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SOME IDEAL-CONVERGENT GENERALIZED DIFFERENCE SEQUENCES IN A LOCALLY CONVEX SPACE DEFINED BY A MUSIELAK-ORLICZ FUNCTION

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An ideal I is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. A sequence (x_k) of real numbers is said to be I-convergent to a real number ℓ, if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N}\colon |x_k - \ell| \geq \varepsilon\}$ belongs to I. In this article, we introduce a new class of ideal convergent (shortly I-convergent) sequence spaces using a Musielak-Orlicz function and the difference operator in locally convex spaces. We investigate some linear topological structures and algebraic properties of these spaces. We also establish some relations between these sequence spaces.

Б. Хазарика, А. Еси. *Некоторые сходящиеся по идеалу обощенные разностные последовательности в локально-выпуклом пространстве, определенном функцией Мусиеляка-Орлича* // Мат. Студії. – 2014. – Т.42, №2. – C.195–208.

Идеал I — это семейство подмножеств натурального ряда $\mathbb{N}$, замкнутое относительно конечных объединений и подмножеств своих элементов. Последовательность (x_k) действительных чисел называется I-сходящейся к действительному числу ℓ, если для любого $\varepsilon > 0$ множество $\{k \in \mathbb{N}\colon |x_k - \ell| \geq \varepsilon\}$ принадлежит I. В данной статье мы вводим новый класс идеально сходящихся (коротко — I-сходящихся) пространств последовательностей, используя функцию Муселака-Орлича и разностный оператор в локально выпуклых пространствах. Мы исследуем некоторые линейно-топологические структуры и алгебраические свойства, а также устанавливаем некоторые взаимоотношения между этими пространствами последовательностей. Некоторые идеально сходящиеся обобщеные.

1. Introduction. Throughout the article w, ℓ_∞, c, c_0, denote the classes of *all, bounded, convergent, null* sequences of complex numbers, respectively.

The notion of statistical convergence is a very useful functional tool for studying convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by H. Fast ([7]) and I. J. Schoenberg ([26]) independently for real sequences. Later on it was further investigated from sequence point of view and linked with the summability theory by J. A. Fridy ([8]) and many others. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows.

The natural density of a subset E of $\mathbb{N}$ is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \limsup_{n\to\infty} \frac{1}{n} |\{k \leq n\colon k \in E\}|,$$

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where the vertical bars denote the cardinality of the respective set.

Definition 1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$: $\delta(\{k \in \mathbb{N} \colon |x_k - \ell| \geq \varepsilon\}) = 0$. In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$; here S denotes the set of all statistically convergent sequences.

The notion of I-convergence (I denoting an ideal of the subsets of the set $\mathbb{N}$), which is a generalization of statistical convergence, was introduced by P. Kostyrko et al ([18]) and further studied by many others (see [12], [13], [27], [28], [29]). Recently, B. Hazarika ([14]) introduced the notion of generalized difference ideal convergent sequences and obtained some interesting results.

Before proceeding let us recall a few concepts, which we shall use throughout this paper.

Let X be a non-empty set. Then a family of sets $I \subset 2^X$ (2^X being the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$ we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\varnothing \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$ we have $B \in F$. An ideal I is called the non-trivial ideal if $I \neq \varnothing$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A \colon A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} \colon x \in X\} \subset I$. A non-trivial ideal I is maximal if there does not exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of 2^X can be found in P. Kostyrko, et.al ([18]). Recall that a sequence $x = (x_k)$ of points in $\mathbf{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbb{N} \colon |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([18]). In this case we write $I - \lim x_k = \ell$.

The notion of difference sequence space was introduced by H. Kizmaz ([17]), who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$. The notion was further generalized by M. Et and R. Colak ([5]) introducing the sequence spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$, $c_0(\Delta^s)$. For a nonnegative integer s, the generalized difference sequence spaces are defined as follows. For a given sequence space Z we have

$$Z(\Delta^s) = \{x = (x_k) \in w \colon (\Delta^s x_k) \in Z\},$$

where $\Delta^s x_k = \Delta^{s-1} x_k - \Delta^{s-1} x_{k+1}$, $\Delta^0 x_k = x_k$, for all $k \in \mathbb{N}$. The difference operator is equivalent to the following binomial representation

$$\Delta^s x_k = \sum_{\nu=0}^{s} \binom{s}{\nu} (-1)^\nu x_{k+\nu}.$$

Taking $s = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$, introduced and studied by H. Kizmaz ([17]). Difference sequence spaces were further studied by many others (see [3], [6]).

Let r, s be non-negative integers and $v = (v_k)$ be a sequence of non-zero scalars. Then for Z, a given sequence space, recently H. Dutta ([4]) introduced the following sequence spaces

$$Z(\Delta^s_{(vr)}) = \{x = (x_k) \in w \colon (\Delta^s_{(vr)} x_k) \in Z\}, \text{ for } X = c, c_0, \ell_\infty,$$

where $\Delta^s_{(vr)} x_k = \Delta^{s-1}_{(vr)} x_k - \Delta^{s-1}_{(vr)} x_{k+r}$, $\Delta^0_{(vr)} x_k = v_k x_k$, for all $k \in \mathbb{N}$. The difference operator is equivalent to the following binomial representation

$$\Delta^s_{(vr)} x_k = \sum_{i=0}^{s} \binom{s}{i} (-1)^i v_{k-ir} x_{k-ir}.$$

In this expansion it is important to note that we take $v_{k-ir} = 0$ and $x_{k-ir} = 0$ for non-positive values of $k - ir$.

