УДК 517.5

A. O. Kuryliak, O. B. Skaskiv, D. Yu. Zikrach

ON BOREL'S TYPE RELATION FOR THE LAPLACE-STIELTJES INTEGRALS

A. O. Kuryliak[1], O. B. Skaskiv[2], D. Yu. Zikrach[3]. *On Borel's type relation for the Laplace-Stieltjes integrals*, Mat. Stud. **42** (2014), 134–142.

Let $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function of the form $F(x) = \int_{\mathbb{R}_+} f(u)e^{xu}\nu(du)$, where ν is a Borel measure with unbounded support, f some measurable positive function, $\mu_*(x,F) = \max\{f(u)e^{xu}\colon u \in \operatorname{supp}\nu\}$. We obtain necessary and sufficient conditions for the relation $\ln F(x) \le (1+o(1))\ln\mu_*(x,F)$ to be held as $x \to +\infty$, for each function F outside some set E of zero lower linear density.

А. О. Куриляк, О. Б. Скаскив, Д. Ю. Зикрач. *О сотноношении типа Бореля для интегралов Лапласа-Стильтьеса* // Мат. Студії. – 2014. – Т.42, №2. – С.134–142.

В статье получены необходимые и достаточные условия для того, чтобы соотношение $\ln F(x) \le (1+o(1))\ln\mu_*(x,F)$ имело место для каждой функции $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ вида $F(x) = \int_{\mathbb{R}_+} f(u)e^{xu}\nu(du)$ при $x \to +\infty$ вне некоторого множества E нулевой нижней линейной плотности, где ν — борелева мера с неограниченным носителем, f — некоторая положительная измеримая функция, $\mu_*(x,F) = \max\{f(u)e^{xu}\colon u \in \operatorname{supp}\nu\}$.

1. Introduction and the main result. Let $\mathbb{R}_+ = [0,+\infty)$, ν be a nonnegative measure on $\mathbb{R}_+$ with unbounded support $\operatorname{supp}\nu$ and $f(x)$ an arbitrary nonnegative ν-measurable function on $\mathbb{R}_+$. By $\mathcal{I}(\nu)$ we denote the class of functions $F\colon \mathbb{R} \to \mathbb{R}$ of the form

$$F(x) = \int_{\mathbb{R}_+} f(u)e^{xu}\nu(du). \tag{1}$$

For $F \in \mathcal{I}(\nu)$ and $x \in \mathbb{R}$ we denote

$$\mu_*(x,F) = \sup\{f(u)e^{xu}\colon u \in \operatorname{supp}\,\nu\}.$$

We remark that the condition $(\forall x \in \mathbb{R})\colon\ \mu_*(x,F) < +\infty$ is fulfilled if and only if (for example see [1, 2])

$$\varliminf_{\substack{u\to+\infty\\ u\in\operatorname{supp}\,\nu}} \frac{-\ln f(u)}{u} = +\infty.$$

Denote by L the class of nonnegative continuous functions $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) \to +\infty$ as $t \to +\infty$ and by L^+ the subclass of functions $\psi \in L$ such that $\psi(t) \nearrow +\infty$ as $t \to +\infty$.

2010 *Mathematics Subject Classification:* 30B50.
Keywords: Laplace–Stieltjes integral; exceptional set; asymptotic estimate.

The following result can be deduced from [3] (for entire Dirichlet series see in [5]).

Theorem A ([3]). *Let $F \in \mathcal{I}(\nu, \Phi)$. If*

$$\int_0^{+\infty} \frac{d \ln \nu_0(t)}{t} < +\infty, \quad \nu_0(t) := \nu((0,t]), \tag{2}$$

then the relation

$$\ln F(x) \le (1 + o(1)) \ln \mu_*(x, F) \tag{3}$$

holds as $x \to +\infty$ ($x \notin E$), where $E \subset \mathbb{R}_+$ is some set of finite Lebesgue measure on $\mathbb{R}_+$, i.e. meas $E = \int_E dx < +\infty$.

