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V. Dilnyi

ON THE SOLUTIONS OF A CONVOLUTION EQUATION IN A SEMI-STRIP

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We consider a convolution type equation for the Smirnov spaces in a semi-strip. An estimation of a solution in terms of analytic extension is obtained.

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Рассматривается уравнение типа свертки для пространств Смирнова в полуполосе. Получена оценка решения в терминах аналитического продолжения.

Let $H^p(\mathbb{C}_+), 1 \le p < +\infty$, be the Hardy space of analytic in the half-plane $\mathbb{C}_+ = \{z\colon \mathrm{Re} z > 0\}$ functions with the condition

$$||f||_* := \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dy \right\}^{1/p} < +\infty.$$

Properties of these spaces are described in details in [1]. There it is shown, particularly, that each $H^p(\mathbb{C}_+), 1 \le p < +\infty$, is a Banach space with respect to the above norm.

The following result is very famous (see [2, 3]).

Theorem A. *Let $q \in L_2(-\infty; 0)$ and $Q(z) = \int_{-\infty}^0 q(t)e^{tz}dt$. Then the following conditions are equivalent:*

1) the equation

$$\int_{-\infty}^{0} \psi(t+\tau)q(t)dt = 0, \quad \tau \le 0, \tag{1}$$

has a nontrivial solution in $\psi \in L_2(-\infty; 0)$;

2) the system $\{Q(z)e^{\tau z}\colon \tau \le 0\}$ is not complete in $H^2(\mathrm{C}_+)$;

3) Q is not outer for $H^2(\mathbb{C}_+)$.

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A function $Q \in H^2(\mathbb{C}_+)$ is said to be outer for $H^2(\mathbb{C}_+)$ if $G(z) \neq 0$ for all $z \in \mathbb{C}_+$,

$$\overline{\lim_{x\to+\infty}} \frac{\ln|Q(x)|}{x} = 0$$

and the singular boundary function of Q is constant.

The necessity part of above theorem is based on the following result.

Theorem B. *Suppose $q \in L_2(-\infty;0)$, $Q(z) = \int_{-\infty}^0 q(t)e^{tz}dt$. A function $\psi \in L_2(-\infty;0)$ is a solution of equation (1) if and only if the function $Q(iy)\Psi(iy)$, where $\Psi(z) = \int_{-\infty}^0 \psi(t)e^{-tz}dt$ is an angular boundary function on $i\mathbb{R}$ of some function $P \in H^1(\mathbb{C}_+)$.*

Let $H_\sigma^p(\mathbb{C}_+), \sigma \geq 0, 1 \leq p < +\infty$, be the space of analytic functions in $\mathbb{C}_+$ with

$$||f|| := \sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|}dr \right\}^{1/p} < +\infty.$$

Let $E^p[D_\sigma]$ and $E_*^p[D_\sigma], 1 \leq p < +\infty, \sigma > 0$, be the spaces of analytic functions respectively in the domains $D_\sigma = \{z\colon |\operatorname{Im} z| < \sigma, \operatorname{Re} z < 0\}$ and $D_\sigma^* = C\backslash\overline{D}_\sigma$, for which

$$\sup \left\{ \int_\gamma |f(z)|^p|dz| \right\}^{1/p} < +\infty,$$

where supremum is taken over all segments γ, that lay in D_σ and D_σ^*, respectively. The spaces $E^p[D_\sigma]$ and $E_*^p[D_\sigma]$ are studied in [4], where it has been shown that the functions f that belong to these spaces have a. e. on ∂D_σ the angular boundary values which will be denoted by $f(z)$ and $f \in L^p[\partial D_\sigma]$.

In [5, 6] the following equation is considered

$$\int_{\partial D_\sigma} f(w+\tau)g(w)dw = 0, \quad \tau \leq 0, \quad g \in E_*^2[D_\sigma]. \tag{2}$$

In [7] the following analogue of Theorem A is obtained.

