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ESTIMATES FOR THE MAXIMUM MODULUS OF ANALYTIC CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS ON SOME SEQUENCES

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Let φ be the characteristic function of a probability law F analytic in $\mathbb{D}_R=\{z\colon |z|<R\}$, $0<R\le+\infty$, $M(r,\varphi)=\max\{|\varphi(z)|\colon |z|=r\}$ and $W_F(x)=1-F(x)+F(-x)$, $x\ge 0$. We obtain upper estimates for $\underline{\lim}_{r\uparrow R}(\ln M(r,\varphi))/\Phi(r)$ for some positive convex on $(0,R)$ function Φ under certain conditions on W_F.

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Пусть φ — характеристическая функция вероятностного закона F, аналитическая в $\mathbb{D}_R=\{z\colon |z|<R\}$, $0<R\le+\infty$, $M(r,\varphi)=\max\{|\varphi(z)|\colon |z|=r\}$ и $W_F(x)=1-F(x)+F(-x)$, $x\ge 0$. Для некоторой положительной выпуклой на $(0,R)$ функции Φ при определенных условиях на W_F получены оценки сверху для $\underline{\lim}_{r\uparrow R}(\ln M(r,\varphi))/\Phi(r)$.

1. Introduction and preliminary results. A non-decreasing left continuous on $(-\infty,+\infty)$ function F is said to be a *probability law* ([1, p. 10]) if $\lim_{x\to+\infty}F(x)=1$ and $\lim_{x\to-\infty}F(x)=0$, and the function $\varphi(z)=\int_{-\infty}^{+\infty}e^{izx}dF(x)$ defined for real z is called the *characteristic function of this law* ([1, p. 12]). If φ has an analytic extension to the disk $\mathbb{D}_R=\{z\colon |z|<R\}$, $0<R\le+\infty$, then we call φ to be the analytic in $\mathbb{D}_R$ characteristic function of the law F. In the sequel we always assume that $\mathbb{D}_R$ is the maximal disk of the analyticity of φ. It is known ([1, p. 37–38]) that φ is the analytic in $\mathbb{D}_R$ characteristic function of a law F if and only if $W_F(x)=:1-F(x)+F(-x)=O(e^{-rx})$ as $0\le x\to+\infty$ for every $r\in[0,R)$. Hence it follows that $\underline{\lim}_{x\to+\infty}\frac{1}{x}\ln\frac{1}{W_F(x)}=R$. If we put $M(r,\varphi)=\max\{|\varphi(z)|\colon |z|=r\}$ and $\mu(r,\varphi)=\sup\{W_F(x)e^{rx}\colon x\ge 0\}$ for $0\le r<R$ then ([1, p. 55], see also [2]) $\mu(r,\varphi)\le 2M(r,\varphi)$. Therefore, the estimates from below for $\ln M(r,\varphi)$ follow from the same estimates for $\ln\mu(r,\varphi)$. Further we assume that $\ln\mu(r,\varphi)\uparrow+\infty$ as $r\uparrow R$, that is

$$\overline{\lim_{x\to+\infty}}\, W_F(x)e^{Rx}=+\infty. \tag{1}$$

On the other hand ([1, p. 52]), $M(r,\varphi)\le I(r,\varphi)+1+W_F(0)$ for all $r\in[0,R)$, where $I(r,\varphi)=\int_0^\infty W_F(x)e^{rx}dx$. Since it is possible to estimate $I(r,\varphi)$ via $\mu(r,\varphi)$ we will obtain

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the corresponding estimates for $M(r,\varphi)$ via $\mu(r,\varphi)$. Therefore, the investigation of the relationship between the growth of $M(r,\varphi)$ and the decrease of $W_F(x)$ reduces to the study of the behavior of $\mu(r,\varphi)$.

For entire characteristic functions the relationship between the growth of $M(r,\varphi)$ and the decrease of $W_F(x)$ in terms of the order and the type is investigated by B. Ramachandran ([3], see also [1, p. 54]). N. I. Jakovleva ([4–5]) obtained such a relationship in terms of generalized orders. Some additions tothe results of N. I. Jakovleva are obtained by B. V. Vynnyts'kyi ([6]) and M. Dewess ([7]). V. M. Sorokivs'kyi ([8]) investigated the relationship between the growth of $M(r,\varphi)$ and the decrease of $W_F(x)$ for analytic functions in the disk $\mathbb{D}_1$. The most general results are obtained ([9]) for entire as well as analytic in $\mathbb{D}_R$, $R<+\infty$, characteristic functions.

For the lower order

$$\lambda[\varphi]=\varliminf_{r\to+\infty}\frac{\ln\ln M(r,\varphi)}{\ln r}$$

of an entire characteristic function φ N. I. Jakovleva ([4]) obtained a lower estimate. She proved that if

$$\varliminf_{x\to+\infty}\frac{\ln x}{\ln\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)}\ge\lambda \quad\text{then}\quad \varliminf_{r\to+\infty}\frac{1}{\ln r}\ln\frac{\ln M(r,\varphi)}{r}\ge\lambda.$$

This result is generalized in [10]; namely, it is proved that if there exists an increasing to $+\infty$ sequence (x_k) such that $x_{k+1}=O(x_k)$ as $k\to+\infty$ and

$$\ln x_k\ge\lambda\ln\left(\frac{1}{x_k}\ln\frac{1}{W_F(x_k)}\right)\quad\text{then}\quad\varliminf_{r\to+\infty}\frac{1}{\ln r}\ln\frac{\ln M(r,\varphi)}{r}\ge\lambda.$$

Various generalizations of this result are obtained in [10] for entire as well as for analytic in $\mathbb{D}_R$, $R<+\infty$, characteristic functions.