In [4] (see also [5]) it was proved that, for every positive measure ν such that $\ln \nu_0(t) = O(t)$ $(t \to +\infty)$ and $\int_0^{+\infty} t^{-1} d \ln \nu_0(t) = +\infty$ there exist a function $F \in \mathcal{I}(\nu)$ and a positive constant $d > 0$ such that the inequality $\ln F(x) \ge (1 + d) \ln \mu_*(x, F)$ holds for all $x \ge x_0$, i.e. condition (2), in some sense, is a necessary condition for the conclusion of Theorem A.

Let $\Phi \in L^+$. By $\mathcal{I}(\nu, \Phi)$ we denote the class of functions $F \in \mathcal{I}(\nu)$ such that

$$(\exists c > 0)\colon \quad \ln F(x) \le \Phi(cx) \quad (x \ge x_0),$$
$$\mathcal{I}^*(\nu, \Phi) := \{F \in \mathcal{I}(\nu)\colon (\exists c > 0)(\exists x_j \to +\infty)[\, \ln F(x) \le \Phi(cx) \ (x = x_j,\ j \ge 1)]\}.$$

Proposition 7 of [6, p.135–137] implies the following assertion.

Theorem B. *Let $\Phi \in L^+$, $F \in \mathcal{I}(\nu, \Phi)$. If*

$$(\forall\, \eta > 0)\colon \quad \lim_{R \to +\infty} \frac{1}{R} \int_0^{\eta \Phi(R)} \frac{d \ln \nu_0(t)}{t} = 0, \tag{4}$$

then relation (3) *holds as $x \to +\infty$ ($x \notin E$), where E is a set of zero linear density, i.e.*

$$\mathcal{D}E := \overline{\lim_{R \to +\infty}} \frac{1}{R} \operatorname{meas}(E \cap [0, R]) = 0.$$

From [8, Theorem 1] one can deduce the next statement.

Theorem 1. *Let $\Phi_0(x) = x\Phi(x)$, $\Phi \in L^+$, $F \in \mathcal{I}(\nu, \Phi_0)$. If condition* (4) *is satisfied then relation* (3) *holds as $\sigma \to +\infty$ ($\sigma \notin E$), where E is some set of zero linear density, i.e. $\mathcal{D}E = 0$.*

Remark 1. If $t\Phi(t) = O(\Phi(2t))$ $(t \to +\infty)$, then $\mathcal{I}(\nu, \Phi) = \mathcal{I}(\nu, \Phi_0)$ with $\Phi_0(x) := x\Phi(x)$. But, in general, $\mathcal{I}(\nu, \Phi) \subset \mathcal{I}(\nu, \Phi_0)$, $\mathcal{I}(\nu, \Phi) \ne \mathcal{I}(\nu, \Phi_0)$.

From Proposition 8 in [6, p. 137–138] we can obtain the following assertion.

Theorem C. *Let $\Phi \in L^+$, $F \in \mathcal{I}^*(\nu, \Phi)$. If condition* (4) *is satisfied, then there exists a measurable set $E \subset \mathbb{R}_+$ such that*

$$\underline{\mathcal{D}}E := \lim_{x \to +\infty} \frac{1}{x} \operatorname{meas}(E \cap [0, x]) = 0 \tag{5}$$

and relation (3) *holds as $x \to +\infty$ ($x \in \mathbb{R}_+ \setminus E$).*

From [8, Theorem 2] we get the next assertion.

Theorem 2. *Let $\Phi \in L^+, F \in \mathcal{I}^*(\nu, \Phi)$. If condition* (4) *is satisfied then there exists a measurable set $E \subset \mathbb{R}_+$ such that $\underline{D}E = 0$ and relation* (3) *holds as $x \to +\infty$ $(x \in \mathbb{R}_+ \setminus E)$.*

Conjecture 1. *Condition* (4) *is a necessary condition for the conclusions of Theorems 1 and 2.*

In this paper we prove the following theorem.

