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L. I. Kolyasa, A. Z. Mokhon'ko, V. D. Mokhon'ko

ASYMPTOTIC PROPERTIES OF MEROMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A NEIGHBORHOOD OF A LOGARITHMIC SINGULARITY

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We obtain asymptotic estimates of the moduli of meromorphic solutions with a logarithmic singularity at ∞ of the differential equation

$$\sum_{k+s=m} f^k f_1^s v_{ks}(z) z^{\tau_{ks}} \mathrm{Ln}^{\varkappa_{ks}} z = \sum_{|K|<m} b_K(z) f^{k_0} f_1^{k_1} \dots f_p^{k_p},$$

$$f' = f_1, \dots, f^{(p)} = f_p,\ K = (k_0, k_1, \dots, k_p),\ |K| = k_0 + k_1 + \dots + k_p;$$

$$\tau_{m-s,s} - s \leqslant \tau_{m-n,n} - n,\ s < n = \max\{s\colon k+s=m, c_{ks} \neq 0\};$$

where $v_{ks}(z), b_K(z)$ are analytic functions such that $\forall \alpha, \beta, -\infty < \alpha < \beta < +\infty$,

$$|b_K(re^{i\theta})| < r^{\tau_K},\ v_{ks}(re^{i\theta}) = c_{ks} + o(1),\ r \to +\infty,\ \alpha \leqslant \theta \leqslant \beta;\ \tau_{ks},\ \varkappa_{ks},\ \tau_K \in \mathbb{R},\ c_{ks} \in \mathbb{C}.$$

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Получены асимптотические оценки модуля мероморфных с логарифмической особой точкой в ∞ решений дифференциальных уравнений

$$\sum_{k+s=m} f^k f_1^s v_{ks}(z) z^{\tau_{ks}} \mathrm{Ln}^{\varkappa_{ks}} z = \sum_{|K|<m} b_K(z) f^{k_0} f_1^{k_1} \dots f_p^{k_p},$$

$$f' = f_1, \dots, f^{(p)} = f_p,\ K = (k_0, k_1, \dots, k_p),\ |K| = k_0 + k_1 + \dots + k_p;$$

$$\tau_{m-s,s} - s \leqslant \tau_{m-n,n} - n,\ s < n = \max\{s : k+s=m, c_{ks} \neq 0\};$$

где $v_{ks}(z),\ b_K(z)$ — аналитические функции, такие что $\forall \alpha, \beta, -\infty < \alpha < \beta < +\infty$,

$$|b_K(re^{i\theta})| < r^{\tau_K},\ v_{ks}(re^{i\theta}) = c_{ks} + o(1),\ r \to +\infty,\ \alpha \leqslant \theta \leqslant \beta;\ \tau_{ks},\ \varkappa_{ks},\ \tau_K \in \mathbb{R},\ c_{ks} \in \mathbb{C}.$$

Let us consider the differential equation

$$\sum_{k+s=m} f^k f_1^s v_{ks}(z) z^{\tau_{ks}} \mathrm{Ln}^{\varkappa_{ks}} z = \sum_{|K|<m} b_K(z) f^{k_0} f_1^{k_1} \dots f_p^{k_p}, \tag{1}$$

$$f' = f_1, \dots, f^{(p)} = f_p,\ k_0, k_1, \dots, k_p \in \mathbb{N} \cup \{0\},\ K = (k_0, k_1, \dots, k_p),$$
$$|K| = k_0 + k_1 + \dots + k_p,\ \tau_{ks},\ \varkappa_{ks} \in \mathbb{R},$$

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where $v_{ks}(z), b_K(z), z \in G = \{z\colon r_0 \leqslant |z| < +\infty\}$, such that $\forall \alpha, \beta,\ -\infty < \alpha < \beta < +\infty$,

$$|b_K(re^{i\theta})| < r^{\tau_K},\ v_{ks}(re^{i\theta}) = c_{ks} + o(1),\ r \to \infty,\ \alpha \leqslant \theta \leqslant \beta,\ r \geqslant r_0,$$

$\tau_K \in \mathbb{R},\ c_{ks} \in \mathbb{C};\ v_{ks}(z),\ b_K(z),\ z \in G$, are analytic functions, for example, for some K,

$$b_K(z) = M_K z^{\tau_K - \varepsilon} \mathrm{Ln}^{\varkappa_K} z,\ \varepsilon > 0,\ M_K \in \mathbb{C}.$$

We assume that $v_{ks}(z) \equiv 0$ if $c_{ks} = 0$ and

$$\exists k_*, s_* \in \mathbb{N} \cup \{0\}\colon k_* + s_* = m, c_{k_* s_*} \neq 0. \tag{2}$$

Denote by

$$n = \max\{s\colon k + s = m, c_{ks} \neq 0\},\ q = \min\{s\colon k + s = m, c_{ks} \neq 0\}. \tag{3}$$

Theorem 1. *Suppose that in the differential equation* (1) $\tau_{m-s,s} - s \leqslant \tau_{m-n,n} - n, s < n$, *or* $n = 0$. *If* $f(z), z \in G$, *is a meromorphic function with a logarithmic singularity at* ∞, *has the order of growth* $\mu, \mu < +\infty$, *(if in* (1) $p = 1$, *then the condition* $\mu < +\infty$ *is not necessary) and is a solution of the differential equation* (1), *then* $(\forall \varepsilon > 0)(\forall \xi, \psi, -\infty < \xi < \psi < +\infty)$ $(\exists d > 0)$ *one has*

$$(r > d \wedge \xi \leqslant \theta \leqslant \psi) \Rightarrow \ln f(re^{i\theta}) = \ln^{\upsilon+1}(re^{i\theta})\Big(\frac{y}{\upsilon + 1} + g(re^{i\theta})\Big), \tag{4}$$

where $|g(re^{i\theta})| < \varepsilon,\ \upsilon > 0,\ \mathrm{Re} y > 0,\ re^{i\theta} \notin E,\ E$ *is a set of disks with a finite sum of radii, or* $\exists \Delta \subset (r_0, +\infty)$*:*

$$\ln|f(re^{i\theta})| < \varepsilon \ln^{\upsilon+1} r,\ r > r(\theta),\ r \notin \Delta, \tag{5}$$

Δ *is a set of segments with a finite sum of lengths.*

Example 1. The function $f(z) = e^{\ln^2 z}, z \neq 0$, is a solution of the differential equation $zf' = 2f \ln z$, and satisfies (4).

Example 2. The Weierstrass elliptic function $\wp(z), z \in \mathbb{C}$, is a meromorphic function which has the order of growth $\mu = 2$ ([1, V.2, p. 422]), and the function $\wp(z), z \in \mathbb{C}$, is a solution of the differential equation ([1, V.2, p. 362]) $4f^3 = (f')^2 + g_2 f + g_3$. This differential equation is an equation of form (1) with $n = 0$ (see (3)) (in the left-hand side of this equation only one summand $4f^3$ has degree $m = 3$ for the functions f and f', therefore $v_{30}(z) \equiv 4 = c_{30}$. Hence, for all other summands we have

$$f^k f_1^s, k + s = 3,\ s \geqslant 1,\ v_{ks}(z) \equiv 0 = c_{ks},\ n = \max\{s\colon k + s = 3, c_{ks} \neq 0\} = 0$$

is true). For the Weierstrass function a sharper estimate than (5) will be proved

$$|\wp(z)| \leqslant |z|^{\nu+\varepsilon},\ z \in \mathbb{C} \setminus E,\ |z| > d,\ \nu = 12,\ \varepsilon > 0,$$

where E is the set of disks with a finite sum of radii.

Consider the differential equations of Painlevé $f'' = 6f^2 + z$ and $f'' = 2f^3 + zf + a$, $a = \mathrm{const}, z \in \mathbb{C}$. All solutions of these differential equations $f(z), z \in \mathbb{C}$ are transcendental meromorphic functions (*Painlevé a transcendent*) ([2, p. 189]) of finite order of growth

$\mu < \infty$ ([3]). Rewrite these equations in the form (1): $6f^2 = f'' - z$ and $2f^3 = f'' - zf - a$ with $n = 0$ (see (3)).

For all solutions of the first and second Painlevé's equations we will prove the following estimate

$$|f(z)| \leqslant |z|^{4\mu+\varepsilon},\ z \in \mathbb{C} \setminus E,\ |z| > d,\ \varepsilon > 0,$$

where E is a set of disks with a finite sum of radii. According to the above estimate the first and the second Painlevé equations have no entire transcendental solutions.

Let us recall the definition of a meromorphic function with a logarithmic singularity at ∞. By A_l we denote the set of analytic functions in $G = \{z\colon r_0 \le |z| < \infty\}$ for which ∞ is the unique singular point, namely a logarithmic singular point. The set A_l is a commutative ring without divisors of zero (complete ring). The field of quotients of the ring A_l is denoted by M_l (each complete ring can be embedded in some field ([4, p. 52, 58])) $A_l \subset M_l$. If $f \in A_l$, then we shall say that $f(z), z \in G$, is an analytic function with an isolated logarithmic singular point at ∞. If $f \in M_l$, then the function $f(z), z \in G$, is called a *meromorphic function with a logarithmic singularity at* ∞.