For an upper estimate of $\lambda[\varphi]$ it is proved in [4] that if

$$\varlimsup_{r\to+\infty}\frac{\ln\ln M(r,\varphi)}{\ln r}=\varrho>1\quad\text{and}\quad\varlimsup_{x\to+\infty,\ x\in U}\frac{\ln x}{\ln\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)}\le\delta<\varrho-1,$$

where $U=\bigcup_j(a_{2j},\,a_{2j+1})$ and $\varlimsup_{j\to\infty}(a_{2j}/a_{2j+1})\le\delta/(\varrho-1)$ then $\lambda[\varphi]\le1+\delta$.

Below we will show that, in this result the set U of intervals can be replaced with a sequence that increases to $+\infty$ not very quickly.

As in [10], by $\Omega(0,R)$, $0<R\le+\infty$, we denote the class of positive unbounded functions Φ on $[r_0,R)$ for some $r_0\in[0,R)$ such that the derivative Φ' is positive, continuously differentiable and increasing to $+\infty$ on (r_0,R). For $\Phi\in\Omega(0,R)$ let $\Psi(r)=r-\frac{\Phi(r)}{\Phi'(r)}$ be the function associated with Φ in the sense of Newton and ϕ be the inverse function to Φ'. It is known ([11]) that the function Ψ is continuously differentiable on $[r_0,R)$, $\Psi(r)\uparrow R$ as $r\uparrow R$, the function ϕ is continuously differentiable and increasing to R on $(x_0,+\infty)$ and the following lemma is true.

Lemma 1 ([11], Theorem 2.1). *Let $\Phi\in\Omega(0,R)$, $0<R<+\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F, which satisfies (1). Then in order that $\ln\mu(r,\varphi)\le\Phi(r)$ for all $r\in[r_0,R)$ it is necessary and sufficient that $\ln W_F(x)\le-x\Psi(\phi(x))$ for all $x\ge x_0$.*

The following assertion is also true.

Proposition 1. *Let $\Phi \in \Omega(0,R)$, $0<R\le+\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F, which satisfies (1). Then if $\ln\mu(r_k,\varphi)\le\Phi(r_k)$ for some sequence (r_k) increasing to R then*

$$\ln W_F(x_k)\le -x_k\Psi(\phi(x_k)) \tag{2}$$

for all k, where $x_k=\Phi'(r_k)$.

Indeed, the condition $\ln\mu(r_k,\varphi)\le\Phi(r_k)$ implies that $\ln W_F(x)\le\Phi(r_k)-xr_k$ for all $x\ge0$ and $k\ge1$. Therefore,

$$\ln W_F(x_k)=\ln W_F(\Phi'(r_k))\le\Phi(r_k)-r_k\Phi'(r_k)=-\Phi'(r_k)\Psi(r_k)=-x_k\Psi(\phi(x_k)).$$

In view of Proposition 1 the question arises whether inequality (2) for some increasing to $+\infty$ sequence (x_k) implies the estimate $\ln\mu(r_k,\varphi)\le\Phi(r_k)$ for some sequence (r_k) increasing to R. The answer is negative in general because the following statement is true.

Proposition 2. *For every function $\Phi\in\Omega(0,R)$, $0<R\le+\infty$, and increasing to $+\infty$ sequence (x_k) of positive numbers there exists a probability law F such that (2) holds for all $k\ge1$ and $\ln\mu(r,\varphi)>\Phi(r)$ for all $r<R$.*

Proof. Indeed, let $F(x)=0$ for $x\le x_1$ and $F(x)=1-\exp\{-x_k\Psi(\phi(x_k))\}$ for $x\in[x_k,x_{k+1})$, $k\ge1$. Then $\ln W_F(x)=\ln W_F(x_k)=-x_k\Psi(\phi(x_k))$ for all $x\in[x_k,x_{k+1})$ and $k\ge1$. Therefore, if $r\in[\phi(x_k),\phi(x_{k+1})]$ then

$$\begin{aligned}\ln\mu(r,\varphi)&\ge\sup\{\ln W_F(x)+rx\colon x_k\le x<x_{k+1}\}=\\&=\sup\{\ln W_F(x_k)+rx\colon x_k\le x<x_{k+1}\}=-x_k\Psi(\phi(x_k))+rx_{k+1}.\end{aligned} \tag{3}$$

On $[\phi(x_k),\phi(x_{k+1})]$ we consider the function $A(r)=(-x_k\Psi(\phi(x_k))+rx_{k+1})/\Phi(r)$. Then $A'(r)=a(r)/\Phi(r)^2$, where $a(r)=\Phi(r)x_{k+1}-\Phi'(r)(rx_{k+1}-x_k\Psi(\phi(x_k)))$. Since

$$\begin{gathered}a(\phi(x_k))=\Phi(\phi(x_k))x_{k+1}-x_k(\phi(x_k)-x_k\Psi(\phi(x_k)))=\\=x_{k+1}(\Phi(\phi(x_k))-x_k(\phi(x_k)))+x_k^2\Psi(\phi(x_k))=\\=-x_{k+1}x_k\Psi(\phi(x_k))+x_k^2\Psi(\phi(x_k))=-(x_{k+1}-x_k)x_k\Psi(\phi(x_k))<0,\\a(\phi(x_{k+1}))=\Phi(\phi(x_{k+1}))x_{k+1}-x_{k+1}(x_{k+1}\phi(x_{k+1})-x_k\Psi(\phi(x_k)))=x_{k+1}(\Phi(\phi(x_{k+1}))-\\-x_{k+1}\phi(x_{k+1}))+x_kx_{k+1}\Psi(\phi(x_k))=x_{k+1}(x_k\Psi(\phi(x_k))-x_{k+1}\Psi(\phi(x_{k+1})))<0,\\a'(r)=x_{k+1}\Phi'(r)-\Phi''(r)(rx_{k+1}-x_k\Psi(\phi(x_k)))-\Phi'(r)x_{k+1}\le-\Phi''(r)(\phi(x_k)x_{k+1}-\\-x_k\phi(x_k)+\Phi(\phi(x_k)))=-\Phi''(r)((x_{k+1}-x_k)\phi(x_k)+\Phi(\phi(x_k)))<0\end{gathered}$$

we obtain that $a(r)<0$ on $[\phi(x_k),\phi(x_{k+1})]$, the function $A(r)$ decreases on $[\phi(x_k),\phi(x_{k+1})]$ and, thus,

$$A(r)\ge\frac{-x_k\Psi(\phi(x_k))+\phi(x_{k+1})x_{k+1}}{\Phi(\phi(x_{k+1}))}>\frac{-x_{k+1}\Psi(\phi(x_{k+1}))+x_{k+1}\phi(x_{k+1})}{\Phi(\phi(x_{k+1}))}=1.$$