Theorem 3. *Let $\Phi_0(x) = x\Phi(x)$, $\Phi \in L^+$, $F \in \mathcal{I}(\nu, \Phi_0)$. If conditions*

$$(\forall \eta > 0)\colon \quad \ln \nu_0(\eta\Phi(t)) = o(t\Phi(t)) \ (t \to +\infty) \tag{6}$$

and

$$(\forall\ \eta > 0)\colon\ \varliminf_{R\to+\infty} \frac{1}{R} \int\limits_0^{\eta\Phi(R)} \frac{d\ln \nu_0(t)}{t} = 0, \quad \nu_0(t) := \nu((0,t])) \tag{7}$$

are satisfied then relation (3) *holds as $x \to +\infty$ $(x \notin E)$, where E is a set of zero linear lower density, i.e. $\underline{D}E = 0$.*

We remark that condition (7) implies that the equality

$$\varliminf_{t\to\infty} \frac{\ln \nu_0(\eta\Phi(t))}{t\Phi(t)} = 0$$

holds for every $\eta > 0$.

The following theorem shows that condition (7) is a necessary condition for the conclusion of Theorem 3.

Theorem 4. *Let $\Phi \in L^+$. If conditions*

$$(\exists \eta > 0)(\exists b > 0)\colon \quad \varliminf_{R\to+\infty} \frac{1}{R} \int\limits_0^{\eta\Phi(R)} \frac{d\ln \nu_0(t)}{t} > b, \quad \int\limits_0^{+\infty} e^{-\eta t} d\nu_0(t) < +\infty \tag{8}$$

are satisfied then for every $h > 0$ there exists a function $F \in \mathcal{I}(\nu, \Phi_0)$, $\Phi_0(x) = x\Phi(x)$ such that for all $x \geq x_0$ the inequality

$$\ln F(x) \geq (1 + h) \ln \mu_*(x, F) \tag{9}$$

holds.

Conjecture 2. *The assertion of Theorem 3 is valid without condition* (6).

Conjecture 3. *The assertion of Theorem 4 is true without the second condition of* (8).

Remark 2. It is easy to see that the second condition of (8) is satisfied if and only if $\ln \nu_0(t) = O(t)$ $(t \to +\infty)$.

2. Proof of the main results. We define a class of positive functions by setting

$$L(\Phi) = \Big\{\psi \in L^+ \colon \ (\forall b > 0)\Big[\varliminf_{t\to+\infty} \frac{1}{t} \int^{b\Phi(t)} \frac{d\psi^{-1}(x)}{x} = 0\Big], \Phi(t) = o\left(\psi(t\Phi(t))\right) \quad (t \to +\infty)\Big\}.$$

We need the following two lemmas.

Lemma 1 ([7, 11])**.** *Let $\varphi,\ \psi \in L^+$ be two functions such that*

$$A_1(R) \overset{def}{=} \frac{1}{\varphi(R)} \int^{R} \frac{d\psi^{-1}(t)}{t} = o(1) \quad (R \to +\infty,\ R \in G),$$

$G \subset \mathbb{R}_+$, and $R = o(\psi(R\varphi(R)))$ $(R \to +\infty)$. Then

$$A_2(R) \overset{def}{=} \frac{1}{\varphi(R)} \int^{R\varphi(R)} \frac{dx}{\psi(x)} = o(1) \quad (R \to +\infty,\ R \in G).$$

Remark 3. It is easy to see that the conditions $R = o(\psi(R\varphi(R)))$ $(R \to +\infty)$ and

$$(\forall b > 0):\ \ \psi^{-1}(R) = o(R\varphi(bR)) \quad (R \to +\infty)$$

are equivalent.

