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ON L-SEPARATEDNESS AND L-REGULARITY OF THE CEDER PRODUCTS

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We study properties of the Ceder product $X \times_b Y$ of topological spaces X and Y with distinguished point $b \in Y$, recently introduced by the authors. Important examples of the Ceder product are the Ceder plane, the Alexandroff double circle and the Alexandroff duplicate. In particular, we detect Ceder products which are L-separated or L-regular (these notions generalize the separation axioms T_i for $i \in \{1, 2, 2\frac{1}{3}, 2\frac{1}{2}, 3, 3\frac{1}{2}\}$).

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Изучаются свойства введенного авторами понятия произведения Сидра $X \times_b Y$ топологических пространств X и Y, где $b \in Y$. Примерами произведения Сидра служат плоскость Сидра, двойная окружность Александрова, или ее обобщение, удвоение по Александрову. В частности, изучаются условия, при которых произведение Сидра будет L-отделимым или L-регулярным пространством (эти недавно введенные понятия включают аксиомы отделимости T_i при $i \in \{1, 2, 2\frac{1}{3}, 2\frac{1}{2}, 3, 3\frac{1}{2}\}$).

1. Introduction. In [1, ex. 9.1], J. Ceder gave an example of a stratifiable space, which is not metrizable. We call this space the *Ceder plane* and denote it by $\mathbb{M}$. In [2] Ceder's construction was generalized by introducing a general notion of the Ceder product $X \times_b Y$ of two topological spaces with a distinguished point $b \in Y$. Important examples of the Ceder product are the Ceder plane $\mathbb{M} = \mathbb{R} \times_0 [0, +\infty)$, the Alexandroff double circle [3, p. 204] and the Alexandroff duplicate of a given topological space. Topological properties of the Alexandroff duplicate $AD(X)$ of a topological space X have been studied in many papers (see for example [4, 5]). In particular, A. Caserta and S. Watson [4, Corollary 3.7] characterized metrizable subspaces of $AD(X)$.

In [2] it was proved that, if the spaces X and Y are stratifiable then the Ceder product $X \times_b Y$ is stratifiable. Some conditions under which the Ceder products are T_i-spaces for $i \in \{0, 1, 2, 3\}$ were announced in [6] and proved in [7]. Paper [7] also contains a characterization of spaces X, Y with metrizable Ceder product $X \times_b Y$.

In this paper we study separation axioms in Ceder products of topological spaces. We accept a general approach of [8] based on the notions of L-separated and L-regular spaces

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for a linearly ordered set L. Varying the linearly ordered set L we get the classical separation axioms T_i, $i \in \{1, 2, 2\frac{1}{3}, 2\frac{1}{2}, 3, 3\frac{1}{2}\}$ as partial cases.

2. The Ceder product. Recall that a family $\mathcal{B}$ of open subsets of X is a base for a topological space $(X, \mathcal{T})$ if and only if $\mathcal{B} \subseteq \mathcal{T}$ and for any $G \in \mathcal{T}$ and every point $x \in G$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq G$. It is known [3, p. 33], that any base $\mathcal{B}$ for X has the following properties:

(B1) for any $B_1, B_2 \in \mathcal{B}$ and every point $x \in B_1 \cap B_2$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$;

(B2) for any $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

Moreover, if a family $\mathcal{B}$ of subsets of X has the properties (B1) and (B2), then the family

$$\mathcal{T} = \{G \in 2^X : (\forall x \in G)(\exists B \in \mathcal{B})(x \in B \subseteq G)\}$$

is a topology on X, $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{B}$ is a base for the topological space $(X, \mathcal{T})$ [3, p. 46].

For a base $\mathcal{B}$ of the topology of a space X and a point $x \in X$ the family $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a neighborhood base of the topology at x.

Let X and Y be topological spaces, $F \subseteq X$ and $b \in Y$. For $U \subseteq X$ and $V \subseteq Y$ we put

$$\dot{V} = V \setminus \{b\}, \ U \overset{F}{\times} V = (U \times V) \setminus (F \times \dot{V}).$$

If $F = \{x\}$, then we put $U \overset{x}{\times} V = U \overset{\{x\}}{\times} V$. Consider a family $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ of subsets of $X \times Y$, where

$$\mathcal{B}_1 = \{\{x\} \times \dot{V} : x \in X, \ V \text{ is open in } Y\}$$

and

$$\mathcal{B}_2 = \Big\{U \overset{F}{\times} V : U \text{ is open in } X, \ V \text{ is open in } Y, \ b \in V \text{ and } F \subseteq X \text{ is a finite set}\Big\}.$$

It is easy to see that the family $\mathcal{B}$ has properties $(B1)$ and $(B2)$. Thus the family

$$\mathcal{T} = \{W \in 2^{X \times Y} : (\forall p \in W)(\exists B \in \mathcal{B})(p \in B \subseteq W)\}$$

is a topology on $X \times Y$ and $\mathcal{B}$ is a base for the topological space $(X \times Y, \mathcal{T})$. Note that a neighborhood base at a point $p = (x, y)$, $y \neq b$, consists of all sets $\{x\} \times \dot{V}$, where V is open in Y and $y \in V$; a neighborhood base at a point $p = (x, b)$ consists of all sets $U \overset{F}{\times} V$, where U is an open neighborhood of x in X and V is an open neighborhood of b in Y and $F \subseteq X$ is a finite set such that $x \in F$. Since $U \overset{F}{\times} V = U \overset{F_0}{\times} V$, where $F_0 = F \cap U$, we can assume that $F \subseteq U$.