Therefore, in view of (3) and of the definition of $A(r)$ for $r\in[\phi(x_k),\phi(x_{k+1})]$ we have $(\ln\mu(r,\varphi))/\Phi(r)=A(r)>1$. $\square$

2. Main results. Under additional assumptions on the decrease of W_F (i. e. on the growth of $\ln\mu(r,\varphi)$) it is possible to get from (2) estimates on $\ln\mu(r_k,\varphi)$ from above for some sequence $(r_k)\uparrow R$. Here we will suggest two related solutions of this problem. One of them is based on results from [12].

For $\Phi\in\Omega(0,R)$ and $\Phi'(x_0)\le a<b<+\infty$ we put

$$G_1(a,b,\Phi)=\frac{ab}{b-a}\int_a^b\frac{\Phi(\varphi(t)}{t^2}dt,\quad G_2(a,b,\Phi)=\Phi\left(\frac{1}{b-a}\int_a^b\varphi(t)dt\right).$$

It is known ([13]) that $G_1(a,b,\Phi)<G_2(a,b,\Phi)$, and in [12] the following lemma is proved.

Lemma 2. *Let (x_k) be an increasing to $+\infty$ sequence of positive numbers, $\Phi\in\Omega(0,R)$, $0<R<+\infty$, and $\mu_D(r)$ be the maximal term of formal Dirichlet series*

$$D(s)=\sum_{k=1}^{\infty}\exp\{-x_k\Psi(\phi(x_k))+sx_k\},\quad s=r+it.$$

Then

$$\varlimsup_{r\uparrow R}\frac{\ln\mu_D(r)}{\Phi(r)}=1,\quad \varlimsup_{r\uparrow R}\frac{\ln\ln\mu_D(r)}{\ln\Phi(r)}=1,\tag{4}$$

$$\varliminf_{r\uparrow R}\frac{\ln\mu_D(r)}{\Phi(r)}=\varliminf_{k\to\infty}\frac{G_1(x_k,x_{k+1},\Phi)}{G_2(x_k,x_{k+1},\Phi)}\tag{5}$$

and if

$$\ln\mu_D(r)+\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2}-1\right)\ln\Phi(r)\ge 0,\quad r\in[r_0,R),\tag{6}$$

then

$$\varliminf_{r\uparrow R}\frac{\ln\ln\mu_D(r)}{\ln\Phi(r)}=\varliminf_{k\to\infty}\frac{\ln G_1(x_k,x_{k+1},\Phi)}{\ln G_2(x_k,x_{k+1},\Phi)}.\tag{7}$$

Using Lemma 2 we prove the following theorem.

Theorem 1. *Let $\Phi\in\Omega(0,R)$, $0<R\le+\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F, which satisfies (1). We suppose that $\ln\mu(r,\varphi)\le\Phi(r)$ for all $r\in[r_0,R)$ and $\ln W_F(x_k)-\ln W_F(x_{k+1})=O(1)$, $k\to\infty$, for some increasing to $+\infty$ sequence $X=(x_k)$ of positive numbers. Then*

$$\varliminf_{r\uparrow R}\frac{\ln\mu(r,\varphi)}{\Phi(r)}\le\varliminf_{k\to\infty}\frac{G_1(x_k,x_{k+1},\Phi)}{G_2(x_k,x_{k+1},\Phi)}\tag{8}$$

and if

$$Q(r)+\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2}-1\right)\ln\Phi(r)\ge q>-\infty,\quad r\in[r_0,R),\tag{9}$$

where $Q(r)\equiv 0$ if $R<+\infty$ and $Q(r)\equiv\ln r$ if $R=+\infty$, then

$$\varliminf_{r\uparrow R}\frac{\ln\ln\mu(r,\varphi)}{\ln\Phi(r)}\le\varliminf_{k\to\infty}\frac{\ln G_1(x_k,x_{k+1},\Phi)}{\ln G_2(x_k,x_{k+1},\Phi)}.\tag{10}$$

Proof. We put $x_0 = 0$ and $\mu(r,\varphi;X) = \max\{W_F(x_k)e^{rx_k} : k \ge 1\}$. Then the condition $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \to \infty$ yields that

$$\ln\mu(r,\varphi) = \sup_{x\ge 0}(\ln W_F(x) + rx) = \max_{k\ge 0}\sup_{x_k\le x<x_{k+1}}(\ln W_F(x) + rx) \le$$
$$\le \max_{k\ge 0}(\ln W_F(x_k) + rx_{k+1}) = \max_{k\ge 0}(\ln W_F(x_{k+1}) + rx_{k+1} + \ln W_F(x_k) - \ln W_F(x_{k+1})) \le$$
$$\le \max_{k\ge 0}(\ln W_F(x_{k+1}) + rx_{k+1}) + \text{const} \le \ln\mu(r,\varphi;X) + \text{const}. \tag{11}$$