Lemma 2. *Let $\Phi_1 \in L,\ \psi \in L(\Phi_1)$. If $g(x)$ is a positive differentiable nondecreasing function on $[0, +\infty)$ such that $g(x) \le x\Phi_1(x)$ $(x \ge x_0)$, then for the set $E = \{x \ge 0 : g'(x) \ge \psi(g(x))\}$ we have*

$$\frac{1}{R} meas(E \cap [0, R]) \to 0 \quad (R = R_j \to +\infty)$$

for some sequence $0 < R_j \uparrow +\infty$ $(1 \le j \uparrow +\infty)$.

Proof. The condition $\psi \in L(\Phi_1)$, $\Phi_1 \in L^+$ implies that there exists a sequence (R_j) such that $0 < R_j \uparrow +\infty$ $(1 \le j \uparrow +\infty)$ and

$$\frac{1}{R} \int_0^{\Phi_1(R)} \frac{d\psi^{-1}(x)}{x} \to 0 \quad (R = R_j \to +\infty).$$

Therefore, using Lemma 1 we obtain

$$\frac{1}{R}\text{meas}(E \cap [0, R]) \le \frac{1}{R} \int_{E\cap[0,R]} \frac{g'(x)}{\psi(g(x))} dx \le \frac{1}{R} \int_0^{g(R)} \frac{du}{\psi(u)} \le \frac{1}{R} \int_0^{R\Phi_1(R)} \frac{du}{\psi(u)} = o(1),$$

$(R = R_j \to +\infty)$. □

Proof of Theorem 3. From conditions (6) and (7) it follows that there exists a function $\psi \in L^+$ such that

$$(\forall b > 0):\quad \psi^{-1}(b\Phi(R)) = o(R\Phi(R)), \qquad \varliminf_{R\to+\infty} \frac{1}{R} \int_0^{b\Phi(R)} \frac{d\psi^{-1}(t)}{t} = 0, \tag{10}$$

$$\ln \nu_0(R) = o\left(\psi^{-1}(R)\right) \quad (R \to +\infty), \tag{11}$$

i.e., by Remark 3 we have $\psi \in L(\Phi)$.

For any fixed $x > 0$ we obtain

$$\int\limits_{u>2(\ln F(x))'} f(u)e^{ux}\nu(du) \le \int\limits_{u\ge 2(\ln F(x))'} \frac{u}{2(\ln F(x))'} f(u)e^{ux}\nu(du) \le$$
$$\le \frac{1}{2(\ln F(x))'} \int\limits_0^{+\infty} uf(u)e^{ux}\nu(du) = \frac{F(x)}{2}.$$

Therefore, $F(x) \le \int_{u\le 2(\ln F(x))'} f(u)e^{ux}\nu(du) + F(x)/2$ and

$$F(x) \le 2 \int\limits_{u\le 2(\ln F(x))'} f(u)e^{ux}\nu(du) \le 2\mu_*(x)\nu_0\big(2(\ln F(x))'\big) \tag{12}$$

for every $x > 0$.

Applying Lemma 2 with $g(x) = \ln F(x)$, $\Phi_1(x) = \Phi(x)$ and $\psi(t) = \frac{1}{2}\psi_1(t)$ we get

$$g'(x) \le \frac{1}{2}\psi_1(g(x))$$

for all $x \in \mathbb{R}_+ \setminus E$, $\underline{D}E = 0$. Hence using (11) and (12) we obtain

$$\ln F(x) \le \ln 2 + \ln \mu_*(x) + \ln \nu_0\big(2g'(x))\big) \le \ln 2 + \ln \mu_*(x) + \ln \nu_0\big(\psi_1(g(x))\big) \le$$
$$\le \ln 2 + \ln \mu_*(x) + o(\ln F(x))$$

as $x \to +\infty$ $(x \in \mathbb{R}_+ \setminus E)$. Thus $(1+o(1)) \ln F(x) \le \ln \mu_*(x, F)$ as $x \to +\infty$ $(x \in \mathbb{R}_+ \setminus E)$. □