For $s = 1$ and $v_k = (1, 1, 1, \ldots)$ we get the spaces $\ell_\infty(\Delta_{(r)})$, $c(\Delta_{(r)})$ and $c_0(\Delta_{(r)})$. For $r = 1$ and $v_k = (1, 1, 1, \ldots)$ we get the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. For $s = r = 1$ $v_k = (1, 1, 1, \ldots)$ then we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = (a_{nk}), (n, k \in \{1, 2, 3, \ldots\})$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ converges for each n. If $x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y$ we say that A defines a (matrix) transformation from X to Y and we denote it by $A\colon X \to Y$.

An Orlicz function is a function $M\colon [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see [19]).

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x p(t)dt$$

where p is the known kernel of M, right differentiable for $t \geq 0,\ p(0) = 0,\ p(t) > 0$ for $t > 0$ and $p(t) \to \infty$ as $t \to \infty$.

If the convexity of an Orlicz function M is replaced with the condition $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function; it was characterized by W. H. Ruckle ([24]). An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Let M be an Orlicz function which satisfies the Δ_2-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

Two Orlicz functions M_1 and M_2 are said to be *equivalent* if there exist positive constants α, β and x_0 such that $M_1(\alpha) \leq M_2(x) \leq M_1(\beta)$ for all x with $0 \leq x < x_0$.

J. Lindenstrauss and L. Tzafriri ([21]) studied some Orlicz type sequence spaces defined as follows

$$\ell_M = \Big\{(x_k) \in w\colon \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty,\ \text{for some } \rho > 0\Big\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\Big\{\rho > 0\colon \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

A sequence $\mathbf{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (for details see [2], [11], [15], [16]). Also a Musielak-Orlicz function $\phi = (\phi_k)$ is called a *complementary function* to a Musielak-Orlicz function $\mathbf{M}$ if $\phi_k(t) = \sup\{|\ t|s - M_k(s)\colon s \geq 0\}$, for $k \in \{1, 2, 3, \ldots\}$. For a given Musielak-Orlicz function $\mathbf{M}$, the Musielak-Orlicz sequence space $l_{\mathbf{M}}$ and its subspace $h_{\mathbf{M}}$ are defined as follows

$$l_{\mathbf{M}} = \{x = (x_k) \in w\colon I_{\mathbf{M}}(cx) < \infty,\ \text{for some } c > 0\};$$
$$h_{\mathbf{M}} = \{x = (x_k) \in w\colon I_{\mathbf{M}}(cx) < \infty,\ \text{for all } c > 0\},$$

where $I_{\mathbf{M}}$ is the convex modular defined by $I_{\mathbf{M}} = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in l_{\mathbf{M}}$. We consider $l_{\mathbf{M}}$ equipped with the Luxemburg norm $\|x\| = \inf\{k > 0\colon I_{\mathbf{M}}(\frac{x}{k}) \leq 1\}$ or equipped with the Orlicz norm $\|x\|^0 = \inf\{\frac{1}{k}(1 + I_{\mathbf{M}}(kx))\colon k > 0\}$.

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \le p_k \le \sup_k p_k = G$, $D = \max\{1, 2^{G-1}\}$ then $|a_k + b_k|^{p_k} \le D(|a_k|^{p_k} + |b_k|^{p_k})$ for all $k \in \mathbb{N}$ and $a_k, b_k \in \mathbf{C}$. Also $|a|^{p_k} \le \max\{1, |a|^G\}$ for all $a \in \mathbb{C}$.

Subsequently Orlicz function was used to define sequence spaces by S. Parashar and D. B. Choudhary ([23]) and many others (see [1], [20], [22], [30]).

Remark 1. It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$, for all λ with $0 < \lambda < 1$.

Definition 2. A sequence space E is said to be *solid (or normal)* if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \le 1$, for all $k \in \mathbb{N}$.

Let $K = \{k_1 < k_2 < \ldots\} \subseteq \mathbb{N}$ and E be a sequence space. The K-step space of E is the sequence space $\lambda_K^E = \{(x_{k_n}) \in w \colon (x_n) \in E\}$.

The canonical preimage of a sequence $\{(x_{k_n})\} \in \lambda_K^E$ is the sequence $\{y_n\} \in w$ defined by

$$y_n = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise.} \end{cases}$$

The canonical preimage of a step space λ_K^E is the set of canonical preimages of all elements in λ_K^E, i.e. y belongs to the canonical preimage of λ_K^E if and only if y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 3. A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

Lemma 1. *Every normal space is monotone.*

Throughout the paper by X we denote a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous seminorms q. Also we denote by I a non-trivial admissible ideal of
mathbbN.

2. I-convergence in a locally convex space. In this section we define I-convergence in a locally convex space X and investigate some basic properties.

Definition 4. A sequence $x = (x_k)$ in X is said to be *I-convergent* to $\ell \in X$ if for all $q \in Q$ and all $\varepsilon > 0$, $\{k \in \mathbb{N} \colon q(x_k - \ell) \ge \varepsilon\} \in I$. In this case we can write I_q-$\lim x_k = \ell$. We denote $I_q = \{k \in \mathbb{N} \colon q(x_k - \ell) \ge \varepsilon\}$.

Further, since X is Hausdorff, the limit of an ideal convergent sequence is unique.

Definition 5. Let $\mathbf{M}$ be a Musielak-Orlicz function. We say that a sequence $x = (x_k)$ *converges in $w^I(\boldsymbol{M})$* if and only if there exists $\ell \in X$ such that, for all $q \in Q$ and all $\varepsilon > 0$,

$$\Big\{n \in \mathbb{N} \colon \frac{1}{n} \sum_{k=1}^{n} \Big[M_k\Big(\frac{q(x_k - \ell)}{r}\Big)\Big] \ge \varepsilon\Big\} \in I \text{ for some } r > 0. \tag{1}$$

If (1) holds then we write $x_k \to \ell((w^I(\mathbf{M})))$. Condition (1) provides a definition of strong ideal summability for a sequence in a locally convex space.

