary 1. *If $f(z) \in RT$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f, then*

$$|H_3(1)| \leq \frac{1}{3}\left[\frac{181}{40} + \left(\frac{13}{6}\right)^{\frac{5}{2}}\right].$$

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