Proof of Theorem 4. Following [4] we put

$$N_0(t) = \int\limits_1^t \frac{\nu_0(x)}{x} dx, \quad \nu_0(t) = \nu(0; t], \quad B = \frac{1}{1+h}, \quad h > 0,$$

$$\psi(u) = -Bu \int\limits_1^u \frac{\ln\left(N_0(0,5(t+1))/\ln(t+1)\right)}{t^2} dt, \quad f(u) = \begin{cases} \exp\{\psi(u)\}, & u \ge 1, \\ 1, & 0 < u \le 1. \end{cases}$$

We prove that a function F defined by the integral of the form (1). Indeed, the condition $\int_0^{+\infty} e^{-\eta t} d\nu_0(t) < +\infty$ implies that

$$F(x) = \int\limits_0^{+\infty} f(u)e^{xu}\nu(du) = \int\limits_0^{+\infty} f(u)e^{xu} d\nu_0(u) \le \mu_*(x+\eta, F) \cdot \int\limits_0^{+\infty} e^{-\eta u} d\nu_0(u). \tag{13}$$

Now, for each fixed $x \in \mathbb{R}_+$ we consider the function $\psi_0(u, x) = \psi(u) + xu$. It is easy to see that $\psi_0(u, x)$ is a concave function of $u \ge 1$ for each fixed $x \in \mathbb{R}_+$ and has a unique point of maximum $\overline{u} = u(x) \in [1, +\infty)$. We can find this point from the equation

$$\frac{\partial \psi}{\partial u} = -B \int\limits_1^u t^{-2} \ln\left(\frac{N_0(0,5(t+1)}{\ln(t+1)}\right) dt - \frac{B}{u} \ln\left(\frac{N_0(0,5(u+1))}{\ln(u+1))}\right) + x = 0,$$

and also $\psi(u,x) \ge \psi(1,x) = x \ge 0$ $(1 \le u \le \overline{u},\ x \ge 0)$. Hence

$$\ln \mu_*(x,F) = \sup\{\ln f(u) + xu\colon u \in \operatorname{supp}\nu\} \le \max\{\psi(u) + ux\colon u \ge 1\} =$$
$$= \psi(\overline{u}) + \overline{u}x = B\ln\left(\frac{N_0(0,5(u+1))}{\ln(u+1))}\right) \le B\ln\nu_0(0,5(\overline{u}+1)) \le B\ln\nu_0(\overline{u}) < +\infty, \tag{14}$$

and $F \in \mathcal{I}(\nu)$. On the other hand, for $x \ge 0$ we obtain

$$F(x) \ge \int\limits_0^{\overline{u}} f(u)e^{xu}\nu(du) \ge \int\limits_0^{\overline{u}} \nu(du) = \nu_0(\overline{u}) - \nu_0(0) = \nu_0(\overline{u}).$$

Using inequality (14) we have

$$\ln F(x) \ge \ln\nu_0(\overline{u}) \ge \frac{1}{B}\ln\mu_*(x,F) = (1+h)\cdot\ln\mu_*(x,F) \quad (x \ge x_0).$$

The first condition of (8) yields

$$(\exists\eta > 0)(\exists b > 0)\colon \qquad \varliminf_{R\to+\infty} \frac{1}{R}\int\limits_0^{\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt > b. \tag{15}$$

Indeed, if we assume that

$$(\forall\delta > 0)\colon \qquad \varliminf_{R\to+\infty} \frac{1}{R}\int\limits_0^{\delta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt = 0$$

then for any fixed $\eta > 0$ there exists a sequence $R_j \to +\infty$ $(j \to +\infty)$ such that

$$\max\left\{\frac{1}{R}\int\limits_0^{2\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt,\ \frac{1}{R}\int\limits_{\eta\Phi(R)}^{2\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt\right\} = \frac{1}{R}\int\limits_0^{2\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt \to 0$$