On the other hand,

$$\ln\mu(r,\varphi) = \max_{k\ge 0}\sup_{x_k<x\le x_{k+1}}(\ln W_F(x) + rx) \ge \max_{k\ge 0}(\ln W_F(x_{k+1}) + rx_{k+1}) \ge \ln\mu(r,\varphi;X)$$

and since $\ln\mu(r,\varphi) \le \Phi(r)$ we have $\ln\mu(r,\varphi;X) \le \Phi(r)$ for $r \in [r_0, R)$. Therefore, by Lemma 1 $\ln W_F(x_k) \le -x_k\Psi(\phi(x_k))$ for all $k \ge k_0$. Hence it follows that $\ln\mu(r,\varphi;X) \le \ln\mu_D(r)$ for $r \in [r_0,R)$. Therefore, by Lemma 2 from (5) we obtain

$$\varlimsup_{r\uparrow R}\frac{\ln\mu(r,\varphi;X)}{\Phi(r)} \le \varlimsup_{k\to\infty}\frac{G_1(x_k,x_{k+1},\Phi)}{G_2(x_k,x_{k+1},\Phi)} \tag{12}$$

and if condition (6) holds then (7) implies

$$\varlimsup_{r\uparrow R}\frac{\ln\ln\mu(r,\varphi;X)}{\ln\Phi(r)} \le \varlimsup_{k\to\infty}\frac{\ln G_1(x_k,x_{k+1},\Phi)}{\ln G_2(x_k,x_{k+1},\Phi)}. \tag{13}$$

We remark that (9) implies (6), because if $R < +\infty$ then (4) implies $\ln\mu_D(r) \uparrow +\infty$ as $r \uparrow R$, and if $R = +\infty$ then $(\ln\mu_D(r))/r \to \infty$ as $r \to +\infty$, that is $\ln\ln\mu_D(r) - \ln r \to +\infty$ as $r \to +\infty$.

Inequalities (8) and (10) follow from (11)–(13). □

If $R = +\infty$ then the condition $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \to \infty$, can be replaced with some weaker condition provided that the function $\Phi \in \Omega(0,+\infty)$ grows not very quickly.

Let L be the class of positive continuous functions α on $(-\infty,+\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \le x_0$, $0 < \alpha(x) \uparrow +\infty$ as $x_0 \le x \to +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$, and $\alpha \in L_{si}$ if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0,+\infty)$.

Theorem 2. *Let $\Phi \in \Omega(0,+\infty)$ and φ be an entire characteristic function of a probability law F and $\ln\mu(r,\varphi) \le \Phi(r)$ for all $r \ge r_0$. Then:*

1) *if $\Phi \in L^0$ and $\ln W_F(x_k) = (1+o(1))\ln W_F(x_{k+1})$, $k \to \infty$, for some increasing to $+\infty$ sequence $X = (x_k)$ of positive numbers then inequality (8) holds;*
2) *if $\ln\Phi \in L_{si}$,*

$$\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1\right)\ln\Phi(r) + \ln r \ge q > -\infty, \quad r \ge r_0, \tag{14}$$

and if there exists an increasing to $+\infty$ sequence $X = (x_k)$ of positive numbers such that $\ln W_F(x_k) \le a\ln W_F(x_{k+1})$ for some $a \in (0,1)$ and all $k \ge k_0$ then inequality (10) holds.

Proof. We begin with the first part. Since $\ln W_F(x_k) \le (1-\varepsilon)\ln W_F(x_{k+1})$ for each $\varepsilon \in (0,1)$ and all $k \ge k_0 = k_0(\varepsilon)$, instead of (11) now we have

$$\ln\mu(r,\varphi) \le \max_{k\ge 0}(\ln W_F(x_k) + rx_{k+1}) =$$
$$= \max\left\{\max_{0\le k\le k_0}(\ln W_F(x_k) + rx_{k+1}),\ \max_{k\ge k_0}\left(\frac{\ln W_F(x_k)}{\ln W_F(x_{k+1})}\ln W_F(x_{k+1}) + rx_{k+1}\right)\right\} \le$$
$$\le \max\{rx_{k_0+1},\ \max_{k\ge k_0}((1-\varepsilon)\ln W_F(x_{k+1}) + rx_{k+1})\} \le (1-\varepsilon)\max_{k\ge 0}(\ln W_F(x_k) + x_k r/(1-\varepsilon)) +$$
$$+ rx_{k_0+1} \le (1-\varepsilon)\ln\mu(r/(1-\varepsilon),\varphi;X) + rx_{k_0+1}. \tag{15}$$

Therefore, from (12) we obtain

$$\varliminf_{r\to+\infty}\frac{\ln\mu(r,\varphi)}{\Phi(r)} \le (1-\varepsilon)\varliminf_{r\to+\infty}\frac{\ln\mu(r/(1-\varepsilon),\varphi;X)}{\Phi(r)} \le$$
$$\le (1-\varepsilon)\varliminf_{r\to+\infty}\frac{\ln\mu(r,\varphi;X)}{\Phi(r)}\varlimsup_{r\to+\infty}\frac{\Phi(r/(1-\varepsilon))}{\Phi(r)} \le (1-\varepsilon)A(\varepsilon)\varliminf_{k\to\infty}\frac{G_1(x_k,x_{k+1},\Phi)}{G_2(x_k,x_{k+1},\Phi)}, \tag{16}$$

where $A(\varepsilon) = \varlimsup_{r\to+\infty}\frac{\Phi(r/(1-\varepsilon))}{\Phi(r)}$. For $\Phi \in L^0$ in [14] it is proved that $A(\varepsilon) \searrow 1$ as $\varepsilon \downarrow 0$. Therefore, (16) implies (8).

For the proof of the second part we remark that now instead of (15) we have $\ln\mu(r,\varphi) \le a\ln\mu(r/a,\varphi;X)$, and (14) implies (9). Therefore, from (13) we obtain

$$\varliminf_{r\to+\infty}\frac{\ln\ln\ \mu(r,\varphi)}{\ln\Phi(r)} \le \varliminf_{r\to+\infty}\frac{\ln\ln\ \mu(r/a,\varphi;X)}{\ln\Phi(r/a)}\varlimsup_{r\to+\infty}\frac{\ln\Phi(r/a)}{\ln\Phi(r)} \le \varliminf_{k\to\infty}\frac{\ln G_1(x_k,x_{k+1},\Phi)}{\ln G_2(x_k,x_{k+1},\Phi)}. \ \square$$

If the function $\ln W_F(x)$ is smooth enough, then it is possible to get an estimate of $\ln\mu(r,\varphi)$.