In ([5, p. 12]) an equivalent definition of a meromorphic function is considered. This definition is based on a concept of analytic extension.

Let $f \in M_l$. For any $\alpha, \beta,\ -\infty < \alpha < \beta < +\infty$ (it is possible that $\beta - \alpha > 2\pi$) we denote by $f(z)$,

$$z \in g_{\alpha,\beta} = \left\{z = re^{i\theta}\colon \alpha \le \theta \le \beta,\ r_0 \le r < +\infty\right\}, \tag{6}$$

a single-valued branch of the function $f \in M_l$ (see [5, p. 12]).

We consider Nevanlinna's characteristics of the function $f(z), z \in g_{\alpha,\beta}$ ([6, p. 40]). Write $\ln^+ x = \max(\ln x, 0)$, $x \ge 0$; $k = \pi/(\beta - \alpha) > 0$. Let $b_l = |b_l| \exp(i\theta_l)$ be the poles of the function $f(z),\ z \in g_{\alpha,\beta}$. We put

$$A_{\alpha,\beta}(r,f) = \frac{k}{\pi}\int_{r_0}^{r}\left(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}}\right)\left(\ln^+\left|f\left(te^{i\alpha}\right)\right| + \ln^+\left|f\left(te^{i\beta}\right)\right|\right)dt,$$

$$B_{\alpha,\beta}(r,f) = \frac{2k}{\pi r^k}\int_{\alpha}^{\beta}\ln^+\left|f\left(re^{i\theta}\right)\right|\sin k(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2k\int_{r_0}^{r} c_{\alpha,\beta}(t,f)\left(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\right)dt, \tag{7}$$

where

$$c_{\alpha,\beta}(t,f) = c_{\alpha,\beta}(t,\infty) = \sum_{r_0<|b_l|\le t,\ \alpha\le\theta_l\le\beta} \sin k(\theta_l - \alpha),$$

is the counting function of the poles; each pole is counted according to its multiplicity,

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f), \quad r_0 \le r < \infty. \tag{8}$$

For any single-valued branch $f(z), z \in g_{\alpha,\beta}$ of the function $f \in M_l$ we define

$$\rho_{\alpha,\beta} = \varlimsup_{r\to+\infty}\frac{\ln^+ S_{\alpha,\beta}(r,f)}{\ln r}. \tag{9}$$

The value $\mu=\mu[f]=\sup\{\rho_{\alpha,\beta}\colon -\infty<\alpha<\beta<+\infty\}$ is called the order of growth of the function $f(z), z\in G$.

In particular if $\rho_{\alpha_j,\alpha_j+\pi}$ is the order of growth of the single-valued branch $f(z),\ z\in g_{\alpha_j,\alpha_j+\pi}=\{re^{i\theta}\colon r_0\leqslant r, \alpha_j\leqslant\theta\leqslant\alpha_j+\pi\}$, of the function $f\in M_l$, then $\rho_{\alpha_j,\alpha_j+\pi}\leqslant\mu$.

Preliminary construction. Let $f\in M_l$ be a solution of the differential equation (1); if in (1) $p=1$, then the function f has the order of growth $\mu, 0\leqslant\mu<+\infty$ ([7]); if $p\geqslant 2$, then (according to the assumption of the theorem) the function f has the order of growth $\mu<+\infty$.

Let $\{c_q\}$ be the set of all zeros and poles of the meromorphic solution $f(z),\ z\in g_{\alpha,\alpha+\pi}$ of the differential equation (1). For any $\sigma>0$ and for each $c_q=|c_q|e^{i\theta_q}$ we consider the disk with the center at c_q and radius $\delta_q=|c_q|^{-\mu-1-\frac{\sigma}{5}}\sin(\theta_q-\alpha);\alpha<\theta_q<\alpha+\pi,\sigma>0$. Let E_* be the set of points that belong to these disks. Then by [8]

$$\left|\frac{f_j(z)}{f(z)}\right|<\frac{K|z|^{2j(\mu+1+\frac{\sigma}{4})}}{\sin^{2j}(\theta-\alpha)},\ z=re^{i\theta}\in g_{\alpha,\alpha+\pi}\backslash E_*,\ \sigma>0;\ \sum\delta_q<M=\text{const}, c_q\in\{c_q\}. \quad (10)$$

For each $c_q\in\{c_q\}$ we consider the interval $[|c_q|-\delta_q,|c_q|+\delta_q]$. Let Δ be the set of points that belong to these intervals. According to (10) E_* is the set of disks with a finite sum of radii, mes$\Delta\leqslant\sum 2\delta_q<+\infty$.

We divide (1) by $f^m(z)$. After simple transformations and new notation for coefficients and exponents we may rewrite differential equation (1) in the form ($z\in g_{\alpha,\alpha+\pi}$)

$$\left(\frac{zf'(z)}{f(z)}\right)^n+\sum_{j=1}^{n-q}\left(\frac{zf'(z)}{f(z)}\right)^{n-j}v_j(z)z^{d_j}\ln^{h_j}z=\omega(z),\ v_j(z)=c_j+o(1), \quad (11)$$

$$\omega(z)=\sum_{|K|\leqslant m-1}b_K(z)z^{n-\tau_{m-n,m}}(\ln z)^{-\varkappa_{m-n,m}}\frac{(f_1/f)^{k_1}\dots(f_p/f)^{k_p}}{f^{m-|K|}}, \quad (12)$$

$c_j\in\mathbb{C}, z\to\infty$. In particular, in (11)

$$d_j=\tau_{m-n+j,n-j}-\tau_{m-n,n}+j\leqslant 0,\ j\in\{1,2\dots,n-q\},\ d_0\overset{\text{def}}{=}0, \quad (13)$$

(according to our assumptions $\tau_{m-s,s}-s\leqslant\tau_{m-n,n}-n, s<n, s=n-j$). Set

$$\Omega=\{z=te^{i\theta}\colon t>(\sin(\theta-\alpha))^{-\frac{4}{\sigma}},\alpha<\theta<\alpha+\pi\},\sigma>0;$$
$$\chi=\max\{k_1+2k_2+\ldots+pk_p\colon |K|<m\}. \quad (14)$$

For any $\delta, 0<\delta<\frac{\pi}{2}$, there exists $d=d(\delta)$ such that

$$Q=\{z\colon |z|\geqslant d,\alpha+\delta\leqslant\arg_\alpha z\leqslant\alpha+\pi-\delta\}\subset\Omega.$$

Considering (10), (14), we obtain ($c=\text{const}$)

$$\left|\frac{f_1(z)}{f(z)}\right|^{k_1}\dots\left|\frac{f_p(z)}{f(z)}\right|^{k_p}<\frac{c|z|^{(2\mu+2+\frac{\sigma}{2})\chi}}{\sin^{2\chi}(\arg_\alpha z-\alpha)}<c|z|^{(2\mu+2+\sigma)\chi},\ z\in\Omega\setminus E_*. \quad (15)$$

Using the new notation

$$\frac{zf'(z)}{f(z)}=L(z),\ c_0=1,\ d_0=0,\ h_0=0,\ v_0(z)\equiv 1,\ z\in g_{\alpha,\alpha+\pi}, \quad (16)$$

we may rewrite differential equation (11) in the form ($q \geqslant 0$)

$$L^n(z) + \sum_{j=1}^{n-q} L^{n-j}(z) v_j(z) z^{d_j} (\ln z)^{h_j} = \omega(z),\ c_{n-q} \neq 0. \tag{17}$$

This equation is said to be *characteristic* for (1). We consider the set

$$F = \{j \colon v_j(z) = c_j + o(1) \not\equiv 0,\ z \in g_{\alpha,\alpha+\pi},\ j \in \{1, 2, \dots, n-q\}\}.$$

Suppose that

$$\exists j \in F \colon d_j > 0 \vee d_j = 0, h_j > 0. \tag{18}$$

1°. If in equation (17)

$$q = 0 \vee q \geqslant 1, d_{n-q} < \max_{j \in F} d_j,$$

then we denote (see (17)) $H = \{(j, d_j) \colon j \in F \cup \{0\}\}$ which is a subset of the plane.

2°. If in equation (17)

$$q \geqslant 1, d_{n-q} = \max_{j \in F} d_j \overset{(18)}{\geqslant} 0,$$

then we append to the set H the point $(n, d_n), d_n \overset{\text{def}}{=} -1$, and obtain the set $\widetilde{H} = \{(j, d_j) \colon j \in F\} \cup \{(0, d_0)\} \cup \{(n, d_n)\}$.