Theorem 1. *Let $A = (a_{nk})$ be a non-negative regular matrix. Let $\boldsymbol{M}$ be a Musielak-Orlicz function. Then $x_k \to \ell(w(\boldsymbol{M}, A, p))$ implies that $x_k \to \ell(I_q(A))$.*

Proof. Let $q \in Q$. Assume that $x_k \to \ell(w(\mathbf{M}, A, p))$, then for some $r > 0$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} = 0 \text{ for } \ell \in \mathbb{C}.$$

Let $\varepsilon > 0$ be given. We define $K(\varepsilon) = \{k \in \mathbb{N} \colon q(x_k - \ell) \geq \varepsilon\}$ and we write

$$\sum_{k=1}^{\infty} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} = \sum_{k\in K(\varepsilon)} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} + \sum_{k\notin K(\varepsilon)} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} \geq$$
$$\geq \Big(\sum_{k\in K(\varepsilon)} a_{nk}\Big) \Big[M_k\Big(\frac{\varepsilon}{r}\Big)\Big]^{p_k}.$$

Then we have $x_k \to \ell(I_q(A))$. □

Theorem 2. *Let $A = (a_{nk})$ be a non-negative regular matrix. Let $\boldsymbol{M}$ be a Musielak-Orlicz function. If $x = (x_k) \in \ell_\infty$ and $x_k \to \ell(I_q(A))$, then $x_k \to \ell(w(\boldsymbol{M}, A, p))$.*

Proof. Suppose that $x = (x_k) \in \ell_\infty$ and $x_k \to \ell(I_q(A))$. Then there is a set $K \in F(I_q)$ such that $\lim_{k\in K} q(x_k - \ell) = 0$. Now

$$\sum_{k=1}^{\infty} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} = \sum_{k\in K(\varepsilon)} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} + \sum_{k\notin K(\varepsilon)} a_{nk} \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} =$$
$$= \sum_{k=1}^{\infty} a_{nk}\chi_K(k) \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k} + \sum_{k=1}^{\infty} a_{nk}\chi_{K^c}(k) \Big[M_k\Big(\frac{q(x_k-\ell)}{r}\Big)\Big]^{p_k}.$$

If we consider the regularity of A, $K^c \in I_q$ and boundedness of (x_k) then the right hand side tends to zero. Hence the left hand side also tends to zero. □

3. New classes of ideal convergent sequences in a locally convex space. In this section we define some new classes of I-convergent sequences by using infinite matrix in a locally convex space X and investigate their linear topological structures. Also we find out some relations related to these spaces.

Recall that a mapping $g \colon X \to \mathbb{R}$ is called a *paranorm* on X if it satisfies the following conditions:

(i) $g(\theta) = 0$ where θ is the zero element of the space;

(ii) $g(x) = g(-x)$;

(iii) $g(x + y) \leq g(x) + g(y)$;

(iv) $\lambda^n \to \lambda (n \to \infty)$ and $g(x^n - x) \to 0 (n \to \infty)$ imply $g(\lambda^n x^n - \lambda x) \to 0 (n \to \infty)$ for all $x, y \in X$. The ordered pair $(X; g)$ is called a paranormed space with respect to the paranorm g.

The main aim of this article is to introduce the following sequence spaces and examine some of their properties.

Let I be an admissible ideal of $\mathbb{N}$, $p=(p_k)$ be a bounded sequence of strictly positive real numbers and $A=(a_{nk})$ be an infinite matrix. Let $\mathbf{M}$ be a Musielak-Orlicz function. Further $w(X)$ denotes the space of all X-valued sequences. For each $\varepsilon>0$, for all $q\in Q$ and for some $\rho>0$ we define the following sequence spaces

$$w^I(A,\Delta^s_{(vr)},\mathbf{M},p,q)=$$
$$=\Big\{(x_k)\in w(X)\colon \Big\{n\in\mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)}x_k-\ell)}{\rho}\Big)\Big]^{p_k}\geq\varepsilon\Big\}\in I,\ \text{for}\ \ \ell\in X\Big\},$$
$$w_0^I(A,\Delta^s_{(vr)},\mathbf{M},p,q)=\Big\{(x_k)\in w(X)\colon \Big\{n\in\mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)}x_k)}{\rho}\Big)\Big]^{p_k}\geq\varepsilon\Big\}\in I\Big\},$$
$$w_\infty^I(A,\Delta^s_{(vr)},\mathbf{M},p,q)=$$
$$=\Big\{(x_k)\in w(X)\colon \exists K>0 s.t.\Big\{n\in\mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)}x_k)}{\rho}\Big)\Big]^{p_k}\geq K\Big\}\in I\Big\},$$
$$[w_\infty(A,\Delta^s_{(vr)},\mathbf{M},p,q)=\Big\{(x_k)\in w(X)\colon \exists K>0 s.t. \sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)}x_k)}{\rho}\Big)\Big]^{p_k}<\infty\Big\}.$$

Some classes are obtained by specializing s,v,r, $\mathbf{M}$ and (p_k). Here are some examples:

(i) If $s=0$ then we obtain the above spaces as $w^I(A,\mathbf{M},p,q)$, $w_0^I(A,\mathbf{M},p,q)$, $w_\infty^I(A,\mathbf{M},p,q)$, and $w_\infty(A,\mathbf{M},p,q)$.