Theorem 3. *Let $\Phi \in \Omega(0,R)$, $0 < R \le +\infty$, and φ be an analytic in $\mathbb{D}_R$ characteristic function of a probability law F such that $\ln W_F(x) = -V(x)$ for all $x \ge a$, where the function V is positive, continuously differentiable and $V'(x) \uparrow R$ as $0 < x \uparrow +\infty$. If conditions (1) and (2) hold for some sequence (x_k) of positive numbers then $\ln\mu(r_k,\varphi) \le \Phi(r_k) + ar_k$, where $r_k = V'(x_k)$.*

Proof. Clearly

$$\ln\mu(r,\varphi) \le \max\left\{\sup_{0\le x\le a}(\ln W_F(x) + rx),\ \sup_{x\ge a}(\ln W_F(x) + rx)\right\} \le ar + \max_{x\ge a}(-V(x) + rx)$$

and $\max_{x\ge a}(-V(x) + rx) = (-V(x) + rx)\big|_{x=v(r)}$, where $v(r)$ is the inverse function to V'. Therefore,

$$\ln\mu(r_k,\varphi) \le -V(v(r_k)) + r_k v(r_k) + ar_k = -V(v(V'(x_k))) + V'(x_k)v(V'(x_k)) + ar_k =$$
$$= V(x_k) + r_k x_k + ar_k \le \max_{j\ge 1}(V(x_j) + x_j r_k) + ar_k = \max_{j\ge 1}(\ln W_F(x_j) + x_j r_k) + ar_k \le$$
$$\le \max_{j\ge 1}(-x_j\Psi(\phi(x_j)) + x_j r_k) + ar_k \le \max_{x\ge a}(-x\Psi(\phi(x)) + xr_k) + ar_k \le \Phi(r_k) + ar_k,$$

because $(x\Psi(\phi(x)))' = \phi(x)$ and $(-x\Psi(\phi(x)) + xr)|_{x=\Phi'(r)} = \Phi(r)$. $\square$

3. Corollaries. Examining the scale of growth in Theorems 1–2 it is possible to get a number of results for analytic in $\mathbb{D}_R$ characteristic functions. Here we will restrict ourselves only by three cases which arise often in mathematical literature. The most often used characteristics of growth for analytic in $\mathbb{D}_R$, $0 < R < +\infty$ functions φ are the order $\varrho_*[\varphi]$, the lower order $\lambda_*[\varphi]$ and (if $0 < \varrho_*[\varphi] < +\infty$) the type $T_*[\varphi]$ and the lower type $t_*[\varphi]$, which are defined by the formulas

$$\varrho_*[\varphi] = \varlimsup_{r\uparrow R} \frac{\ln\ln M(r,\varphi)}{\ln(1/(R-r))},\ \ \lambda_*[\varphi] = \varliminf_{r\uparrow R} \frac{\ln\ln M(r,\varphi)}{\ln(1/(R-r))}, \tag{17}$$

$$T_*[\varphi] = \varlimsup_{r\uparrow R} (R-r)^{\varrho_*[\varphi]} \ln M(r,\varphi),\ \ t_*[\varphi] = \varliminf_{r\uparrow R} (R-r)^{\varrho_*[\varphi]} \ln M(r,\varphi). \tag{18}$$

We will show that in these formulas $\ln M(r,\varphi)$ can be replaced with $\ln\mu(r,\varphi)$. Indeed ([1, p. 55])

$$\ln\mu(r,\varphi) \le \ln M(r,\varphi) + \ln 2. \tag{19}$$

On the other hand ([1, p. 52]), if $0 < \eta(r) < R - r$, then

$$M(r,\varphi) \le \int_0^\infty W_F(x)e^{rx}dx + 1 + W_F(0) = \int_0^\infty W_F(x)e^{(r+\eta(r))x}e^{-\eta(r)x}dx + 1 + W_F(0) \le$$
$$\le \frac{1}{\eta(r)}\mu(r+\eta(r),\varphi) + 1 + W_F(0),$$

that is

$$\ln M(r,\varphi) \le \ln\mu(r+\eta(r),\varphi) - \ln\eta(r) + o(1), \quad r\uparrow R. \tag{20}$$

We choose $\eta(r) = (R-r)^2$. Then for $r > R-1$ from (20) we obtain

$$\ln M(r,\varphi) \le \ln\mu(r+(R-r)^2,\varphi) + 2\ln(1/(R-r)) + o(1), \quad r\uparrow R. \tag{21}$$

Since $\frac{R-r+(R-r)^2}{R-r} \to 1$, $\frac{\ln(R-r+(R-r)^2)}{\ln(R-r)} \to 1$, $\frac{\ln\ln(1/(R-r))}{\ln(1/(R-r)} \to 0$ and $(R-r)^{\varrho_*[\varphi]}\ln(1/(R-r) \to 0$ as $r\uparrow R$, from (19) and (21) it follows that in formulas (17) and (18) $\ln M(r,\varphi)$ can be replaced with $\ln\mu(r,\varphi)$.