Consider the case 1°. By the points of H, let us construct the Newton diagram (N. D.) of equation (17) (of the set H) and consider the convex hull of the set H. The polygon is the boundary of the hull. The points $(0, d_0)$ and $(n-q, d_{n-q})$ divide the polygon into two polygonal lines. The top line is the N. D.. Let the *vertices of the N. D.* have abscissas

$$i_0, i_1, \dots, i_T,\ 0 = i_0 < i_1 < \ldots < i_T = n - q, \tag{19}$$

then

$$\sum_{s \in \{1,\dots,T\}} (i_s - i_{s-1}) = i_T - i_0 = n - q,\ i_T = n - q. \tag{20}$$

We denote

$$\rho_s = \frac{d_{i_s} - d_{i_{s-1}}}{i_s - i_{s-1}},\ s \in \{1, 2, \dots, T\}, \tag{21}$$

where ρ_s are *angular coefficients of the segments of the N. D.*, $\rho_1 > \rho_2 > \ldots > \rho_T$. We set

$$\rho_s(n - j) + d_j \overset{\text{def}}{=} l_{j,s}, j \in F. \tag{22}$$

The properties of the N. D. imply for the points of the set H

$$l_{i_{s-1},s} = l_{i_s,s} = \max_{j \in F} l_{j,s} \overset{\text{def}}{=} l_s. \tag{23}$$

Let a constant A satisfy the condition (see (21), (23))

$$A > \max(0, \max_{s \in \{1,2,\dots,T\}} -l_s). \tag{24}$$

If the conditions in 2° hold then the point $(n, d_n), d_n \overset{\text{def}}{=} -1$, is added to the set H. As a result, we have the set $\widetilde{H} = \{(j, d_j) \colon j \in F \cup \{0\}\} \cup (n, d_n)$. In this case we construct the

N. D. by the points of $\widetilde{H}$ of equation (17) and consider the convex hull of $\widetilde{H}$. The polygon is the boundary of the hull. The points $(0, d_0)$ and (n, d_n) divide the polygon into two polygonal lines. The top line is the N. D. of the set $\widetilde{H}$ (of equation (17)). The vertices of the N. D. of the set $\widetilde{H}$ have abscissas

$$i_0, i_1, \ldots, i_T, i_{T+1}, 0 = i_0 < i_1 < \ldots < i_T = n - q < i_{T+1} = n,$$

that are different from the abscissas of the N. D. vertices of the set H by just one additional point $i_{T+1} = n$. Similarly to (21) we consider angular coefficients of the edges of the N. D. of the set $\widetilde{H}$

$$\rho_s = \frac{d_{i_s} - d_{i_{s-1}}}{i_s - i_{s-1}}, \; s \in \{1, 2, \ldots, T, T+1\}, \; \rho_1 > \rho_2 > \ldots > \rho_T > \rho_{T+1}. \tag{25}$$

Due to $2°$ and the properties of the convex hull we obtain that $\rho_1 > \rho_2 > \ldots > \rho_T \geqslant 0$; $\rho_{T+1} < 0$. Similarly to (22) we set

$$\rho_s(n-j) + d_j = l_{j,s}, \; s \in \{1, \ldots, T, T+1\}, \; j \in F. \tag{26}$$

From the properties of the N. D. we get for the points of the set $\widetilde{H}$ that

$$l_{i_{s-1},s} = l_{i_s,s} = \max_{j \in F} l_{j,s} \stackrel{\text{def}}{=} l_s, \; s \in \{1, \ldots, T, T+1\}. \tag{27}$$

Therefore the estimate of the number A (24) becomes

$$A > \max(0, \max_{s \in \{1,\ldots,T\}+1} -l_s). \tag{28}$$

In [9] the following lemma is proved.

Lemma 1. *Let the coefficients $\omega(z), v_j(z), z \in \Phi \subset g_{\alpha,\beta}, \; j \in \{1, 2, \ldots, \; n-q\}$, in equation (17) be defined on an unbounded set Φ, such that $|\omega(z)| < |z|^{-A}$, (where the constant A satisfy (24) in the case $1°$ and (28) in the case $2°$), as well as: a) $v_j(z) \equiv 0$ if $c_j = 0$, b) $\forall \varepsilon > 0 \exists \sigma = \sigma(\delta) \forall z \in \Phi \cap \{z = re^{i\theta} \colon r \geqslant \sigma, \alpha \leqslant \theta \leqslant \beta\} \Rightarrow$*

$$v_j(z) = (c_j + g_j(z)), c_j \in \mathbb{C}, |g_j(z)| < \frac{\delta^2}{2n}.$$

If $\forall j \in F = \{j \colon v_j(z) \not\equiv 0, z \in \Phi, j \in \{1, 2, \ldots, n-q\}\}$ in equation (17): the degrees $d_j < 0 \vee d_j = 0, h_j \leqslant 0$, then all solutions of equation (17) are bounded in $\Phi \cap \{z = re^{i\theta} \colon r \geqslant \sigma, \alpha \leqslant \theta \leqslant \beta\}$.

Let $\exists j \in F \colon d_j > 0 \vee d_j = 0, h_j > 0$. Then $\rho_s, s \in \{1, \ldots, T\}$, are angular coefficients of the segments of the Newton diagram of the set $H = \{(j, d_j) \colon j \in F \cup \{0\}\}$. Moreover, there exist integers $\xi_{s,0}, \xi_{s,1}, \ldots, \xi_{s,p_s}, s \in \{1, \ldots, T\}$, $0 = \xi_{1,0} < \xi_{1,1} < \ldots < \xi_{1,p_1} < \ldots < \xi_{s,0} < \xi_{s,1} < \ldots < \xi_{s,p_s} < \ldots < \xi_{T,0} < \xi_{T,1} < \cdots < \xi_{T,p_T} = n - q$, and corresponding numbers υ_{sk}, defined by the points $(j, h_j), j \in F$, such that there exist $\xi_{s,k} - \xi_{s,k-1}$ solutions

$$x_j(z) = (y_j + o(1)) z^{\rho_s} (\ln z)^{\upsilon_{sk}}, \; y_j \neq 0, \; z \in \Phi, \; z \to \infty,$$
$$s \in \{1, \ldots, T\}, \; k \in \{1, 2, \ldots, p_s\}, \; j \in \{\xi_{s,k-1} + 1, \; \xi_{s,k-1} + 2, \ldots, \xi_{s,k}\}, \tag{29}$$

of equation (17). *The total sum of such solutions is*

$$\sum_{s\in\{1,\dots,T\}}\sum_{k\in\{1,2,\dots,p_s\}}(\xi_{s,k}-\xi_{s,k-1})=\sum_{s\in\{1,\dots,T\}}(\xi_{s,p_s}-\xi_{s,0})=\sum_{s\in\{1,\dots,T\}}(i_s-i_{s-1})=i_T-i_0=n-q$$

(see (20)*). Therefore, if* $q=0$, *then all* n *solutions of equation* (17) *are of form* (29). *In the case* $q\geqslant 1, d_{n-q}<\max_{j\in F}d_j$, q *solutions of* (17) *are of the form*

$$\begin{gathered}x_j(z)=o(1)z^{\rho_T}(\ln z)^{\upsilon_{T,p_T}},\ \rho_T<0,\ z\in\Phi,\ z\to\infty,\\ j\in\{\xi_{T,p_T}+1,\xi_{T,p_T}+2,\dots,n\};\ \xi_{T,p_T}=n-q,\end{gathered}\tag{30}$$

and in the case $q\geqslant 1, d_{n-q}=\max_{j\in F}d_j$, q *solutions of* (17) *are of the form*

$$x_j(z)=o(z^{\rho_{T+1}}),\ z\to\infty,\ \rho_{T+1}<0,\ j\in\{n-q+1,n-q+2,\dots,n\}.\tag{31}$$

Lemma 2. *Let conditions of Lemma 1 hold true and* $\Phi\subset g_{\alpha\beta},\Phi$ *be an unbounded closed (open) set. By* Φ_0 *we denote the connected component of the set* Φ. *If* $\forall j\in F=\{j\colon g_j(z)\not\equiv 0, z\in\Phi\}$ *in equation* (17) *the degrees* $d_j<0\vee d_j=0, h_j\leqslant 0$, *then all solutions of equation* (17) *are bounded in* $\Phi\cap\{z=re^{i\theta}\colon r\geqslant\delta,\alpha\leqslant\theta\leqslant\beta\}$.