as $R = R_j \to +\infty$. From the inequality

$$\frac{1}{R}\int\limits_{\eta\Phi(R)}^{2\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt \ge \frac{\ln\nu_0(\eta\Phi(R))}{2\eta R\Phi(R)} \quad (R > 0),$$

we obtain

$$\frac{\ln\nu_0(\eta\Phi(R))}{R\Phi(R)} \to 0 \quad (R = R_j \to +\infty).$$

But

$$0 < b < \frac{1}{R}\int\limits_0^{\eta\Phi(R)} \frac{d\ln\nu_0(t)}{t} = \frac{\ln\nu_0(\eta\Phi(R))}{\eta R\Phi(R)} + \frac{1}{R}\int_0^{\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt \le$$
$$\le \frac{\ln\nu_0(\eta\Phi(R))}{\eta R\Phi(R)} + \frac{1}{R}\int_0^{2\eta\Phi(R)} \frac{\ln\nu_0(t)}{t^2}dt \to 0 \quad (R = R_j \to +\infty).$$

We have got a contradiction.

Remark that $N_0(t) \geq \int_{t/e}^{t} \frac{\nu_0(x)}{x}dx \geq \nu_0(t/e) \geq \nu_0(t/3)$ $(t > 0)$. Then

$$\int\limits_1^y \frac{\ln\left(N_0((t+1)/2)/\ln(t+1)\right)}{t^2}dt \geq \int\limits_1^y \frac{\ln\left(\nu_0\left((t+1)/6\right)/\ln(t+1)\right)}{t^2}dt \geq$$

$$\geq \int\limits_1^y \frac{\ln \nu_0\left(t/6\right)}{t^2}dt - \int\limits_1^y \frac{\ln\ln(t+1)}{t^2}dt \geq \frac{1}{6}\int\limits_0^{y/6} \frac{\ln \nu_0(u)}{u^2}du - c,$$

where $c > 0$ is some constant. Hence by conditions (8) we obtain

$$\int\limits_1^{6\eta\Phi(R)} \frac{\ln\left(N_0((t+1)/2)/\ln(t+1)\right)}{t^2}dt \geq \frac{1}{6}\int\limits_0^{\eta\Phi(R)} \frac{\ln \nu_0(u)}{u^2} - c \geq \frac{bR}{12} \quad (R \geq R_0),$$

and

$$\ln f(u) = \psi(u) \leq -\frac{bB}{12}\cdot u\varphi\Big(\frac{u}{6\eta}\Big) = -c_1 u\varphi(c_2 u) \quad (u \geq 6\eta\Phi(r_0)),$$

where the function φ is the inverse function to Φ, and $c_1, c_2 > 0$. Then for large enough x we have

$$\ln \mu_*(x, F) \leq \max\{\max\{\psi(u) + xu\colon u \geq 6\eta\Phi(r_0)\},\ \max\{\psi(u) + xu\colon 0 \leq u < 6\eta\Phi(r_0)\}\} \leq$$

$$\leq \max\{-c_1 u\varphi(c_2 u) + xu\colon u \geq 6\eta\Phi(r_0)\} \leq \max\Big\{-\frac{c_1}{c_2}v\Phi(v) + \frac{x}{c_2}\Phi(v)\colon v \geq 0\Big\} =$$

$$= \max\Big\{\frac{x - c_1 v}{c_2}\Phi(v)\colon 0 \leq v \leq \frac{x}{c_1}\Big\} \leq \frac{x}{c_2}\Phi\Big(\frac{x}{c_1}\Big).$$

Finally, from (13) it follows that $F \in \mathcal{I}(\nu, \Phi_0)$. □

3. Corollaries. Let $\lambda = (\lambda_n)$ be a sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leq n \uparrow +\infty)$, and $\nu(E) := \sum_{\lambda_n \in E} \delta_{\lambda_n}(E)$ for any bounded set $E \subset \mathbb{R}_+$, where $\delta_\lambda(E) = 1$ at $\lambda \in E$ and $\delta_\lambda(E) = 0$ otherwise. Then for a function $F \in \mathcal{I}(\nu)$ and $x \geq 0$ we have an entire Dirichlet series

$$F(x) = \int\limits_{\mathbb{R}_+} f(u)e^{xu}\nu(du) = \sum_{n=0}^{+\infty} f(\lambda_n)e^{x\lambda_n}.$$