Therefore, if $\varrho_*[\varphi] < +\infty$ ($T_*[\varphi] < +\infty$) then $\ln\mu(r,\varphi) \le \frac{T}{(R-r)^\varrho}$ for all $r \in [r_0(\varepsilon), R)$, where either $\varrho = \varrho_*[\varphi] + \varepsilon$ and $T = 1$ or $\varrho = \varrho_*[\varphi]$ and $T = T_*[\varphi] + \varepsilon$. For a function $\Phi \in \Omega(0,R)$ such that $\Phi(r) = \frac{T}{(R-r)^\varrho}$ for all $r \in [r_0(\varepsilon), R)$ we have

$$\Phi'(r) = \frac{T\varrho}{(R-r)^{\varrho+1}},\ \ \phi(x) = R - (T\varrho/x)^{1/(\varrho+1)},\ \ \frac{\Phi(\phi(x))}{x^2} = T(T\varrho)^{-\varrho/(\varrho+1)}x^{\varrho/(\varrho+1)-2}$$

for $x \ge x_0(\varepsilon)$. Hence it follows that for $k \ge k_0(\varepsilon)$

$$G_1(x_k,x_{k+1},\Phi) = \frac{T(\varrho+1)}{(T\varrho)^{\varrho/(\varrho+1)}} \frac{x_k x_{k+1}}{x_{k+1}-x_k} \left(\frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}}\right), \tag{22}$$

$$G_2(x_k,x_{k+1},\Phi) = T\left(\frac{(\varrho+1)(T\varrho)^{1/(\varrho+1)}}{\varrho} \frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1}-x_k}\right)^{-\varrho}. \tag{23}$$

Further we remark that

$$\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1\right)\ln\Phi(r) = \frac{1}{\varrho}\ln\frac{T}{(R-r)^\varrho} \uparrow +\infty, \quad r\uparrow R,$$

that is (9) holds. Therefore, if $\ln W_F(x_k)-\ln W_F(x_{k+1})=O(1)$, $k\to\infty$, then by Theorem 1 in view of (22)–(23) and of arbitrariness of ε,

$$\lambda_*[\varphi]\le \varrho_*[\varphi]\varliminf_{k\to\infty}\frac{\ln\left(\frac{x_kx_{k+1}}{x_{k+1}-x_k}\left(\frac{1}{x_k^{1/(\varrho+1)}}-\frac{1}{x_{k+1}^{1/(\varrho+1)}}\right)\right)}{\ln\left(\frac{x_{k+1}-x_k}{x_{k+1}^{\varrho/(\varrho+1)}-x_k^{\varrho/(\varrho+1)}}\right)^{\varrho}}, \tag{24}$$

$$t_*[\varphi]\le T_*[\varphi]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}}\varliminf_{k\to\infty}\frac{x_kx_{k+1}}{x_{k+1}-x_k}\left(\frac{1}{x_k^{1/(\varrho+1)}}-\frac{1}{x_{k+1}^{1/(\varrho+1)}}\right)\left(\frac{x_{k+1}^{\varrho/(\varrho+1)}-x_k^{\varrho/(\varrho+1)}}{x_{k+1}-x_k}\right)^{\varrho}. \tag{25}$$

We suppose that

$$\beta=:\varliminf_{k\to\infty}\frac{\ln x_k}{\ln x_{k+1}}<1.$$

Then there exists a number $\beta^*\in(\beta,1)$ and an increasing sequence (k_j) of positive integers such that $\ln x_{k_j}\le\beta^*\ln x_{k_j+1}$, that is $x_{k_j}=o(x_{k_j+1})$ as $j\to\infty$. Therefore, from (24) we obtain

$$\lambda_*[\varphi]\le \varrho_*[\varphi]\varliminf_{j\to\infty}\frac{\ln\left(\frac{x_{k_j}x_{k_j+1}}{x_{k_j+1}-x_{k_j}}\left(\frac{1}{x_{k_j}^{1/(\varrho+1)}}-\frac{1}{x_{k_j+1}^{1/(\varrho+1)}}\right)\right)}{\ln\left(\frac{x_{k_j+1}-x_{k_j}}{x_{k_j+1}^{\varrho/(\varrho+1)}-x_{k_j}^{\varrho/(\varrho+1)}}\right)^{\varrho}}=$$

$$=\varrho_*[\varphi]\varliminf_{j\to\infty}\frac{\ln x_{k_j}^{\varrho/(\varrho+1)}}{\varrho\ln x_{k_j+1}^{1/(\varrho+1)}}=\varrho_*[\varphi]\varliminf_{j\to\infty}\frac{\ln x_{k_j}}{\ln x_{k_j+1}}\le\varrho_*[\varphi]\beta^*,$$

i. e. in view of arbitrariness of β^* we have the inequality $\lambda_*[\varphi]\le\beta\varrho_*[\varphi]$. For $\beta=1$ this inequality is trivial.

Now we suppose that

$$\gamma=:\varliminf_{k\to\infty}\frac{x_k}{x_{k+1}}\in(0,1).$$

Then there exist an increasing sequence (k_j) of positive integers such that $x_{k_j}=(1+o(1))\times$ $\times\gamma x_{k_j+1}$ as $j\to\infty$. Therefore, from (25) we obtain

$$t_*[\varphi]\le T_*[\varphi]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}}\varliminf_{j\to\infty}\frac{x_{k_j}x_{k_i+1}}{x_{k_j+1}-x_{k_j}}\left(\frac{1}{x_{k_j}^{1/(\varrho+1)}}-\frac{1}{x_{k_j+1}^{1/(\varrho+1)}}\right)\left(\frac{x_{k_j+1}^{\varrho/(\varrho+1)}-x_{k_j}^{\varrho/(\varrho+1)}}{x_{k_j+1}-x_{k_j}}\right)^{\varrho}\le$$

$$\le T_*[\varphi]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}}\frac{\gamma}{\gamma-1}\left(\frac{1}{\gamma^{1/(\varrho+1)}}-1\right)\frac{(1-\gamma^{\varrho/(\varrho+1)})^{\varrho}}{(1-\gamma)^{\varrho}}=T_*[\varphi]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}}A(\gamma), \tag{26}$$

where

$$A(\gamma)=:\frac{\gamma^{\varrho/(\varrho+1)}(1-\gamma^{1/(\varrho+1)})(1-\gamma^{\varrho/(\varrho+1)})^{\varrho}}{(1-\gamma)^{\varrho+1}}.$$

It is easy to show that $A(\gamma)\to\frac{\varrho^{\varrho}}{(\varrho+1)^{\varrho+1}}$ as $\gamma\to1$, that is, (26) is transformed in the obvious inequality $t_*[\varphi]\le T_*[\varphi]$ as $\gamma\to1$. If $\gamma=0$ then $x_{k_j}=O(x_{k_j+1})$ as $j\to\infty$ and from (25) we obtain easily that $t_*[\varphi]=0$. This equality follows from (26), because $A(0)=0$. Thus, the following corollary is proved.