$\exists j\in F\colon d_j>0\vee d_j=0, h_j>0$. *Let a continuous (analytic) function* $x(z),\ z\in\Phi$ *is a solution of equation* (17). *If in* (17) $q=0$, *then* $\forall\varepsilon>0\ \exists r_0$

$$\begin{gathered}x(z)=(y+u(z))z^{\rho}\ln^{\upsilon}z,\ y\neq 0,\ |u(z)|<\varepsilon,\ z\in\Phi_0,\ |z|\geqslant r_0,\\ \rho,\upsilon\in\mathbb{R},\ y\in\mathbb{C},\ y=y(\Phi_0),\ \rho=\rho(\Phi_0),\ \upsilon=\upsilon(\Phi_0),\end{gathered}\tag{32}$$

where $u(z)$ *is some continuous (analytic) function;* y, ρ, υ *do not change if* $z\in\Phi_0$, $|z|\geqslant r_0$; y,ρ,υ *are one of the numbers* y_j,ρ_s,υ_{sk}, *respectively, defined in Lemma 1. If* $q\geqslant 1$, *then statement* (32) *is true on* Φ_0 *or*

$$|x(z)|<|z|^{\zeta+\varepsilon},\ \zeta+\varepsilon<0,\ z\in\Phi_0,\ |z|\geqslant r_0,\tag{33}$$

holds, $\zeta=\rho_T\vee\zeta=\rho_{T+1}$ *(see* (30), (31)*),* $\varepsilon>0$ *is sufficiently small.*

Proof of Lemma 2. Let in (17) $q=0$. According to Lemma 1 all the solutions of equation (17) has form (29). The continuous function $x(z), z\in\Phi$, is the solution of equation (17). Hence, at each point $z\in\Phi, |z|\geqslant r_0$, the function $x(z), z\in\Phi$, coincides at least with one of the solutions (29)

$$x(z)=(y_j+o(1))z^{\rho_s}(\ln z)^{\upsilon_{sk}},\ y_j\neq 0,\ z\in\Phi.\tag{34}$$

If $x_t(z)=(y_t+o(1))z^{\rho_l}(\ln z)^{\upsilon_{ln}},\ y_t\neq 0,\ z\in\Phi$, is one of the solutions (29) of equation (17) and r_0 is sufficiently large, then $x(z)\neq x_t(z), z\in\Phi, |z|\geqslant r_0$, when $|y_j-y_t|+|\rho_s-\rho_l|+|\upsilon_{sk}-\upsilon_{ln}|>0$. Hence, taking into consideration continuity of $x(z), z\in\Phi$, and connectedness of the component $\Phi_0,\Phi_0\subset\Phi$, we obtain that (34) is true for all $z\in\Phi_0, |z|\geqslant r_0$. Hence, (32) holds. □

The solution $f\in M_l$ of equation (1) has the order of growth $\mu,\mu<+\infty$; the angular coefficients ρ_s of the segments of the N. D. are defined in (21), (25). Let $\mu_0=\max(\mu,\rho_1)$. We denote (see (28) and (24))

$$\begin{gathered}l=\max_{s\in\{1,\dots,T\}}-l_s\vee l=\max_{s\in\{1,\dots,T\}+1}-l_s,\\ \nu=\max(\mathfrak{y},\max_K\{\tau_K+n-\tau_{m-n,m}+2(\mu_0+1)\chi+l\}),\end{gathered}\tag{35}$$

where a constant $\mathfrak{y}$ is further determined by the solutions of characteristic equation (see (29), (39)). Let us consider the sets

$$\Phi = \{z\colon z \in \Omega \setminus E_*, |f(z)| \geqslant |z|^{\nu+\varepsilon}\},\ \Phi_1 = \{z\colon z \in \Omega \setminus E_*, |f(z)| < |z|^{\nu+\varepsilon}\}. \tag{36}$$

On the set Φ the assumptions of Lemma 1 are satisfied ($|b_K(z)| < |z|^{\tau_K}$)

$$\begin{aligned}|\omega(z)| &\overset{(12)}{=} \sum_{|K|\leqslant m-1} |b_K(z) z^{n-\tau_{m-n,m}} (\ln z)^{-\kappa_{m-n,m}}| \frac{|f_1/f|^{k_1} \dots |f_p/f|^{k_p}}{|f|^{m-|K|}} \leqslant \\ &\leqslant \sum_{|K|\leqslant m-1} |z|^{n-\tau_{m-n,m}+\tau_K+\frac{\sigma}{9}} \frac{|f_1/f|^{k_1} \dots |f_p/f|^{k_p}}{|f|} \overset{(15)}{\leqslant} \\ &\leqslant \sum_{|K|\leqslant m-1} c|z|^{n-\tau_{m-n,m}+\tau_K+(2\mu+2+\sigma)\chi+\frac{\sigma}{9}} \frac{1}{|f|} \overset{(36)}{\leqslant} \\ &\leqslant \sum_{|K|\leqslant m-1} |z|^{n-\tau_{m-n,m}+\tau_K+(2\mu_0+2+\sigma)\chi+\frac{\sigma}{8}-\nu-\varepsilon} \overset{(35)}{\leqslant} |z|^{-l+\sigma\chi+\frac{\sigma}{7}-\varepsilon},\end{aligned} \tag{37}$$

(in (10) we assume that $\sigma < \frac{7\varepsilon}{7\chi+1}$).

If in equation (1) $n = 0$ (see (3)) (hence, corresponding characteristic equation (17) does not depend on L), then in the left-hand side of (1) only one summand $f^m v_{m0}(z) z^{\tau_{m0}} \mathrm{Ln}^{\varkappa_{m0}} z$, $v_{m0}(z) = c_{m0} + o(1)$, has the degree m in f and f'. Then there exists $d > 0$ such that $\Phi \cap \{z\colon |z| > d\} = \varnothing$. If we assume the contrary, then equation (17) has the form $c_{m0} + o(1) = o(1)$, $z \in \Phi$. From here we obtain that $c_{m0} = 0$ which contradicts the assumption (2). Then from (36) it follows

$$|f(z)| < |z|^{\nu+\varepsilon},\ z \in \Omega \setminus E_*, \tag{38}$$

namely, in this case we obtain a sharper estimate than (5). From inequality (38) assertions formulated in example 2 follow.

Let us consider that (17) depend on L ($n \geqslant 1$).

Let Φ_0 be an arbitrary connected components of Φ, $\Phi_0 \subset \Phi$ (see (36)).

Assume for definiteness in (17) $q = 0$. According to Lemma 2 for the continuous function $L(z) = \frac{zf'(z)}{f(z)}$ which is a solution of equation (17): $\forall \delta > 0 \exists r_0$ such that

$$\begin{gathered}\frac{f'(z)}{f(z)} = (y + u(z)) z^{\rho-1} \ln^{\upsilon} z,\ y \neq 0,\ |u(z)| < \frac{\delta}{10},\ z \in \Phi_0,\ |z| \geqslant r_0, \\ \rho, \upsilon \in \mathbb{R},\ y \in \mathbb{C},\ y = y(\Phi_0),\ \rho = \rho(\Phi_0),\ \upsilon = \upsilon(\Phi_0),\end{gathered} \tag{39}$$

is true, where $u(z)$ is some analytic function; y, ρ, υ do not change for $z \in \Phi_0$, y, ρ, υ are one from the numbers $y_j, \rho_s, \upsilon_{sk}$ respectively, defined in Lemma 1.

Let $\mathfrak{y}$ be the greatest possible values of $|y|$ in (39) (see (35)).

Proof of Theorem 1. According to (13) inequalities $0 \geqslant \rho_1 > \rho_2 > \ldots > \rho_T$ hold for angular coefficients of the segments of the N.P.L, defined in (17) (see (21)). Hence, the angular coefficients are not positive. One of the conditions of (39) is true on the set Φ_0. We assume $r_0 > 0$ such that $f(r_0 e^{i\theta}) \neq 0, \infty, \alpha \leqslant \theta \leqslant \alpha + \pi$. Thus

$$0 < c < |f(r_0 e^{i\theta})| < C,\ \alpha \leqslant \theta \leqslant \alpha + \pi,\ c, C = \mathrm{const}. \tag{40}$$

Choose some $\varphi,\ \alpha < \varphi < \alpha+\pi$. Suppose that $z_0 = r_0e^{i\varphi} \notin E_*$. Let us denote by $S(\varphi)$ the curve obtained when the point z moves from the point z_0 along the ray $\{z\colon z = re^{i\varphi}, r \geqslant r_0\}$ enveloping the disks of the set E_*, which are along the arcs $\{z\colon |z - c_q| = \delta_q\}$ (see (10)) so that $\arg_{\alpha-\pi} z \geqslant \varphi$, where $\alpha - \pi \leqslant \arg_{\alpha-\pi} z < \alpha + \pi$.