Denote by $H(\lambda, \Phi)$ the class of entire Dirichlet series with fixed sequence of exponents λ of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n},$$

such that

$$(\exists c > 0)\colon \quad \ln \mathfrak{M}(x, F) \leq \Phi(cx) \quad (x \geq x_0), \quad \mathfrak{M}(x, F) := \sum_{n=0}^{+\infty} |a_n| e^{z\lambda_n}.$$

From Theorem 3 we obtain the following corollary.

Corollary 1. *Let* $\Phi_0(x) = x\Phi(x)$, $\Phi \in L^+$, $F \in H(\lambda, \Phi_0)$. *If conditions*

$$(\forall \eta > 0)\colon \qquad \ln n(\eta\Phi(t)) = o(t\Phi(t))\ (t \to +\infty) \tag{16}$$

and

$$(\forall\ \eta > 0)\colon\ \ \varliminf_{R\to+\infty} \frac{1}{R} \int\limits_0^{\eta\Phi(R)} \frac{d\ln n(t)}{t} = 0, \quad n(t) := \sum_{\lambda_n \le t} 1\ ,$$

are satisfied then the relation

$$\ln M(x, F) = (1 + o(1)) \ln \mu(x, F)$$

holds as $x \to +\infty$, $(x \notin E)$, *where* E *is a set of zero linear lower density, i.e.* $\underline{D}E = 0$, $M(x, F) = \sup\{|F(x + iy)|\colon y \in \mathbb{R}\}$, $\mu(x, F) = \max\{|a_n|e^{x\lambda_n}\colon n \ge 0\}$.

In [9, Theorem 2] we prove the statement of Corollary 1 with the condition

$$\sup\Big\{\frac{\ln n}{\lambda_n}\colon n \ge m\Big\} = O\Big(\frac{\ln m}{\lambda_m}\Big) \quad (m \to +\infty) \tag{17}$$

instead of condition (16). Remark that condition (17) implies $\ln n = O(\lambda_n)$ $(n \to +\infty)$, i.e. $\ln n(t) = O(t)$ $(t \to +\infty)$. Thus condition (16) follows from condition (17).

The statement of Corollary 1 follows also from Theorem 3 in [10].

From Theorem 4 we obtain the following corollary (see also [9, Theorem 2]).

Corollary 2. *Let* $\Phi \in L^+$. *If conditions*

$$(\exists \eta > 0)(\exists b > 0)\colon \qquad \varliminf_{R\to+\infty} \frac{1}{R} \int\limits_0^{\eta\Phi(R)} \frac{d\ln n(t)}{t} > b, \qquad \int\limits_0^{+\infty} e^{-\eta t} dn(t) < +\infty$$

are satisfied then for every $h > 0$ *there exists a function* $F \in H(\lambda, \Phi)$ *such that for all* $x \ge x_0$ *one has*

$$\ln M(x, F) \ge (1 + h) \ln \mu(x, F).$$

Condition (17) implies that $\ln n(t) = O(t)$ $(t \to +\infty)$. Therefore, by Remark 2, $\int_0^{+\infty} e^{-\eta t} dn(t) < +\infty$ for some $\eta > 0$ large enough.

4. Concluding remarks. Let ν be a discrete measure on $\mathbb{R}_+$ with unbounded support. From the results of [12, 13] it follows that the boundedness of the Lebesgue measure of an exceptional set E in Theorem A is the best possible in this case (for similar statements on the class of multiple Dirichlet series see [14] and on the class of Laplace integrals of several variables see [15]). In this connection the following questions arise.