(ii) If $s=1$, $v_k=(1,1,1,\ldots)$ then the above spaces are denoted by $w^I(A,\Delta_{(r)},\mathbf{M},p,q)$, $w_0^I(A,\Delta_{(r)},\mathbf{M},p,q)$, $w_\infty^I(A,\Delta_{(r)},\mathbf{M},p,q)$ and $w_\infty(A,\Delta_{(r)},\mathbf{M},p,q)$.

(iii) If $r=1$, $v_k=(1,1,1,\ldots)$ then the above spaces are denoted by $w^I(A,\Delta^s,\mathbf{M},p,q)$, $w_0^I(A,\Delta^s,\mathbf{M},p,q)$, $w_\infty^I(A,\Delta^s,\mathbf{M},p,q)$ and $w_\infty(A,\Delta^s,\mathbf{M},p,q)$.

(iv) If $M_k(x)=x$ for all $x\in[0,\infty), k\in\mathbb{N}$ then we obtain the above spaces as $w^I(A,\Delta^s_{(vr)}, p,q)$, $w_0^I(A,\Delta^s_{(vr)},p,q)$, $w_\infty^I(A,\Delta^s_{(vr)},p,q)$ and $w_\infty(A,\Delta^s_{(vr)},p,q)$.

(v) If $p=(p_k)=(1,1,1...)$, then the above spaces are denoted by $w^I(A,\Delta^s_{(vr)},\mathbf{M},q)$, $w_0^I(A,\Delta^s_{(vr)},\mathbf{M},q)$, $w_\infty^I(A,\Delta^s_{(vr)},\mathbf{M},q)$ and $w_\infty(A,\Delta^s_{(vr)},\mathbf{M},q)$.

(vi) If we take $A=(C,1)$), i.e., the Cesàro matrix, then the above classes of sequences are denoted by $w^I(\Delta^s_{(vr)},\mathbf{M},p,q)$, $w_0^I(\Delta^s_{(vr)},\mathbf{M},p,q)$, $w_\infty^I(\Delta^s_{(vr)},\mathbf{M},p,q)$ and $w_\infty(\Delta^s_{(vr)}, \mathbf{M},p,q)$.

(vii) If we take $A=(a_{nk})$ is a de la Vallée Poussin mean, i.e.,

$$a_{nk}=\begin{cases}\frac{1}{\lambda_n}, & \text{if } k\in I_n=[n-\lambda_n+1,n];\\ 0, & \text{otherwise,}\end{cases}$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1}\leq\lambda_n+1, \lambda_1=1$, then the above classes of sequences are denoted by $w^I(\lambda,\Delta^s_{(vr)},\mathbf{M},p,q)$, $w_0^I(\lambda,\Delta^s_{(vr)},\mathbf{M},p,q)$, $w_\infty^I(\lambda,\Delta^s_{(vr)},\mathbf{M},p,q)$ and $w_\infty(\lambda,\Delta^s_{(vr)},\mathbf{M},p,q)$.

(viii) By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. As a final illustration let

$$a_{nk} = \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r = (k_{r-1}, k_r]; \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by $w^I(\theta, \Delta^s_{(vr)}, \mathbf{M}, p, q)$, $w_0^I(\theta, \Delta^s_{(vr)}, \mathbf{M}, p, q)$, $w_\infty^I(\theta, \Delta^s_{(vr)}, \mathbf{M}, p, q)$ and $w_\infty(\theta, \Delta^s_{(vr)}, \mathbf{M}, p, q)$.

Theorem 3. *$w^I(A, \Delta^s_{(vr)}, \boldsymbol{M}, p, q)$, $w_0^I(A, \Delta^s_{(vr)}, \boldsymbol{M}, p, q)$ and $w_\infty^I(A, \Delta^s_{(vr)}, \boldsymbol{M}, p, q)$ are topological linear spaces.*

Proof. We will prove the result for the space $w_0^I(A, \Delta^s_{(vr)}, \mathbf{M}, p, q)$ only as the others can be treated in a similar way. Let $x = (x_k)$ and $y = (y_k)$ be two elements of $w_0^I(A, \Delta^s_{(vr)}, \mathbf{M}, p, q)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \Big\{n \in \mathbb{N} \colon \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho_1}\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\} \in I,$$

$$B_{\frac{\varepsilon}{2}} = \Big\{n \in \mathbb{N} \colon \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)} y_k)}{\rho_2}\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\} \in I.$$

Let α, β be two scalars in $\mathbb{R}$. Since Δ^s is linear and the Musielak-Orlicz function $\mathbf{M}$ is continuous, the following inequality holds

$$\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)}(\alpha x_k + \beta y_k))}{|\alpha|\rho_1 + |\beta|\rho_2}\Big)\Big]^{p_k} \leq D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho_1}\Big)\Big]^{p_k} +$$
$$+ D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\Big(\frac{q(\Delta^s_{(vr)} y_k)}{\rho_2}\Big)\Big]^{p_k} \leq$$
$$\leq DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho_1}\Big)\Big]^{p_k} + DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)} y_k)}{\rho_2}\Big)\Big]^{p_k},$$

where $K = \max\{1, (\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2}), (\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2})\}$.

From the above relation we get

$$\Big\{n \in \mathbb{N} \colon \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)}(\alpha x_k + \beta y_k))}{(|\alpha|\rho_1 + |\beta|\rho_2)}\Big)\Big]^{p_k} \geq \varepsilon\Big\} \subseteq$$
$$\subseteq \Big\{n \in \mathbb{N} \colon DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho_1}\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\} \cup$$
$$\cup \Big\{n \in \mathbb{N} \colon DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(\frac{q(\Delta^s_{(vr)} y_k)}{\rho_2}\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\}. \quad (2)$$

Since both sets from the right hand side of (2) belong to I. □

Remark 2. It is easy to verify that the space $w_\infty(A, \Delta^s_{(vr)}, \mathbf{M}, p, q)$ is a linear space.