Theorem C. *Let $g \in E_*^2[D_\sigma]$. Then the following conditions are equivalent:*

1) *equation (2) has a nontrivial solution $f \in E^2[D_\sigma]$;*
2) *the system $\{G(z)e^{\tau z}\colon \tau \leq 0\}$, where $G(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_\sigma} g(w)e^{-zw}dw$, is not complete in $H_\sigma^2(\mathbb{C}_+)$;*
3) *the function G has zero in $\mathbb{C}_+$ or the singular boundary function of G is not constant or*

$$\underline{\lim}_{r\to+\infty} \left(\frac{1}{2\pi} \int_{1<|t|<r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln|G(it)|dt - \frac{\sigma}{\pi}\ln r \right) > -\infty.$$

The singular boundary function h of $G \in H_\sigma^p(\mathbb{C}_+)$ is uniquely defined (up to a constant) at points of continuity t_1, t_2 by the equality

$$h(t_2) - h(t_1) = \lim_{x\to 0+} \int_{t_1}^{t_2} \ln|G(x+iy)|dy - \int_{t_1}^{t_2} \ln|G(iy)|dy.$$

The aim of this article is to obtain the analogue of Theorem B for equation (2). Results of this type can be used to describe the translation invariant subspaces for weighted Hardy spaces (see [3], [7]). We denote by F_j, $j \in \{1;2;3\}$ the functions

$$F_j(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{l_j} f(w) e^{-zw} dw, \quad j \in \{1;2;3\},$$

where l_1, l_3, and l_2 are the legs of ∂D_σ, respectively the rays laying under and above of the real axis, and the segment $[-i\sigma; i\sigma]$, and their orientation corresponds to the positive orientation of D_σ.

Theorem 1. *Suppose $f \in E_2[D_\sigma]$ is a solution of equation (2). Then $F_1(iy)G(iy)e^{\sigma y}$ is an angular boundary function on $i\mathbb{R}$ of a function P analytic in $\mathbb{C}_+$ such that*

$$\sup\left\{ \int\limits_0^{+\infty} \left|P(re^{i\varphi})\right| e^{-\sigma r|\sin\varphi|} dr \colon \varphi \in (-\pi/2+\delta; -\delta) \cup (\delta; \pi/2-\delta) \right\} < +\infty \tag{3}$$

for each $\delta \in (0;\pi/4)$.

For the proof of Theorem 1 we need some auxiliary results (see [5]).

Theorem D. *If $g \in E_*^2[D_\sigma]$, $f \in E^2[D_\sigma]$, then for all $\tau \le 0$ the equality*

$$\int\limits_{\partial D_\sigma} f(w+\tau) g(w) dw = \int\limits_0^{+i\infty} \Phi_1(z) e^{\tau z} dz + \int\limits_{-i\infty}^{0} \Phi_3(z) e^{\tau z} dz + \int\limits_0^{+\infty} \Phi_2(z) e^{\tau z} dz, \tag{4}$$

is valid, where $\Phi_j(it) = F_j(it)G(it)$, $t \in \mathbb{R}$.

Lemma 1. *Suppose $f \in E^2[D_\sigma]$ is a solution of equation (2) and*

$$S(z) = -\frac{1}{2\pi i} \int\limits_0^{+i\infty} \Phi_1(w) \frac{1}{w-z} dw + \frac{1}{2\pi i} \int\limits_{-i\infty}^{0} \Phi_3(w) \frac{1}{w-z} dw - \frac{1}{2\pi i} \int\limits_0^{+\infty} \Phi_2(w) \frac{1}{w-z} dw.$$

Then $S(z) = 0$, $z \in \mathbb{C}_-$.

Proof. Denote by $\mu_1(\tau)$, $\mu_2(\tau)$ and $\mu_3(\tau)$ the summands in the right hand side of (4) respectively. Then by Theorem A $\mu_1(\tau) + \mu_2(\tau) + \mu_3(\tau) = 0$, $\tau \in (-\infty; 0)$ and also $\int_{-\infty}^0 e^{-\tau z}(\mu_1(\tau) + \mu_2(\tau) + \mu_3(\tau)) d\tau = 0$, $\operatorname{Re} z < 0$. But by Fubini's Theorem for $\operatorname{Re} z < 0$

$$\int\limits_{-\infty}^{0} e^{-\tau z} \mu_2(\tau) d\tau = \int\limits_{-\infty}^{0} e^{-\tau z} \int\limits_0^{+\infty} \Phi_2(u) e^{\tau u} du d\tau = \int\limits_0^{+\infty} \Phi_2(u) du \int\limits_{-\infty}^{0} e^{\tau(u-z)} d\tau = -\int\limits_0^{+\infty} \frac{\Phi_2(u)}{u-z} du.$$

Analogously,

$$\int\limits_{-\infty}^{0} e^{-\tau z} \mu_1(\tau) d\tau = -i \int\limits_0^{+\infty} \frac{\Phi_1(iv)}{iv-z} dv, \quad \int\limits_{-\infty}^{0} e^{-\tau z} \mu_2(\tau) d\tau = i \int\limits_{-\infty}^{0} \frac{\Phi_3(iv)}{iv-z} dv.$$