If $\Phi \cap S(\varphi) = \varnothing$, then from (36) it follows

$$|f(z)| < |z|^{\nu+\varepsilon},\ z \in S(\varphi). \tag{41}$$

Let $\Phi \cap S(\varphi) \neq \varnothing$. The set $\Phi \cap S(\varphi)$ is a union of connected components ω_t such that

$$|f(z)| \geqslant |z|^{\nu+\varepsilon},\ z \in \omega_t, \tag{42}$$

Moreover, if z_{1t} is the initial point and z_{2t} is the ending point of ω_t and $|z_{1t}| > r_0$, $|z_{2t}| < +\infty$, then

$$|f(z_{1t})| = |z_{1t}|^{\nu+\varepsilon},\ |f(z_{2t})| = |z_{2t}|^{\nu+\varepsilon}. \tag{43}$$

For $z \in \omega_t$ (39) holds, $y = y(t),\ \rho = \rho(t),\ v = v(t)$. We denote by $\{\omega_t\}$ the set of all ω_t on $S(\varphi)$.

By w_t^-, w_t^0 we denote such segments $w_t \in \{w_t\}$, for which equality (39) holds true with $\rho < 0$, $\rho = 0$, respectively.

Let $\omega_{[z_{1t},z_{2t}]}$ be a parameterization of the "segment" ω_t from z_{1t} to $z_{2t}, \omega_{[z_{1t},z_{2t}]}\colon z = \lambda(s)$, $0 \leqslant s \leqslant 1, z_{1t} = \lambda(0), z_{2t} = \lambda(1)$; $[\omega_{[z_{1t},z_{2t}]}] = \omega_t$ be the carrier. According to the definition of ω_t we choose bijective mapping of the segment $[0,1]$ onto ω_t as the parameterization of the $\omega_{[z_{1t},z_{2t}]}$. This curve consist of the segments of a ray $\{z\colon z = re^{i\varphi}, r \geqslant r_0\}$ (denoted by γ) and arcs of the circles (denoted by ℓ). From the construction ω_t it follows that

$$z = \lambda(s) \in \gamma, s \nearrow\Rightarrow |z| \nearrow, \tag{44}$$

when the point $z, z \in \omega_t$, moves along ω_t on the segment of the line γ.

The sequence $\{c_q\}$ of zeros and poles $f(z), z \in g_{\alpha,\alpha+\pi}$, does not have finite accumulation points and in (10) the series $\sum_{c_q\in\{c_q\}} \delta_q$ is convergent, therefore $\forall\delta > 0\ \exists r(\delta) > 0$

$$\operatorname{mes}\{\Delta \cap [r(\delta), +\infty)\} \leqslant \sum_{c_q\in\{c_q\},\ |c_q|>r(\delta)} 2\delta_q < \frac{\delta}{2\pi}. \tag{45}$$

From (45) and the definition of Δ it follows that

$$\Delta = \bigcup_{j=1}^{\infty}[x_j, y_j], x_j < y_j < x_{j+1}, \sum_{x_j>r(\delta)} y_j - x_j < \frac{\delta}{2\pi}, \tag{46}$$

where

$$\{z\colon y_j < |z| < x_{j+1}\} \cap \ell = \varnothing. \tag{47}$$

Let the point z_{1t} of the "segment" ω_t is such that $y_j < |z_{1t}| < x_{j+1}$; then from (47) it follows that $z_{1t} \in \gamma$; considering the definition of $S(\varphi)$ and (44) we obtain $y_j < |z_{1t}| < x_{j+1} \Rightarrow \forall z \in \omega_t\colon |z_{1t}| \leqslant |z|$.

If $r(\delta) < x_j \leqslant |z_{1t}| \leqslant y_j$, then from (45) it follows that the total length of arcs ℓ is less than δ; however, taking into account (46), $y_j - x_j < \frac{\delta}{2\pi}, |z_{1t}| - \frac{\delta}{2\pi} < x_j$. Since $\{z\colon y_{j-1} <$

$|z| < x_j\} \cap \ell = \varnothing$ from conditions $x_j \leqslant |z_{1t}| \leqslant y_j$ it follows that $\forall z \in \omega_t \colon x_j \leqslant |z|$; therefore $|z_{1t}| - \frac{\delta}{2\pi} < x_j \leqslant |z|$. Hence,

$$y_j < |z_{1t}| < x_{j+1} \Rightarrow \forall z \in \omega_t \colon |z_{1t}| \leqslant |z|, \;\; x_j \leqslant |z_{1t}| \leqslant y_j \Rightarrow \forall z \in \omega_t \colon |z_{1t}| - \frac{\delta}{2\pi} < |z|. \quad (48)$$

Similarly

$$\begin{gathered} y_m < |z_{2t}| < x_{m+1} \Rightarrow \forall z \in \omega_t \colon |z| \leqslant |z_{2t}|, \\ x_m \leqslant |z_{2t}| \leqslant y_m \Rightarrow \forall z \in \omega_t \colon |z| \leqslant y_m < |z_{2t}| + \frac{\delta}{2\pi}. \end{gathered} \quad (49)$$

According to (10), the total length ℓ is less than $2\pi M$. From (39) it follows that for $\zeta \in w_t^-$,

$$\left| \frac{f'(\zeta)}{f(\zeta)} \right| < \left(|y| + \frac{\delta}{10} \right) \frac{1}{|\zeta|}, \; \zeta \in w_t^-, \quad (50)$$

is valid. Integrating (39) along w_t^- and taking the real parts, using (50), (48), (49) and the fact that $|z| \geq r_{1t} \geq r_0$, r_0 is sufficiently large, $\delta > 0$ is sufficiently small, we obtain

$$\begin{gathered} \ln \left| \frac{f(z)}{f(z_{1t})} \right| \leq \left| \int_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta + \int_\ell \frac{f'(\zeta)}{f(\zeta)} d\zeta \right| < \left(|y| + \frac{\delta}{10} \right) \left(\int_{r_{1t}-\delta}^{|z|+\delta} \frac{dx}{x} + \frac{2\pi M}{r_{1t} - \delta} \right) < \\ < \left(|y| + \frac{\delta}{9} \right) \ln |z|, \;\; z \in w_t^-, \;\; |z| \geq r_0, \;\; \delta < \varepsilon. \end{gathered} \quad (51)$$

If $r_{1t} > r_0$, then (43), (51) imply it follows that

$$\ln |f(z_{1t})| = (\nu + \varepsilon) \ln r_{1t}, \;\; \ln |f(z)| \leq \left(|y| + \frac{\delta}{9} \right) \ln |z| + (\nu + \varepsilon) \ln r_{1t},$$

therefore, taking into account (35) and $|y| \leq \nu$, $\delta < \varepsilon$, we obtain

$$\ln |f(z)| \leq (\nu + 2\varepsilon) \ln r, \;\; z \in w_t^-, \;\; z > r(\varphi). \quad (52)$$

If $r_1 = r_0$, then from (40), (35), (42), (51) we obtain $\ln |f(z_{1t})| \leq \ln C$ and

$$(\nu + \varepsilon) \ln r \leq \ln |f(z)| < \left(|y| + \frac{\delta}{9} \right) \ln |z| + \ln C, \;\; r_{1t} \leq |z| \leq r_{2t}, \;\; |y| \leq \nu, \;\; \delta < \varepsilon.$$

It is possible only if $r_{2t} < r_*$, where $r_* = \text{const}$. Finally, in this case (52) holds.

For any θ_1, θ_2, $\alpha - \pi < \theta_1 < \theta_2 < \alpha + \pi$, there exists φ, $\theta_1 < \varphi < \theta_2$

$$\{z \colon \arg_{\alpha-\pi} z = \varphi = \text{const}, \quad z \geq d\} \cap E_* = \varnothing, \quad d = \frac{2\pi M}{\theta_2 - \theta_1}. \quad (53)$$

Therefore, if Π is a set of these values φ, $\alpha < \varphi < \alpha + \pi$, for which (53) holds, then the set Π is dense on $(\alpha, \alpha + \pi)$.

Let $\varphi \in \Pi$. Choose $r_0 > d$ such that $f(r_0) \neq 0, \infty$, $\alpha \leq \theta \leq \alpha + \pi$. Thus (40), (53) hold; the curve $S = S(\varphi)$, defined above, is the ray $S(\varphi) = \{z \colon \arg_{\alpha-\pi} z = \varphi = \text{const}, \quad z \geq d\}$, and the part $\omega_t \subset S(\varphi)$, defined in (42), (43), is a segment of line. By ω_t^- we denote such segments ω_t for which equality (39) holds with $\rho < 0$; let ω_t^0 be the segments ω_t for which equality (39) with $\rho = 0$ holds. If segment $w_t^- \subset S(\varphi)$ exists, then from (39) it follows that

$$\left|\frac{f'(\zeta)}{f(\zeta)}\right| \le \left(|y| + \frac{\delta}{9}\right)\frac{1}{|\zeta|},\ \zeta \in w_t^-. \tag{54}$$