Theorem 4. *The spaces* $w^I(A, \Delta^s_{(vr)}, M, p, q)$, $w^I_0(A, \Delta^s_{(vr)}, M, p, q)$, $w^I_\infty(A, \Delta^s_{(vr)}, M, p, q)$ *and* $w_\infty(A, \Delta^s_{(vr)}, M, p, q)$ *are paranormed spaces with the paranorm* $g_{\Delta^s_{(vr)}}$ *defined by*

$$g_{\Delta^s_{(vr)}}(x) = \sum_{k=1}^{s} q(x_k) + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k M\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho}\Big) \leq 1, \text{ for some } \rho > 0\Big\},$$

where $H = \max\{1, \sup_k p_k\}$.

Proof. Clearly, $g_{\Delta^s_{(vr)}}(-x) = g_{\Delta^s_{(vr)}}(x)$ and $g_{\Delta^s_{(vr)}}(\overline{\theta}) = 0$. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $w_\infty(A, \Delta^s_{(vr)}, M, p, q)$. Then we put

$$A_1 = \Big\{\rho > 0 \colon \sup_k M\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho}\Big) \leq 1\Big\},\ A_2 = \Big\{\rho > 0 \colon \sup_k M\Big(\frac{q(\Delta^s_{(vr)} y_k)}{\rho}\Big) \leq 1\Big\}.$$

Let $\rho_1 \in A_1$ and $\rho_2 \in A_2$. If $\rho = \rho_1 + \rho_2$ then we obtain

$$M\Big(\frac{q(\Delta^s_{(vr)}(x_k + y_k))}{\rho}\Big) \leq \frac{\rho_1}{\rho_1 + \rho_2} M\Big(\frac{q(\Delta^s_{(vr)} x_k)}{\rho_1}\Big) + \frac{\rho_2}{\rho_1 + \rho_2} M\Big(\frac{q(\Delta^s_{(vr)} y_k)}{\rho_2}\Big).$$

Thus we have

$$\sup_k \Big[M\Big(\frac{q(\Delta^s_{(vr)}(x_k + y_k))}{\rho}\Big)\Big]^{p_k} \leq 1,$$

$$g_{\Delta^s_{(vr)}}(x + y) = \sum_{k=1}^{s} q(x_k + y_k) + \inf\Big\{(\rho_1 + \rho_2)^{\frac{p_k}{H}} : \rho_1 \in A_1, \rho_2 \in A_2\Big\} \leq$$

$$\leq \sum_{k=1}^{s} q(x_k) + \inf\Big\{\rho_1^{\frac{p_k}{H}} : \rho_1 \in A_1\Big\} + \sum_{k=1}^{s} q(y_k) + \inf\Big\{\rho_2^{\frac{p_k}{H}} : \rho_2 \in A_2\Big\} = g_{\Delta^s_{(vr)}}(x) + g_{\Delta^s_{(vr)}}(y).$$

Let $\lambda^s \to \lambda$ where $\lambda^s, \lambda \in \mathbb{C}$ and let $g_{\Delta^s_{(vr)}}(x^s - x) \to 0$ as $s \to \infty$. To prove that $g_{\Delta^s_{(vr)}}(\lambda^s x^s - \lambda x) \to 0$ as $s \to \infty$ we set

$$A_3 = \Big\{\rho_s > 0 \colon \sup_k \Big[M\Big(\frac{q(\Delta^s_{(vr)}(x^s_k))}{\rho_s}\Big)\Big]^{p_k} \leq 1\Big\},$$

$$A_4 = \Big\{\rho'_s > 0 \colon \sup_k \Big[M\Big(\frac{q(\Delta^s_{(vr)}(x^s_k - x_k))}{\rho'_s}\Big)\Big]^{p_k} \leq 1\Big\}.$$

If $\rho_s \in A_3$ and $\rho'_s \in A_4$ and by the continuity of the Orlicz function M we observe that

$$M\Big(\frac{q(\Delta^s_{(vr)}(\lambda^s x^s_k - \lambda x_k))}{|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s}\Big) \leq M\Big(\frac{q(\Delta^s_{(vr)}(\lambda^s x^s_k - \lambda x^s_k))}{|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s}\Big) + M\Big(\frac{q(\Delta^s_{(vr)}(\lambda x^s_k - \lambda x_k))}{|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s}\Big) \leq$$

$$\leq \frac{|\lambda^s - \lambda|\rho_s}{|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s} M\Big(\frac{q(\Delta^s_{(vr)}(x^s_k))}{\rho_s}\Big) + \frac{|\lambda|\rho'_s}{|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s} M\Big(\frac{q(\Delta^s_{(vr)}(x^s_k - x_k))}{\rho'_s}\Big).$$