Hence

$$0=-\int\limits_0^{+i\infty}\Phi_1(w)\frac{1}{w-z}dw+\frac{1}{2\pi i}\int\limits_{-i\infty}^{0}\Phi_3(w)\frac{1}{w-z}dw-\frac{1}{2\pi i}\int\limits_0^{+\infty}\Phi_2(w)\frac{1}{w-z}dw,\quad \operatorname{Re} z<0.$$

□

Lemma 2. *The function S has the angular boundary values almost everywhere (a. e.) on $\partial\mathbb{C}_+$ from $\mathbb{C}_+$. These values are equal to $\Phi_1(iv)$ for $v>0$ and $-\Phi_3(iv)$ for $v<0$.*

Proof. By Lemma 1 we have $S(-\bar{z})=0$, $z\in\mathbb{C}_+$, that is,

$$-\frac{1}{2\pi i}\int\limits_0^{+i\infty}\Phi_1(w)\frac{1}{w+\bar{z}}dw+\frac{1}{2\pi i}\int\limits_{-i\infty}^{0}\Phi_3(w)\frac{1}{w+\bar{z}}dw-\frac{1}{2\pi i}\int\limits_0^{+\infty}\Phi_2(w)\frac{1}{w+\bar{z}}dw=0.$$

Hence

$$S(z)=S(z)-S(-\bar{z})=-\frac{1}{2\pi i}\int\limits_0^{+i\infty}\Phi_1(w)\left(\frac{1}{w-z}-\frac{1}{w+\bar{z}}\right)dw+$$

$$+\frac{1}{2\pi i}\int\limits_{-i\infty}^{0}\Phi_3(w)\left(\frac{1}{w-z}-\frac{1}{w+\bar{z}}\right)dw-\frac{1}{2\pi i}\int\limits_0^{+\infty}\Phi_2(w)\left(\frac{1}{w-z}-\frac{1}{w+\bar{z}}\right)dw=$$

$$=-\frac{1}{2\pi i}\int\limits_0^{+i\infty}\Phi_1(w)\frac{2x}{(w-z)\,(w+\bar{z})}dw+\frac{1}{2\pi i}\int\limits_{-i\infty}^{0}\Phi_3(w)\frac{2x}{(w-z)\,(w+\bar{z})}dw-$$

$$-\frac{1}{2\pi i}\int\limits_0^{+\infty}\Phi_2(w)\frac{2x}{(w-z)\,(w+\bar{z})}dw.$$

The latter integral ([1]) is equal to zero everywhere on an imaginary axis except for, probably, point $z=0$. Obviously,

$$-\frac{1}{2\pi i}\int\limits_0^{+i\infty}\Phi_1(w)\frac{2x}{(w-z)\,(w+\bar{z})}dw=\frac{1}{\pi}\int\limits_0^{+i\infty}\Phi_1(iv)\frac{x}{((v-y)^2+x^2)}dv$$

is the Poisson integral formula, hence has angular boundary values a. e. on $\partial\mathbb{C}_+$ from $\mathbb{C}_+$ and values of the latter integral is equal to $\Phi_1(iv)$ for $v>0$ and 0 for $v<0$. Analogously we can prove that the angular boundary values on $\partial\mathbb{C}_+$ from $\mathbb{C}_+$ of the function

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{0}\Phi_3(w)\frac{2x}{(w-z)\,(w+\bar{z})}dw$$

is equal a. e. to $-\Phi_3(iv)$ for $v<0$ and 0 for $v>0$. □

Let $E^p(D)$ be the Smirnov space (see [8]) over a domain D.

Proof of Theorem 1. We claim that the function

$$P(z) = e^{-i\sigma z} \begin{cases} S(z), & z \in \mathbb{C}(0;\pi/2); \\ S(z) - \Phi_2(z), & z \in \mathbb{C}(-\pi/2;0) \end{cases}$$

is the desired function. Indeed, by Lemma 2 the angular boundary values of the function S a.e. on $\partial\mathbb{C}_+$ from $\mathbb{C}_+$ is equal to $\Phi_1(iv)$, $v > 0$, and $-\Phi_3(iv)$, $v < 0$. But $\Phi_1(iv) = -\Phi_3(iv) - \Phi_2(iv)$, $v \in \mathbb{R}$, hence the angular boundary values of the function P on $\partial\mathbb{C}_+$ from $\mathbb{C}_+$ a.e. is equal to $\Phi_1(iv)$.