Integrating (39) along the arc $w^-_{[z_{1t},z]}$ of the segment $w^-_{[z_{1t},z_{2t}]}$: $z = te^{i\varphi}$, $|z_{1t}| \le t \le |z_{2t}|$, and extracting the real part and taking into account (54) we obtain

$$\ln\left|\frac{f(z)}{f(z_{1t})}\right| < \left(|y| + \frac{\delta}{9}\right)\int\limits_{r_{1t}}^{r}\frac{ds}{s} < \left(|y| + \frac{\delta}{9}\right)\ln\frac{r}{r_{1t}},\ r_{1t} \le |z| \le r_{2t},\ |z| = r,\ z \in w_t,$$

therefore

$$\left|\frac{f(z)}{f(z_{1t})}\right| < \left(\frac{r}{r_{1t}}\right)^{|y|+\frac{\delta}{9}},\ 0 < \delta < \varepsilon. \tag{55}$$

It will be proved that in this case

$$\exists r_*\colon \{z\colon\ \arg_{\alpha-\pi} z = \varphi \in \Pi,\ |z| \ge r_*\}\bigcap w_t^- = \varnothing \tag{56}$$

is true. Let there exists the "segment" $\omega_t^0 \subset S(\varphi)$. Since $\omega_t^0 \subset \Phi_0$, where Φ_0 is the connected component, then from (39) it follows that ($\rho = 0$)

$$\frac{f'(z)}{f(z)} = (y + u(z))z^{-1}\ln^{\upsilon} z,\ y \ne 0,\ |u(z)| < \frac{\delta}{10},\ z \in \Phi_0,\ |z| \geqslant r_0. \tag{57}$$

If $z = |z|e^{i\varphi}$, $\alpha < \varphi < \alpha + \pi$, then $\ln z = \ln|z|(1 + \frac{i\varphi}{\ln|z|})$ and taking into account Taylor series expansion formula we obtain

$$\ln^{\upsilon} z = \ln^{\upsilon}|z|\left(1 + \frac{i\varphi}{\ln|z|}\right)^{\upsilon} = \ln^{\upsilon}|z|\left(1 + O\left(\frac{1}{\ln|z|}\right)\right),\ \alpha < \varphi < \alpha + \pi,\ z \to \infty.$$

Thus from (57) we have

$$\begin{aligned}\frac{f'(z)}{f(z)} &= (y + u(z))\left(1 + O\left(\frac{1}{\ln|z|}\right)\right)\frac{\ln^{\upsilon}|z|}{z} = \\ &= (y + u_1(z))\frac{\ln^{\upsilon}|z|}{z},\ |u_1(z)| < \frac{\delta}{9}, z \in \Phi_0,\ |z| \geqslant r_0,\ y \ne 0.\end{aligned} \tag{58}$$

Let in (58) $\upsilon > 0$. Integrating (58) over the arc $\omega^0_{[z_{1t},z]}$ of the segment

$$\omega^0_{[z_{1t},z_{2t}]}\colon z = te^{i\varphi},\ |z_{1t}| \le t \le |z_{2t}|,\ (|z_{1t}| = r_{1t},\ |z_{2t}| = r_{2t},\ |z| = r)$$

and taking the real part, we obtain

$$\begin{aligned}\ln\frac{f(z)}{f(z_{1t})} &= \frac{y + \omega(z)}{\upsilon + 1}(\ln^{\upsilon+1}|z| - \ln^{\upsilon+1}|z_{1t}|),\ y = |y|e^{i\beta},\ |\omega(z)| < \frac{\delta}{9}, \\ \ln\left|\frac{f(z)}{f(z_{1t})}\right| &= \frac{|y|\cos\beta + g(z)}{\upsilon + 1}(\ln^{\upsilon+1}|z| - \ln^{\upsilon+1}|z_{1t}|),\ |z_{1t}| \le |z| \le |z_{2t}|,\end{aligned} \tag{59}$$

where $w(z)$, $g(z)$ are some functions, $|w(z)| < \frac{\delta}{9}$, $|g(z)| < \frac{\delta}{9}$, $z \in \omega_t^0$, $\upsilon > 0$; hence, β takes a finite amount of possible values.

1. If $\cos\beta = 0$ in (59), then (40) and (43) imply that

$$\ln|f(z)| < \frac{\delta}{5}\ln^{\upsilon+1}|z|,\ z \in \omega_t^0,\ r_{1t} \leqslant |z| \leqslant r_{2t} \leqslant +\infty,\ \upsilon > 0. \tag{60}$$

2. Suppose that $\cos\beta < 0$ in (59). Choose $\delta > 0$ such that $|y|\cos\beta + \frac{\delta}{2} < 0$ (β is one of a finite amount of possible values). Then the right-hand side of (59) is negative. If $r_0 < r_{1t}$, then from (43) and (59) it follows that

$$\ln|f(re^{i\varphi})| < \ln|f(r_{1t}e^{i\varphi})| = (\nu+\varepsilon)\ln r_{1t},\ r_{1t} < r \leqslant r_{2t} \leq +\infty.$$

Therefore, we have a contradiction with (42). If $r_0 = r_{1t}$, then taking into account (59) and (40) we obtain

$$\ln|f(re^{i\varphi})| < \ln|f(r_{1t}e^{i\varphi})| < \ln C,\ r_{1t} < r \leqslant r_{2t}.$$

The latter statement together with (42) imply that $r_{2t} < r_* = \text{const}$, thus

$$\exists r_*\colon \{z\colon \arg_{\alpha-\pi} z = \varphi \in \Pi, |z| \geqslant r_*\} \cap \omega_t^0 = \varnothing,\ \cos\beta < 0. \tag{61}$$

3. Let $\cos\beta > 0$ in (59) and $r_0 < r_{1t} < r_{2t} < +\infty$. Choose δ such that $0 < \delta < |y|\cos\beta$. Then taking into consideration (43) we may rewrite (59) as follows

$$(\nu+\varepsilon)\ln\frac{r_{2t}}{r_{1t}} = \ln\left|\frac{f(r_{2t}e^{i\varphi})}{f(r_{1t}e^{i\varphi})}\right| > |y|\cos\beta\frac{\ln^{\upsilon+1} r_{2t} - \ln^{\upsilon+1} r_{1t}}{2(\upsilon+1)},$$

or

$$c(\ln r_{2t} - \ln r_{1t}) > \ln^{\upsilon+1} r_{2t} - \ln^{\upsilon+1} r_{1t}, r_{1t} < r_{2t}, \upsilon > 0, \tag{62}$$

where $c = \frac{2(\upsilon+1)(\nu+\varepsilon)}{|y|\cos\beta}$. The function $\ln^{\upsilon+1} r - c\ln r \uparrow +\infty$ if $r > e^{(\frac{c}{\upsilon+1})^{\frac{1}{\upsilon}}}$. Therefore, (62) is impossible if $r_{1t} > r_* = \text{const}$. If $r_0 = r_{1t} < r_{2t} < +\infty$ in (59), then the proof is similar. Hence

$$\exists r_*\colon \{z\colon \arg_{\alpha-\pi} z = \varphi \in \Pi, |z| \geqslant r_*\} \cap \omega_t^0 = \varnothing,\ \cos\beta > 0,\ r_{2t} < +\infty. \tag{63}$$

4. Suppose that $r_{2t} = +\infty$ (the segment ω_t^0 has an infinite length) and $\cos\beta > 0$ in (59). We consider $|z| = r$ is so large that in (59): $|g(z)| < \frac{\delta}{9} < \frac{|y|\cos\beta}{3}$, $z \in \omega_t^0 \subset S(\varphi)$. Therefore

$$\ln|f(re^{i\varphi})| = \frac{|y|\cos\beta + g(z)}{\upsilon+1}\ln^{\upsilon+1}|z| + O(1),\ r_{1t} \leq r \leq +\infty. \tag{64}$$

Let take φ_1 such that $\varphi < \varphi_1 < \alpha+\pi$. We have $\omega_t^0 \subset \Phi_0$. Let us prove that

$$\exists d\colon \{z\colon \varphi \leqslant \arg_\alpha z \leqslant \varphi_1, |z| \geqslant d\} \setminus E_{**} \subset \Phi_0, \tag{65}$$

where E_{**} is a set of disks with a finite sum of radii.

Let denote by H_r the curve obtained when the point z moves from the point $z \in S(\varphi)$, $|z| = r$, along the arc $\{re^{i\theta}\colon \varphi \leqslant \theta \leqslant \alpha+\pi\}$ enveloping the disks with centers at c_q (see (10)) along the arcs $\{z\colon |z - c_q| = \delta_q\}$. We denote these arcs by $\varkappa$. Let $z = r(x)e^{i\theta(x)}$, $0 \leqslant x \leqslant 1$ be the equation of H_r. We choose arcs $\varkappa$ such that in the equation $H_r\colon r(x) \geqslant r$ holds.