From the above inequality it follows that

$$\sup_k \Big[M\Big(\frac{q(\Delta^s_{(vr)}(\lambda^s x^s_k - \lambda x_k))}{|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s}\Big)\Big]^{p_k} \leq 1$$

and consequently

$$g_{\Delta^s_{(vr)}}(\lambda^s x^s - \lambda x) = \sum_{k=1}^{s} q\Big(\lambda^s x^s_k - \lambda x_k\Big) + \inf\Big\{(|\lambda^s - \lambda|\rho_s + |\lambda|\rho'_s)^{\frac{p_k}{H}} : \rho_s \in A_3, \rho'_s \in A_4\Big\} \leq$$

$$\leq |\lambda^s - \lambda| \sum_{k=1}^{s} q\Big(x^s_k\Big) + |\lambda^s - \lambda|^{\frac{p_k}{H}} \inf\Big\{(\rho_s)^{\frac{p_k}{H}} : \rho_s \in A_3\Big\} +$$

$$+ |\lambda| \sum_{k=1}^{s} q\Big(x^s_k - x_k\Big) + |\lambda|^{\frac{p_k}{H}} \inf\Big\{(\rho'_s)^{\frac{p_k}{H}} : \rho'_s \in A_4\Big\} \leq$$

$$\leq \max\Big\{|\lambda^s - \lambda|, (|\lambda^s - \lambda|)^{\frac{p_k}{H}}\Big\} g_{\Delta^s_{(vr)}}(x^s) + \max\Big\{|\lambda|, (|\lambda|)^{\frac{p_k}{H}}\Big\} g_{\Delta^s_{(vr)}}(x^s - x). \qquad (3)$$

Note that $g_{\Delta^s_{(vr)}}(x^s) \leq g_{\Delta^s_{(vr)}}(x) + g_{\Delta^s_{(vr)}}(x^s - x)$ for all $s \in \mathbb{N}$.

Hence by our assumption the right hand side of (3) tends to 0 as $s \to \infty$. □

Theorem 5. *Let M, M_1 and M_2 be Orlicz functions. Then the following holds*

$$w^I_0(A, \Delta^s_{(vr)}, M_1, p, q) \cap w^I_0(A, \Delta^s_{(vr)}, M_2, p, q) \subseteq w^I_0(A, \Delta^s_{(vr)}, M_1 + M_2, p, q).$$

Proof. Let $x = (x_k) \in w^I_0(A, \Delta^s_{(vr)}, M_1, p, q) \cap w^I_0(A, \Delta^s_{(vr)}, M_2, p, q)$. Then the result follows from the inequality

$$\sum_{k=1}^{\infty} a_{nk} \Big[(M_1 + M_2)\Big(\frac{q\Big(\Delta^s_{(vr)} x_k\Big)}{\rho}\Big)\Big]^{p_k}$$

$$\leq D \sum_{k=1}^{\infty} a_{nk} \Big[M_1\Big(\frac{q\Big(\Delta^s_{(vr)} x_k\Big)}{\rho}\Big)\Big]^{p_k} + D \sum_{k=1}^{\infty} a_{nk} \Big[M_2\Big(\frac{q\Big(\Delta^s_{(vr)} x_k\Big)}{\rho}\Big)\Big]^{p_k}. \qquad □$$

Theorem 6. *The inclusions $Z(A, \Delta^{s-1}_{(vr)}, M, p, q) \subset Z(A, \Delta^s_{(vr)}, M, p, q)$, are strict for $s \geq 1$. In general $Z(A, \Delta^j_{(vr)}, M, p, q) \subset Z(A, \Delta^s_{(vr)}, M, p, q)$, for $j \in \{0, 1, 2, \ldots, s-1\}$ and the inclusions are strict, where $Z \in \{w^I_0, w^I, w^I_\infty\}$.*

Proof. We shall give the proof for the space $w^I_0(A, \Delta^{s-1}_{(vr)}, M, p, q)$ only. The other spaces can be treated by similar arguments. Let $x = (x_k)$ be any element of the space $w^I_0(A, \Delta^{s-1}_{(vr)}, M, p, q)$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$\Big\{n \in \mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk} \Big[M\Big(\frac{q(\Delta^{s-1}_{(vr)} x_k)}{\rho}\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I.$$

Since M is non-decreasing and convex, it follows that

$$\sum_{k=1}^{\infty} a_{nk} \Big[M\Big(\frac{q(\Delta^s_{(vr)} x_k)}{2\rho}\Big)\Big]^{p_k} = \sum_{k=1}^{\infty} a_{nk} \Big[M\Big(\frac{q(\Delta^{s-1}_{(vr)} x_{k+1} - \Delta^{s-1}_{(vr)} x_k)}{2\rho}\Big)\Big]^{p_k} \leq$$

$$\leq D \sum_{k=1}^{\infty} \Big[\frac{1}{2} M\Big(\frac{q(\Delta^{s-1}_{(vr)} x_{k+1})}{\rho}\Big)\Big]^{p_k} + D \sum_{k=1}^{\infty} a_{nk} \Big[\frac{1}{2} M\Big(\frac{q(\Delta^{s-1}_{(vr)} x_k)}{\rho}\Big)\Big]^{p_k} \leq$$

$$\leq DH \sum_{k=1}^{\infty} a_{nk} \Big[M\Big(\frac{q(\Delta^{s-1}_{(vr)} x_{k+1})}{\rho}\Big)\Big]^{p_k} + DH \sum_{k=1}^{\infty} a_{nk} \Big[M\Big(\frac{q(\Delta^{s-1}_{(vr)} x_k)}{\rho}\Big)\Big]^{p_k},$$

where $H = \max\{1, (\frac{1}{2})^G\}$. Thus we have

$$\Big\{n \in \mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k)}{2\rho}\Big)\Big]^{p_k} \geq \varepsilon\Big\} \subseteq$$
$$\subseteq \Big\{n \in \mathbb{N}\colon DH\sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^{s-1}_{(vr)}x_{k+1})}{\rho}\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\} \cup$$
$$\cup \Big\{n \in \mathbb{N}\colon DH\sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^{s-1}_{(vr)}x_k)}{\rho}\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\}. \tag{4}$$