From Lemma 1 we have

$$S(z) = S(z) + S(-z) =$$
$$= -\int_0^{+i\infty} f_1(w)K_1(w;z)dw + \int_{-i\infty}^{0} f_3(w)K_1(w;z)dw - \int_0^{+\infty} f_2(w)K_1(w;z)dw, \quad z \in \mathbb{C}_+,$$

where $K_1(w;z) := \frac{1}{\pi i}\frac{w}{(w-z)(w+z)} = \frac{1}{2\pi i}\left(\frac{1}{w-z} + \frac{1}{w+z}\right)$. Also

$$\frac{\pi}{2}\int_0^{+\infty} \left|K_1(it;re^{i\varphi})\right| dr = \int_0^{+\infty} \frac{|t|\,dr}{\sqrt{t^2 - 2tr\sin\varphi + r^2}\sqrt{t^2 + 2tr\sin\varphi + r^2}}dr.$$

If $t \neq 0$, then

$$\int_0^{+\infty} \frac{|t|\,dr}{\sqrt{t^2 - 2tr\sin\varphi + r^2}\sqrt{t^2 + 2tr\sin\varphi + r^2}}dr = \int_0^{+\infty} \frac{du}{\sqrt{1 - 2u\sin\varphi + u^2}\sqrt{1 + 2u\sin\varphi + u^2}}.$$

For $|\varphi| < \pi/2 - \delta$ we have

$$\int_0^{+\infty} \frac{du}{\sqrt{1 - 2u\sin\varphi + u^2}\sqrt{1 + 2u\sin\varphi + u^2}} \leq \int_0^{+\infty} \frac{du}{2\sqrt{(1 - 2u|\sin\varphi| + u^2)^2}} \leq$$
$$\leq \frac{1}{2(1 - \sin(\pi/2 - \delta))}\int_0^{+\infty} \frac{du}{1 + u^2} \leq \frac{\pi}{4(1 - \sin(\pi/2 - \delta))}.$$

Also

$$\frac{\pi}{2}\int_0^{+\infty} \left|K_1(t;re^{i\varphi})\right| dr = \int_0^{+\infty} \frac{t}{|(t-z)(t+z)|}dr =$$
$$= \int_0^{+\infty} \frac{t}{\sqrt{t^2 + 2tr\cos\varphi + r^2}\sqrt{t^2 - 2tr\cos\varphi + r^2}}dr =$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{u^2 + 2u\cos\varphi + 1}\sqrt{u^2 - 2u\cos\varphi + 1}}du < +\infty,$$

if $\varphi \in (-\pi/2; \pi/2) \setminus (-\delta; \delta)$.

Therefore by the Fubini theorem $\int_0^{+\infty} dr \int_0^{+i\infty} |\Phi_j(w) K_1(w; re^{i\varphi})| \, dw < +\infty, \ j \in \{1, 2, 3\}$. Hence

$$\sup \left\{ \int_0^{+\infty} \left| S(re^{i\varphi}) \right| dr \colon \varphi \in (-\pi/2 + \delta; -\delta) \cup (\delta; \pi/2 - \delta) \right\} < +\infty.$$

Also

$$\sup \left\{ \int_0^{+\infty} \left| \Phi_2(re^{i\varphi}) \right| e^{-2\sigma r |\sin \varphi|} dr \colon \varphi \in (-\pi/2 + \delta; -\delta) \right\} < +\infty,$$

therefore Theorem 1 is proved. □

We do not know, whether estimation (4) for the case $\delta = 0$ is true, $P \in H^1_\sigma(\mathbb{C}_+)$. Remark that in this case the assumptions of Theorem 1 are sufficient for the existence of a nontrivial solution of equation (2).

Remark. Analogously we can prove that $F_3(iy)G(iy)e^{-\sigma y}$ is an angular boundary function on $i\mathbb{R}$ of such analytic in $\mathbb{C}_+$ function P^*, that

$$\sup \left\{ \int_0^{+\infty} \left| P^*(re^{i\varphi}) \right| e^{-\sigma r |\sin \varphi|} dr \colon \varphi \in (-\pi/2 + \delta; -\delta) \cup (\delta; \pi/2 - \delta) \right\} < +\infty$$

for each $\delta \in (0; \pi/4)$.

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Ivan Franko National University of Lviv
dilnyi@ukr.net

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