The point $z=r(0)e^{i\theta(0)}$ is the starting point of the curve H_r, $z=r(0)e^{i\theta(0)}\in\omega_t^0\subset\Phi_0$. Let $x_*\in[0,1]$, x_* be the largest value, such that the curve

$$h_r=\{z\colon z\in H_r, z=r(x)e^{i\theta(x)}, 0\leqslant x\leqslant x_*\}\subset\Phi_0,$$

($[h_r]\subset\Phi_0$, where $[h_r]$ is the set of points of the arc h_r), Φ_0 is the connected component of Φ. The point $z=r(0)e^{i\theta(0)}\in\Phi_0\subset\Phi$ and (64) is true at this point. From the definition of Φ it follows that $0<x_*$. Suppose that $\theta(x_*)<\varphi_1$. From the definition of the connected component $\Phi_0\subset\Phi$ and the definition of the point x_* it follows that

$$|f(r(x_*)e^{i\theta(x_*)})|=(r(x_*))^{\nu+\varepsilon}. \tag{66}$$

The curve h_r consists of arcs of the circle $\{z\colon |z|=r\}$ and of the arcs $\varkappa$. The total length of the arcs $\varkappa$ is not greater than $2\pi M$ (see (10)), therefore $r\leqslant r(x)<r+2\pi M$, $r(x)e^{i\theta(x)}\in[h_r]$.

The length h_r is less than $\pi r+2\pi M$. Therefore, integrating (58) along h_r, $[h_r]\subset\Phi_0$, and extracting the real parts we obtain

$$\ln\left|\frac{f(r(x_*)e^{i\theta(x_*)})}{f(z)}\right|=\operatorname{Re}\int\limits_{h_r}(y+u(\zeta))\frac{\ln^{\upsilon}\zeta}{\zeta}d\zeta=O(\ln^{\upsilon}r),\ r\to+\infty,$$

because

$$\left|\int\limits_{h_r}(y+u(\zeta))\frac{\ln^{\upsilon}\zeta}{\zeta}d\zeta\right|\leqslant\left(|y|+\frac{\delta}{10}\right)2\int\limits_{h_r}\frac{\ln^{\upsilon}|\zeta|}{|\zeta|}ds\leqslant$$
$$\leqslant\left(|y|+\frac{\delta}{10}\right)2\frac{\ln^{\upsilon}r}{r}(\pi r+2\pi M)=O(\ln^{\upsilon}r).$$

Therefore, from (64) and (66) it follows that ($|g(z)|<\frac{\delta}{9}<\frac{|y|\cos\beta}{3}$, $z\in\omega_t^0$)

$$(\nu+\varepsilon)\ln r=\ln|f(z)|+O(\ln^{\upsilon}r)>\frac{|y|\cos\beta}{2(\upsilon+1)}\ln^{\upsilon+1}r,\ \upsilon>0,\ \cos\beta>0.$$

But this is impossible for a sufficiently large r. Therefore $\theta(x_*)\geqslant\varphi_1$ and (65) is proved. Similarly, we can show that $\forall\varphi_0,\varphi_1,\alpha<\varphi_0<\varphi_1<\alpha+\pi,\exists d:$

$$B=\{z\colon\varphi_0\leqslant\arg_{\alpha}z\leqslant\varphi_1,|z|\geqslant d\}\setminus E_{**}\subset\Phi_0, f(de^{i\theta})\neq0,\infty, \tag{67}$$

where E_{**} is the set of disks with a finite sum of radii, $\alpha\leqslant\theta\leqslant\alpha+\pi$.

Let us denote by $\Gamma(z)$ the curve obtained when the point z moves in due order along the segment $\{z\colon z\in S(\varphi), d\leqslant|z|\leqslant r\}\subset B$ and along the curve H_r. Let $z_1=de^{i\varphi}$ be the starting point and $z=r(x)e^{i\theta(x)}\in B$ be the ending point of $\Gamma(z)$ (see above). Integrating (57) over $\Gamma(z)$, we obtain

$$\ln\frac{f(z)}{f(z_1)}=y(\ln^{\upsilon+1}z-\ln^{\upsilon+1}z_1)(\upsilon+1)^{-1}+\int\limits_{\Gamma(z)}\frac{u(\zeta)\ln^{\upsilon}\zeta}{\zeta}d\zeta,\ z\in B,$$
$$\left|\int\limits_{\Gamma(z)}\frac{u(\zeta)\ln^{\upsilon}\zeta}{\zeta}d\zeta\right|\leqslant\frac{2\delta}{3}\left(\frac{\ln^{\upsilon+1}r}{\upsilon+1}+\frac{\ln^{\upsilon}r}{r}(\pi r+2\pi M)\right). \tag{68}$$

The estimate of integral in (68) is uniform by $\arg_\alpha z$ in B. Since values of φ_0, φ_1, $\alpha < \varphi_0 < \varphi_1 < \alpha + \pi$ were chosen arbitrarily, then from (68) it follows that estimate (4) is true on the set $\{re^{i\theta}\colon \alpha + \varepsilon < \theta < \alpha + \pi - \varepsilon, r \geqslant d\}, \varepsilon > 0$.

5. Suppose that in (57) $v \leq 0$. Integrating (57) along ω_t^0 and extracting the real parts we obtain

$$\ln\left|\frac{f(z)}{f(z_{1t})}\right| < \left(|y| + \frac{\delta}{9}\right)\int\limits_{r_{1t}}^{r} \frac{ds}{s} < \left(|y| + \frac{\delta}{9}\right)\ln\frac{r}{r_{1t}},\ r_{1t} \leqslant |z| \leqslant r_{2t},\ |z| = r,\ z \in \omega_t,$$

or

$$\left|\frac{f(z)}{f(z_{1t})}\right| < \left(\frac{r}{r_{1t}}\right)^{|y|+\frac{\delta}{9}},\ 0 < \delta < \varepsilon. \tag{69}$$

Exactly the same estimate holds in the case $\rho < 0$ (see (55)). Let $r_0 < r_{1t} < r_{2t} < +\infty$. From (69) and (43) we obtain

$$\left|\frac{f(z_{2t})}{f(z_{1t})}\right| = \left(\frac{r_{2t}}{r_{1t}}\right)^{\nu+\varepsilon} < \left(\frac{r_{2t}}{r_{1t}}\right)^{|y|+\frac{\delta}{9}},\ 0 < \delta < \varepsilon,$$

but this is impossible because $|y| \leqslant \nu,\ \delta < \varepsilon$ (see (35)). If $r_0 < r_{1t} < r_{2t} = +\infty$, then from (69) and (42) we obtain

$$r^{\nu+\varepsilon} \leqslant |f(re^{i\varphi})| < |f(r_1 e^{i\varphi})|\left(\frac{r}{r_{1t}}\right)^{|y|+\frac{\delta}{9}},\ r > r_{1t},$$

but this is also impossible because $|y| \leqslant \nu,\ \delta < \varepsilon$. Let $r_0 = r_{1t}$. From (40), (42) and (69) we obtain

$$r^{\nu+\varepsilon} \leqslant |f(re^{i\varphi})| < C\left(\frac{r}{r_{1t}}\right)^{|y|+\frac{\delta}{9}},\ r_0 \leqslant r \leqslant r_{2t}. \tag{70}$$

Since $|y| \leqslant \nu,\ \delta < \varepsilon$, then (70) is possible if $r_{2t} < r_*$ where r_* is a constant. Therefore

$$\exists r_*\colon \{z\colon \arg_{\alpha-\pi} z = \varphi \in \Pi, |z| \geqslant r_*\} \cap \omega_t^0 = \varnothing,\ v \leq 0. \tag{71}$$

Simultaneously we have proved (56).

The similar arguments could be applied if for any $\varphi \in \Pi$ the ray

$$S(\varphi) = \{z\colon \arg_{\alpha-\pi} z = \varphi = \text{const},\ z \geq d\}$$

contains the segment w_t^0 of an infinite length on which $\cos\beta > 0$ in (59), then the statement (4) of theorem 1 holds. Otherwise $\forall\varphi \in \Pi$ on the ray $S(\varphi)$ one of the ratio (41), (60), (61), (63), (56), (71) holds. Hence

$$\forall\varphi \in \Pi\colon \ln|f(z)| < \frac{\delta}{5}\ln^{v+1}|z|,\ z \in \{z\colon \arg_{\alpha-\pi} z = \varphi,\ z \geq r(\varphi)\}. \tag{72}$$

Lets consider any $\{z\colon \arg z = \varphi\}$, $\varphi \notin \Pi$. We denote

$$S_1 = \{z\colon \arg z = \varphi,\ |z| \notin \Delta,\ \text{mes}\Delta < \infty\}, \tag{73}$$

where Δ is the set of points that belong to the intervals $[|c_q| - \delta_q, |c_q| + \delta_q]$, $c_q \in \{c_q\}$ (see (10)). Let us prove that on S_1 estimate (5) holds.