Question 1. *Let* ν *be an absolutely continuous or singular measure. Whether the same is true for these cases?*

Question 2. *Is the description of an exceptional set in theorems 1–3 best possible?*

REFERENCES

1. Posiko O.S., Skaskiv O.B., Sheremeta M.M. *Estimates of the Laplace-Stieltjes integral*// Mat. Stud. – 2004. – V.21, №2. – P. 179–186. (in Ukrainian)
2. Posiko O.S., Sheremeta M.M. *Asymptotic estimates for Laplace-Stieltjes integrals*// Ukr. Math. Visn. – 2005. – V.2, №4. – P. 541–549 (in Ukrainian); English transl. in Ukr. Math. Bull., V.2, №4. – P. 547–555.
3. Skaskiv O.B. *On certain relations between the maximum modulus and the maximal term of an entire Dirichlet series*// Mat. zametki. – 1999. – V.66, №2. – P. 282–292 (in Russian); English transl. in Math. Notes. – 1999. – V.66, №2. – P. 223–232.
4. Skaskiv O.B., Trakalo O.M. *On the stability of the maximum term of the entire Dirichlet series*// Ukr. Mat. Zh. – 2005. – V.57, №4. – P. 571–576 (in Ukrainian); English transl. in Ukr. Math. J. – 2005. – V.57, №4. – P. 686–693.
5. Skaskiv O.B. *Behavior of the maximum term of a Dirichlet series that defines an entire function*// Mat. Zametki. – 1985. – V.37, №1. – P. 41–47 (in Russian); English transl. in Math. Notes. – 1985. – V.37, №1 – P. 24–28.
6. Skaskiv O.B., Trakalo O.M. *Asymptotic estimates for Laplace integrals*// Mat. Stud. – 2002. – V.18, №2. – P. 125–146. (in Ukrainian)
7. Skaskiv O.B., Asymptotic property of analytic functions represented by power series end Dirichlet series: Doctoral thesis, Lviv, 1995. – 299 p.
8. Zikrach D.Yu., Skaskiv O.B. *Asymptotic external estimation of the exeptional sets of Laplace-Stieltjes integrals*// Nauk. Visn. Chernivets'kogo Univ. Math. – 2011. – V.1, №3. – P. 38–43.
9. Skaskiv O.B. *On the central exponent of absolutely convergent Dirichlet series*// Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky. – 2000. – №10. – P. 27–30.
10. Skaskiv O.B. *A generalized Borel relation for entire Dirichlet series*// Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky. – 2004. – №6. – P. 32–36.
11. Zikrach D.Yu., Skaskiv O.B. *On the description of an exceptional set in Borel's relation for multiple Dirichlet series with upper restriction on the growth*// Mat. Stud. – 2009. – V.32, №2. – P. 139–147. (in Ukrainian)
12. Salo T.M., Skaskiv O.B., Trakalo O.M. *On the best possible description of exceptional set in Wiman–Valiron theory for entire function*// Mat. Stud. – 2001. – V.16, №2. – P. 131–140.
13. Filevych P.V. *Asymptotic relations between the means of Dirichlet series and their applications*// Mat. Stud. – 2002. – V.19, №2. – P. 127–140.
14. Skaskiv O.B., Zikrach D.Yu. *The best possible description of exceptional set in Borel's relation for multiple Dirichlet series*// Mat. Stud. – 2008. – V.30, №2. – P. 189–194.
15. Skaskiv O.B., Zikrach D.Yu. *On the best possible description of an exceptional set in asymptotic estimates for Laplace–Stieltjes integrals* // Mat. Stud. – 2011. – V.35, №2. – P. 131–141.

[1,2]Ivan Franko National University of Lviv,
[1]andriykuryliak@gmail.com
[2]matstud@franko.lviv.ua
[3]Ukrainian Academy of Printing,
zikrach.dm@gmail.com

Received 27.02.2013