Corollary 1. *Let the characteristic function φ of a probability law F be analytic in $\mathbb{D}_R$, $0 < R < +\infty$, have order $\varrho_*[\varphi]$ and lower order $\lambda_*[\varphi]$. Assume that $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$, $k \to \infty$, for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\underline{\lim}_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}} = \beta$, where β is some nonnegative constant. Then $\lambda_*[\varphi] \le \beta\varrho_*[\varphi]$. If, moreover, φ has type $T_*[\varphi]$ and lower type $t_*[\varphi]$ and $\underline{\lim}_{k\to\infty} \frac{x_k}{x_{k+1}} = \gamma$ then $\tau_*[\varphi] \le T_*[\varphi] \frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} A(\gamma)$.*

For an entire characteristic function φ of order $\varrho[\varphi] \in (1, +\infty)$ the quantities

$$T[\varphi] = \overline{\lim_{r\to+\infty}} \frac{\ln M(r,\varphi)}{r^{\varrho[\varphi]}}, \quad t[\varphi] = \underline{\lim}_{r\to+\infty} \frac{\ln M(r,\varphi)}{r^{\varrho[\varphi]}} \tag{27}$$

are called the *type* and the *lower type* of φ. From (20) for $\eta(r) = 1$ we obtain

$$\ln M(r,\varphi) \le \ln \mu(r+1,\varphi) + o(1), \ r \to +\infty.$$

Combining this with (19) we conclude that in (27) $\ln M(r,\varphi)$ can be replaced with $\ln \mu(r,\varphi)$. Therefore, we choose $\Phi \in \Omega(0,+\infty)$ such that $\Phi(r) = Tr^\varrho$ for $r \ge r_0 = r_0(\varepsilon)$, where either $\varrho = \varrho[\varphi] + \varepsilon$ and $T = 1$ or $\varrho = \varrho[\varphi]$ and $T = T[\varphi] + \varepsilon$. Then $\Phi \in L^0$, $\ln \Phi \in L_{si}$ and $\left(\frac{\Phi(r)\Phi''(r)}{(\Phi'(r))^2} - 1\right) \ln \Phi(r) = \frac{\varrho-1}{\varrho} \ln \Phi(r) \to +\infty$ as $r \to +\infty$. It is known [15] that for this function

$$G_1(x_k, x_{k+1}, \Phi) = (\varrho-1)T^{-1/(\varrho-1)}\varrho^{-\varrho/(\varrho-1)} \frac{x_k x_{k+1}}{x_{k+1} - x_k} \left(x_{k+1}^{1/(\varrho-1)} - x_k^{1/(\varrho-1)}\right)$$

and

$$G_2(x_k, x_{k+1}, \Phi) = (\varrho-1)^\varrho T^{-1/(\varrho-1)}\varrho^{-\varrho^2/(\varrho-1)} \left(\frac{x_{k+1}^{\varrho/(\varrho-1)} - x_k^{\varrho/(\varrho-1)}}{x_{k+1} - x_k}\right)^\varrho.$$

Therefore, if $x_{k_j} = (1+o(1))\gamma\, x_{k_j+1}$ as $j \to \infty$, where $0 < \gamma < 1$, then

$$\lim_{j\to\infty} \frac{G_1(x_{k_j}, x_{k_j+1}, \Phi)}{G_2(x_{k_j}, x_{k_j+1}, \Phi)} = \frac{\varrho^\varrho}{(\varrho-1)^{\varrho-1}} \frac{\gamma(1-\gamma)^{\varrho-1}(1-\gamma^{1/(\varrho-1)})}{(1-\gamma^{\varrho/(\varrho-1)})^\varrho}$$

and if $\ln x_{k_j} \le \beta^* \ln x_{k_j+1}$, where $0 < \beta^* < 1$, then $x_{k_j} = o(x_{k_j+1})$, $j \to \infty$, and

$$\underline{\lim}_{j\to\infty} \frac{\ln G_1(x_{k_j}, x_{k_j+1}, \Phi)}{\ln G_2(x_{k_j}, x_{k_j+1}, \Phi)} \le \frac{\beta^*(\varrho-1)+1}{\varrho}.$$

So, as in the proof of Corollary 1, using Theorem 2 in view of arbitrariness of β^* we obtain the following corollary.

Corollary 2. *Let the entire characteristic function φ of a probability law F have the order $\varrho[\varphi] > 1$, the lower order $\lambda[\varphi]$, the type $T[\varphi]$ and the lower type $t[\varphi]$. Then:*

1) if $\ln W_F(x_k) \le a \ln W_F(x_{k+1})$, $0 < a < 1$, for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\underline{\lim}_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}} = \beta$ then $\lambda[\varphi] - 1 \le \beta(\varrho[\varphi] - 1)$;

2) if $\ln W_F(x_k) = (1+o(1)) \ln W_F(x_{k+1})$ as $k \to \infty$ for some increasing to $+\infty$ sequence (x_k) of positive numbers such that $\underline{\lim}_{k\to\infty} \frac{x_k}{x_{k+1}} = \gamma$ then $t[\varphi] \le T[\varphi] \frac{\varrho^\varrho}{(\varrho-1)^{\varrho-1}} A_1(\gamma)$, where $A_1(\gamma) = \frac{\gamma(1-\gamma)^{\varrho-1}(1-\gamma^{1/(\varrho-1)})}{(1-\gamma^{\varrho/(\varrho-1)})^\varrho}$.