Since both sets from the right hand side of (3.4) belong to I, we get

$$\Big\{n \in \mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k)}{2\rho}\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I.$$

If follows from the example below that the inclusion is strict. □

Example 1. Let $A = (C, 1)$, $M(x) = x$, for all $x \in [0, \infty), p_k = 1, v_k = 1$ for all $k \in \mathbb{N}$ and $r = 1$. Consider a sequence $x = (x_k) = (k^s)$. Then $x = (x_k)$ belongs to $w_0^I(A, \Delta^s_{(vr)}, M, p, q)$ but does not belong to $w_0^I(A, \Delta^{s-1}_{(vr)}, M, p, q)$, because $\Delta^s_{(vr)}x_k = 0$ and $\Delta^{s-1}_{(vr)}x_k = (-1)^{s-1}$ $(s-1)!$.

Theorem 7. *(a) Let* $0 < \inf p_k \leq p_k \leq 1$, *then*

$$w^I(A, \Delta^s_{(vr)}, M, p, q) \subset w^I(A, \Delta^s_{(vr)}, M, q);\ w_0^I(A, \Delta^s_{(vr)}, M, p, q) \subset w_0^I(A, \Delta^s_{(vr)}, M, q).$$

(b) If $1 < p_k \leq \sup p_k < \infty$, *then*

$$w^I(A, \Delta^s_{(vr)}, M, q) \subset w^I(A, \Delta^s_{(vr)}, M, p, q);\ w_0^I(A, \Delta^s_{(vr)}, M, q) \subset w_0^I(A, \Delta^s_{(vr)}, M, p, q).$$

Proof. (a) Let $x = (x_k) \in w^I(A, \Delta^s_{(vr)}, M, p, q)$. Since $0 < \inf p_k \leq p_k \leq 1$, we have

$$\sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k - \ell)}{\rho}\Big)\Big] \leq \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k - \ell)}{\rho}\Big)\Big]^{p_k}$$

and therefore

$$\Big\{n \in \mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q\Big(\Delta^s_{(vr)}x_k - \ell\Big)}{\rho}\Big)\Big] \geq \varepsilon\Big\} \subseteq$$
$$\subseteq \Big\{n \in \mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q\Big(\Delta^s_{(vr)}x_k - \ell\Big)}{\rho}\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I.$$

(b) Let $1 < p_k \leq \sup p_k < \infty$, and let $x = (x_k) \in w^I(A, \Delta^s_{(vr)}, M, q)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$\sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k - \ell)}{\rho}\Big)\Big] \leq \varepsilon < 1$$

for all $n \geq N$. This implies that

$$\sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k-\ell)}{\rho}\Big)\Big]^{p_k} \leq \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k-\ell)}{\rho}\Big)\Big].$$

Thus we have

$$\Big\{n\in\mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q\Big(\Delta^s_{(vr)}x_k-\ell\Big)}{\rho}\Big)\Big]^{p_k}\geq\varepsilon\Big\}\subseteq$$
$$\subseteq\Big\{n\in\mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q\Big(\Delta^s_{(vr)}x_k-\ell\Big)}{\rho}\Big)\Big]\geq\varepsilon\Big\}\in I.$$

□

Corollary 1. *Let $A=(C,1)$ be a Cesáro matrix and let M be an Orlicz function.*

(a) If $0<\inf p_k\leq p_k\leq 1$, then

$$w^I(\Delta^s_{(vr)},M,p,q)\subset w^I(\Delta^s_{(vr)},M,q);\;\; w^I_0(\Delta^s_{(vr)},M,p,q)\subset w^I_0(\Delta^s_{(vr)},M,q).$$

(b) If $1<p_k\leq\sup p_k<\infty$, then

$$w^I(\Delta^s_{(vr)},M,q)\subset w^I(\Delta^s_{(vr)},M,p,q);\;\; w^I_0(\Delta^s_{(vr)},M,q)\subset w^I_0(\Delta^s_{(vr)},M,p,q).$$

Theorem 8. *The sequence spaces $w^I_0(A,\Delta^s_{(vr)},M,p,q)$ and $w^I_\infty(A,\Delta^s_{(vr)},M,p,q)$ are normal as well as monotone.*

Proof. We give the proof for only $w^I_0[(A,\Delta^s_{(vr)},M,p,q)$. Let $x=(x_k)\in w^I_0(A,\Delta^s_{(vr)},M,p,q)$ and $\alpha=(\alpha_k)$ be a sequence of scalars such that $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$. Then for a given $\varepsilon>0$ we have

$$\Big\{n\in\mathbb{N}\colon \sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}(\alpha_k x_k))}{\rho}\Big)\Big]^{p_k}\geq\varepsilon\Big\}\subseteq$$
$$\subseteq\Big\{n\in\mathbb{N}\colon E\sum_{k=1}^{\infty} a_{nk}\Big[M\Big(\frac{q(\Delta^s_{(vr)}x_k)}{\rho}\Big)\Big]^{p_k}\geq\varepsilon\Big\}\in I,$$

where $E=\max\{1,|\alpha_k|^G\}$.