If $\Phi \cap S_1 = \varnothing$, then on S_1 inequality (41) is true. Let $\Phi \cap S_1 \neq \varnothing$. The set $\Phi \cap S_1$ can be represented as a sum of "maximal" segments

$$w_t(\varphi) = \{z\colon\ \arg z = \varphi, r_{1t} \leq |z| \leq r_{2t}\} \subset \Phi \cap S_1$$

such that

$$|f(z)| \geq |z|^{\nu+\varepsilon},\ z \in w_t(\varphi) \subset \Phi_0,\ \varphi \notin \Pi \tag{74}$$

where Φ_0 is the connected component of Φ. According to the latter (see (42)), we define the segments $w_t^-(\varphi)$, $w_t^0(\varphi)$. On the segment $w_t^-(\varphi)$ estimate (52) holds (this estimate is proved for all $\varphi \in (0,\pi)$).

We obtain estimates on the segment $w_t^0(\varphi)$. Choose any $\delta_1 > 0$. The set Π (see (53)) is dense on $(0,\pi)$. Then there exist $\psi(1)$, $\psi(2)$, such that

$$\psi(1), \psi(2) \in \Pi \wedge \varphi - \delta_1 < \psi(1) < \varphi < \psi(2) < \varphi + \delta_1. \tag{75}$$

On the rays $S(\psi(j)) = \{z\colon\ \arg z = \psi(j)\}, j \in \{1,2\}$, (72) holds, hence

$$\ln|f(re^{i\psi(j)})| < \delta \ln^{v+1} r,\ r > r(\psi(j)),\ j \in \{1,2\};\ v > 0. \tag{76}$$

Consider the segment $w_t(\varphi) \subset S_1$. Let $z = re^{i\varphi} \in w_t(\varphi) \in \Phi_0 \subset \Phi$. By $\theta(1)$ we denote the least value and by $\theta(2)$ we denote the greatest value such that

$$\theta(1) \leq \varphi \leq \theta(2) \wedge \{z\colon \theta(1) \leq \arg z \leq \theta(2), |z| = r\} \subset \Phi_0.$$

Let

$$\zeta = \max(\theta(1), \psi(1)), \lambda = \min(\theta(2), \psi(2)).$$

From the definition of $\theta(1), \theta(2)$, and from (75) it follows that

$$\{z\colon \zeta \leq \arg z \leq \lambda, |z| = r\} \subset \Phi_0,\ \varphi - \delta_1 < \zeta \leq \varphi \leq \lambda < \varphi + \delta_1. \tag{77}$$

Let $z = re^{i\varphi} \in w_t^0(\varphi) \subset S_1$. Thus $w_t^0(\varphi) \subset \Phi_0$ and statement (57) holds on Φ_0. Integrating (57) along the arc $\{z\colon \zeta \leq \arg z \leq \lambda, |z| = r\} \subset \Phi_0$ from the point $re^{i\zeta}$ to the point $z = re^{i\varphi}$, and extracting real parts and taking into account $0 \leq \varphi - \zeta < \delta_1$, we obtain

$$\ln\left|\frac{f(z)}{f(r\exp(i\zeta))}\right| \leq (|y| + \delta)\,\delta_1 \ln^v r,\ z = re^{i\varphi}, \tag{78}$$

where $\zeta = \max(\theta(1), \psi(1))$. If $\psi(1) \leq \theta(1)$, then $\zeta = \theta(1)$. Thus from the definition of $\theta(1)$ and from the definition of the connected component Φ_0 equality $|f(re^{i\zeta})| = r^{\nu+\varepsilon}$ follows. If $\theta(1) < \psi(1)$, then $\zeta = \psi(1)$ and from (76)

$$\ln|f(re^{i\zeta})| = \ln|f(re^{i\psi(1)})| < \delta \ln^{v+1} r,\ v > 0.$$

Therefore from (78) we obtain

$$\ln|f(z)| \leq \delta \ln^{v+1} r + \delta_1 (|y| + \delta) \ln^v r,\ z = re^{i\varphi} \in w_t^0(\varphi).$$

The latter statement, (52) and the estimate $|f(z)| < |z|^{\nu+\varepsilon},\ z \in \Phi_1$ (see (36)), yield that on the ray $S_1(\varphi \notin \Pi)$ estimate (5) holds.

Choose some $\alpha, \xi, \psi, -\infty < \alpha < \xi < \psi < +\infty$. There exists $l \in \mathbb{N}$ such that $\alpha < \xi < \psi < \alpha + \frac{l\pi}{2}$. Let $\alpha_j \stackrel{\text{def}}{=} \alpha + \frac{j\pi}{2}, j \in \{0, 1, \ldots, l\}$. Then $(0 < \varepsilon < \max(\xi - \alpha, \frac{\pi}{4}))$

$$[\xi, \psi] \subset \bigcup_{j=0}^{l} [\alpha_j + \varepsilon, \alpha_j + \pi - \varepsilon]. \tag{79}$$

The function $f \in M_l$ is the solution of equation (1). So the single-valued branch $f(z)$, $z \in g_{\alpha_j, \alpha_j+\pi}$, $j \in \{0, 1, \ldots, l\}$, is the solution of equation (1) (this solution satisfies the conditions of Theorem 1). Therefore (4) or (5) is true for each branch $f(z), z \in g_{\alpha_j, \alpha_j+\pi}$.

For the branch $f(z), z \in g_{\alpha_j, \alpha_j+\pi}, j \in \{0, 1, \ldots, l\}$, by E_j we denote the set of exceptional disks E_* (see theorem 1) with a finite sum of radii on the part of the Riemann surface $g_{\alpha_j, \alpha_j+\pi} = \{re^{i\theta} \colon r_0 \leqslant r, \alpha_j \leqslant \theta \leqslant \alpha_j + \pi\}$ of the meromorphic function $f(z), z \in G = \{z \colon r_0 \leqslant |z| < +\infty\}$. The union of parts

$$g_{\alpha_j+\varepsilon, \alpha_j+\pi-\varepsilon} \setminus E_j = \{re^{i\theta} \colon r_0 \leqslant r, \alpha_j + \varepsilon \leqslant \theta \leqslant \alpha_j + \pi - \varepsilon\} \setminus E_j$$

of the Riemann surface of the function $f(z),\ z \in G$, we denote by

$$\bigcup_{j=0}^{l} g_{\alpha_j+\varepsilon, \alpha_j+\pi-\varepsilon} \setminus E_j,\ E = \bigcup_{j=0}^{l} E_j$$

be the union of the sets of the disks on the Riemann surface. Since E_j is the set of disks with a finite sum of radii then the sum of radii of the disks that form the set E is also finite.

From (79) it follows that the part of the Riemann surface

$$g_{\xi,\psi} = \{re^{i\theta} \colon r_0 \leqslant r, \xi \leqslant \theta \leqslant \psi\} \subset \bigcup_{j=1}^{l} g_{\alpha_j+\varepsilon, \alpha_j+\pi-\varepsilon}. \tag{80}$$

Let us assume that for the branch $f(z),\ z \in g_{\alpha_0, \alpha_0+\pi} (\alpha_0 = \alpha)$ (4) is true. Namely $\exists d > 0 \colon r > d \wedge \alpha_0 + \varepsilon \leqslant \theta \leqslant \alpha_0 + \pi - \varepsilon \Rightarrow$

$$\begin{aligned} &\ln f(re^{i\theta}) = \ln^{\upsilon+1}(re^{i\theta})\Big(\frac{y}{\upsilon+1} + g(re^{i\theta})\Big),\ re^{i\theta} \notin E_0, \\ &|g(re^{i\theta})| < \varepsilon,\ \upsilon > 0,\ y = |y|e^{i\beta},\ \operatorname{Re} y > 0,\ \cos\beta > 0. \end{aligned} \tag{81}$$

By the construction there exists φ such that $\alpha_1 + \varepsilon < \varphi < \alpha_0 + \pi - \varepsilon < \alpha_1 + \pi - \varepsilon$.

The equality (81) is true, in particular on infinite "ray" $S(\varphi)$ (see definition below (40), $E_0 = E_*$). Thus, we proved that (59), (64) hold with $\cos\beta > 0$ on infinite "ray" $S(\varphi)$ for the branch $f(z), z \in g_{\alpha_1, \alpha_1+\pi}$. Repeating the proof of (68) we obtain that for $f(z)$,

$$z \in g_{\alpha_1, \alpha_1+\pi}, \exists d_1 > 0 \colon r > d_1 \wedge \alpha_1 + \varepsilon \leqslant \theta \leqslant \alpha_1 + \pi - \varepsilon$$

statement (81) is true.

Similarly we prove that for each branch $f(z), z \in g_{\alpha_j, \alpha_j+\pi}, j \in \{0, 1, \ldots, l\}$, $\exists d_j > 0 \colon r > d_j \wedge \alpha_j + \varepsilon \leqslant \theta \leqslant \alpha_j + \pi - \varepsilon$ it follows that (81) is true. Hence, from (80) it follows that (4) is valid, $d = \max d_j,\ j \in \{0, 1, \ldots, l\}$. $\square$

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Lviv Polytechnic National University
kolyasa.lubov@gmail.com

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