If we define the modified order $\varrho_m[\varphi] = \overline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{M(r,\varphi)}{r}$ and the modified lower order $\lambda_m[\varphi] = \underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{M(r,\varphi)}{r}$, then $\varrho_m[\varphi] = \varrho[\varphi] - 1$, $\lambda_m[\varphi] = \lambda[\varphi] - 1$ and under the assumptions of item 1) of Corollary 2 we have the inequality $\lambda_m[\varphi] \le \beta\varrho_m[\varphi]$, which is an analog of the inequality from Corollary 1.

If for an entire characteristic function φ the function $\ln M(r,\varphi)$ increases faster than the power functions it is possible to use Theorem 1. We will demonstrate this by the example of R-order $\varrho_R[\varphi]$, lower R-order $\lambda_R[\varphi]$, R-type $T_R[\varphi]$ and lower R-type $t_R[\varphi]$, which are defined by the formulas

$$\varrho_R[\varphi] = \overline{\lim_{r\to+\infty}} \frac{\ln\ln M(r,\varphi)}{r}, \quad \lambda_R[\varphi] = \underline{\lim_{r\to+\infty}} \frac{\ln\ln M(r,\varphi)}{r},$$

$$T_R[\varphi] = \overline{\lim_{r\to+\infty}} \frac{\ln M(r,\varphi)}{\exp\{r\varrho_R[\varphi]\}}, \quad t_R[\varphi] = \underline{\lim_{r\to+\infty}} \frac{\ln M(r,\varphi)}{\exp\{r\varrho_R[\varphi]\}}.$$

For $\eta(r) = 1/r$ (20) implies the inequality $\ln M(r,\varphi) \le \ln\mu(r+1/r,\varphi) + \ln r + o(1)$, $r \to +\infty$. From here and (19) it follows that in the formulas for $\varrho_R[\varphi]$, $\lambda_R[\varphi]$, $T_R[\varphi]$ and $t_R[\varphi]$, the function $\ln M(r,\varphi)$ can be replaced with the function $\ln\mu(r,\varphi)$. Therefore, we choose $\Phi(r) = Te^{r\varrho}$ for $r \ge r_0 = r_0(\varepsilon)$, where either $\varrho = \varrho_R[\varphi] + \varepsilon$ and $T = 1$ or $\varrho = \varrho_R[\varphi]$ and $T = T_R[\varphi] + \varepsilon$. It is known ([15]) that for this function

$$G_1(x_k, x_{k+1}, \Phi) = \frac{1}{\varrho} \frac{x_k x_{k+1}}{x_{k+1} - x_k} \ln \frac{x_{k+1}}{x_k}, \; G_2(x_k, x_{k+1}, \Phi) = \frac{1}{e\varrho} \exp\left\{\frac{x_{k+1}\ln x_{k+1} - x_k \ln x_k}{x_{k+1} - x_k}\right\}.$$

If now $\ln x_{k_j} \le \beta^* \ln x_{k_j+1}$, where $0 < \beta^* < 1$, then $x_{k_j} = o(x_{k_j+1})$, $j \to \infty$, and

$$\underline{\lim_{j\to\infty}} \frac{\ln G_1(x_{k_j}, x_{k_j+1}, \Phi)}{\ln G_2(x_{k_j}, x_{k_j+1}, \Phi)} \le \beta^*$$

and if $x_{k_j} = (1+o(1))\gamma\, x_{k_j+1}$ as $j \to \infty$, where $0 < \gamma < 1$, then

$$\underline{\lim_{j\to\infty}} \frac{G_1(x_{k_j}, x_{k_j+1}, \Phi)}{G_2(x_{k_j}, x_{k_j+1}, \Phi)} = \frac{\gamma}{1-\gamma} \exp\left\{1 + \frac{\gamma\ln\gamma}{1-\gamma}\right\} \ln\frac{1}{\gamma}.$$

Repeating the proof of Corollary 1, we obtain the following corollary.

Corollary 3. *Let an entire characteristic function φ of a probability law F have R-order $\varrho_R[\varphi]$, lower R-order $\lambda_R[\varphi]$, R-type $T_R[\varphi]$ and lower R-type $t_R[\varphi]$. We suppose that $\ln W_F(x_k) - \ln W_F(x_{k+1}) = O(1)$ as $k \to \infty$ for some increasing to $+\infty$ sequence (x_k) of positive numbers. If $\underline{\lim}_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}} = \beta$ then $\lambda_R[\varphi] \le \beta\varrho_R[\varphi]$ and if*

$$\underline{\lim_{k\to\infty}} \frac{x_k}{x_{k+1}} = \gamma \quad \textit{then} \quad t_R[\varphi] \le T_R[\varphi] \frac{\gamma}{1-\gamma} \exp\left\{1 + \frac{\gamma\ln\gamma}{1-\gamma}\right\} \ln\frac{1}{\gamma}.$$

We demonstrate the application of Theorem 3 only for an entire characteristic function of finite R-order. It is easy to verify that for the function $\Phi(r) = e^{r\varrho}$ we have $x\Psi(\phi(x)) = \frac{x}{\varrho}\ln\frac{x}{e\varrho}$. Therefore, under the corresponding assumptions on W_F, Theorem 3 implies that if $\ln W_F(x_k) \le -\frac{x_k}{\varrho}\ln\frac{x_k}{e\varrho}$ then $\ln\mu(r_k,\varphi) \le (1+o(1))e^{\varrho r_k}$, $k \to \infty$, where $r_k = V'(x_k)$. Hence the following corollary follows.

Corollary 4. *Let the characteristic function φ of a probability law F be entire and analytic in $\mathbb{D}_R$. Assume that $\ln W_F(x) = -V(x)$ for all $x \geq a$, where the function V is positive, continuously differentiable and $V'(x) \uparrow R$ as $0 < x \uparrow +\infty$. Then $\lambda_R[\varphi] \leq \varliminf\limits_{x\to+\infty} \frac{x\ln x}{-\ln W_F(x)}$.*

One can obtain analogues of Corollary 3 for other scales of growth, but we are not going to discuss this here.

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