Hence $(\alpha_k x_k)\in w^I_0(A,\Delta^s_{(vr)},M,p,q)$. Thus the space $w^I_0(A,\Delta^s_{(vr)},M,p,q)$ is normal. Also from Lemma 2.1, it follows that $w^I_0(A,\Delta^s_{(vr)},M,p,q)$ is monotone. □

Theorem 9. *The spaces $Z(A,\Delta^s_{(vr)},M,p,q)$ are complete, where $Z=w^I_0,w^I,w^I_\infty,w_\infty$.*

Proof. We prove the result for the space $w^I_0(A,\Delta^s_{(vr)},M,p,q)$, and for the other spaces it will follow by similar arguments. Let $(x^{(i)})$ be a Cauchy sequence in $w^I_0(A,\Delta^s_{(vr)},M,p,q)$, where $x^{(i)}=(x^i_k)_{k=1}^{\infty}=(x^i_1,x^i_2,x^i_3,\ldots)$ for each $k\in\mathbb{N}$. Let $x_0>0$ be fixed and $t>0$ be given such that $\frac{\varepsilon}{x_0 t}>0$ and $x_0 t>1$ for a given $\varepsilon(0<\varepsilon<1)$. Then there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $g_{\Delta^s_{(vr)}}(x^{(i)} - x^{(j)}) < \frac{\varepsilon}{x_0 t}$, for all $i, j \geq n_0$. This implies that for all $i, j \geq n_0$

$$\sum_{k=1}^{s} q\Big(x_k^i - x_k^j\Big) + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k M\Big(\frac{q(\Delta^s_{(vr)} x_k^i - \Delta^s x_k^j)}{\rho}\Big) \leq 1\Big\} < \frac{\varepsilon}{x_0 t}. \tag{5}$$

Thus, it follows that $\sum_{k=1}^{s}\Big(x_k^i - x_k^j\Big) < \varepsilon,$ for all $i, j \geq n_0$

$$\Rightarrow q\Big(x_k^i - x_k^j\Big) < \varepsilon, \text{ for all } i, j \geq n_0, \ k \in \{1, 2, 3, \ldots\}. \tag{6}$$

Hence (x_k^i) is a Cauchy sequence in X for each $k \in \mathbb{N}$. So it is convergent in X. Let $\lim_{k\to\infty} x_k^{(i)} = x_k$ (say) for each $k \in \mathbb{N}$.

Again from (5) we have

$$\inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k M\Big(\frac{q(\Delta^s_{(vr)} x_k^i - \Delta^s_{(vr)} x_k^j)}{\rho}\Big) \leq 1\Big\} < \varepsilon, \text{ for all } i, j \geq n_0.$$

It follows that

$$\sup_k M\Big(\frac{q(\Delta^s x_k^i - \Delta^s_{(vr)} x_k^j)}{\rho}\Big) \leq 1, \text{ for all } i, j \geq n_0. \tag{7}$$

This implies that

$$M\Big(\frac{q(\Delta^s_{(vr)} x_k^i - \Delta^s x_k^j)}{\rho}\Big) \leq 1, \text{ for all } i, j \geq n_0 \text{ and } k \in \mathbb{N}.$$

For $t > 0$ with $M(\frac{tx_0}{2}) \geq 1$, for all $k \in \mathbb{N}$ we have $M(\frac{q(\Delta^s_{(vr)} x_k^i - \Delta^s_{(vr)} x_k^j)}{\rho}) \leq M(\frac{tx_0}{2})$. By the continuity of the Orlicz function M we have $q(\Delta^s_{(vr)} x_k^i - \Delta^s_{(vr)} x_k^j) \leq \frac{tx_0}{2} \cdot \frac{\varepsilon}{tx_0} = \frac{\varepsilon}{2}$.

Hence $(\Delta^s_{(vr)} x_k^i)_i$ is a Cauchy sequence in X for each $k \in \mathbb{N}$. Thus $(\Delta^s_{(vr)} x_k^i)_i$ is convergent in X for each $k \in \mathbb{N}$. Let $\lim_{i\to\infty} \Delta^s_{(vr)} x_k^i = y_k$ (say) for each $k \in \mathbb{N}$.

For $k = 1$ we have

$$\lim_{i\to\infty} \Delta^s_{(vr)} x_1^i = \lim_{i\to\infty} \sum_{\nu=0}^{s} x_{1+\nu}^i = y_1. \tag{8}$$

Similarly we have

$$\lim_{i\to\infty} \Delta^s_{(vr)} x_1^i = \lim_{i\to\infty} x_k^i = y_k. \tag{9}$$

From (8) and (9) we have $\lim_{i\to\infty} x_{s+1}^i$ exists. Let $\lim_{i\to\infty} x_{s+1}^i = x_{s+1}$. Proceeding in this way inductively we have $\lim_{i\to\infty} x_k^i = x_k$ for each $k \in \mathbb{N}$.

Using the continuity of the Orlicz function M, from (7) we have

$$\sup_k M\Big(\frac{q(\Delta^s_{(vr)} x_k^i - x_k)}{\rho}\Big) \leq 1, \text{ for all } i \geq n_0. \tag{10}$$

Then we have

$$\sum_{k=1}^{s} q\Big(x_k^i - x_k\Big) + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k M\Big(\frac{q(\Delta_{(vr)}^s x_k^i - \Delta^s x_k)}{\rho}\Big) \le 1\Big\} < \varepsilon \text{ for all } i \ge n_0.$$

Now it follows that $(x^i - x) \in w_0^I(A, \Delta_{(vr)}^s, M, p, q)$. Since $(x^i) \in w_0^I(A, \Delta_{(vr)}^s, M, p, q)$ and $w_0^I(A, \Delta_{(vr)}^s, M, p, q)$ is a linear space so we have $x = x^i - (x^i - x) \in w_0^I(A, \Delta_{(vr)}^s, M, p, q)$. □

As a corollary, we obtain the following conclusion.

Theorem 10. *The spaces $Z(A, \Delta_{(vr)}^s, M, p, q)$ are FK-space, where $Z \in \{w_0^I, w^I, w_\infty^I, w_\infty\}$.*

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