The *Ceder product* of topological spaces X and Y with distinguished point $b \in Y$, denoted by $P = X \times_b Y$, is defined to be the topological space $(X \times Y, \mathcal{T})$.

Let X be a topological space and $AD(X) = X \times \{0, 1\}$ be the Alexandroff duplicate of X (see [5, 4]). Note that the Alexandroff duplicate $AD(X)$ is the Ceder product $X \times_0 \{0, 1\}_0$ of X and the connected doubleton $\{0, 1\}_0$. The space $\{0, 1\}_0$ is also called the Sierpiński space.

Put $X_b = X \times \{b\}$. Consider the mapping $\varphi : X \to X_b$, $\varphi(x) = (x, b)$. It is easy to see that φ is a homeomorphism of X onto the subspace X_b of the Ceder product $P = X \times_b Y$.

For $a \in X$ we put $Y_a = \{a\} \times Y$ and $\dot{Y}_a = \{a\} \times \dot{Y}$. We consider the mappings $\psi : Y \to Y_a$, $\psi(y) = (a, y)$, and $\psi_0 = \psi|_{\dot{Y}} : \dot{Y} \to \dot{Y}_a$. It is easy to see that ψ_0 is a homeomorphism of the subspace $\dot{Y} = Y \setminus \{b\}$ of Y onto the subspace $\dot{Y}_a$ of the Ceder product P. Note, that if the point b is non-isolated in Y, then the mapping ψ is not a homeomorphism, because the point $\psi(b) = (a, b)$ is isolated in the subspace Y_a.

3. L-separated and L-regular spaces. Let L be a non-empty linearly ordered set, X be a topological space and $x \in X$. An indexed family $(U_\lambda)_{\lambda\in L}$ of open subsets U_λ of a topological space X is called an *L-neighborhood* of a point x in X if $x \in U_\lambda$ for every $\lambda \in L$ and $\overline{U}_\lambda \subseteq U_\mu$ for any elements $\lambda, \mu \in L$ with $\lambda < \mu$. We say that an L-neighborhood $(U_\lambda)_{\lambda\in L}$ of a point x in X *separates* points x and $u \in X$, if $u \notin U_\lambda$ for every $\lambda \in L$. The notation $(U_\lambda)_{\lambda\in L} \prec A$ means that the indexed family $(U_\lambda)_{\lambda\in L}$ is *subordinated* to a set $A \subseteq X$ i.e. $U_\lambda \subseteq A$ for every $\lambda \in L$. It is clear that an L-neighborhood $(U_\lambda)_{\lambda\in L}$ of $x \in X$ separates points x and $u \in X$ if and only if $(U_\lambda)_{\lambda\in L} \prec X \setminus \{u\}$.

Following [8], we define a topological space X to be *L-separated at a point* $x \in X$ if for any point $u \in X \setminus \{x\}$ there exists an L-neighborhood $(U_\lambda)_{\lambda\in L}$ of x in X such that $(U_\lambda)_{\lambda\in L} \prec X \setminus \{u\}$. Next, a topological space X is said to be *L-regular at a point* $x \in X$ if for any neighborhood U of $x \in X$ there exists an L-neighborhood $(U_\lambda)_{\lambda\in L}$ of x in X such that $(U_\lambda)_{\lambda\in L} \prec U$. A topological space X is called *L-separated* (respectively, *L-regular*) if X is *L-separated* (respectively, *L-regular*) at each point $x \in X$.

We add two more properties to the well-known separation axioms $T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}$ (see [3, p. 69]). A topological space X is called a *Urysohn space* if for any two distinct points x and y of X there are neighborhoods U and V of points x and y respectively, with $\overline{U} \cap \overline{V} = \varnothing$. This property is called the *Urysohn separation axiom*, which we denoted by $T_{2\frac{1}{3}}$. A topological space X is called a *functionally Hausdorff* space if for any two distinct points x and y of X there exists a continuous function $f\colon X \to \mathbb{R}$ such that $f(x) \neq f(y)$. We denoted this property by $T_{2\frac{1}{2}}$.

We will identify each positive integer n with the finite ordinal $\{0, 1, \dots, n-1\}$.

The following result from [8] shows that almost all possible separation axioms are partial cases of L-separatedness and L-regularity.

Proposition 1. *A topological space X is:*

(i) a T_1-space if and only if X is 1-separated;

(ii) Hausdorff (= a T_2-space) if and only if X is 2-separated;

(iii) Urysohn (= a $T_{2\frac{1}{3}}$-space) if and only if X is 3-separated;

(iv) functionally Hausdorff (= a $T_{2\frac{1}{2}}$-space) if and only if X is $\mathbb{Q}$-separated;

(v) regular if and only if X is 2-regular if and only if n-regular for every integer $n \geq 2$;

(vi) completely regular if and only if X is $\mathbb{Q}$-regular;

(vii) Tychonoff (= a $T_{3\frac{1}{2}}$-space) if and only if X is a $\mathbb{Q}$-regular T_1-space.

The characterizations in Proposition 1 can be considered as definitions of the corresponding separation axioms.

It is well known [3, p. 114] that any subspace of a T_i-space is a T_i-space for $i \leq 3\frac{1}{2}$. We show that L-separatedness and L-regularity are hereditary properties too.

Proposition 2. *Any subspace E of an L-separated space X is L-separated.*

Proof. The closure of a set A in E will be denoted by $[A]_E$, and the closure of a set A in X will be denoted by $\overline{A}$. It is known [3, p. 111] that $[A]_E = \overline{A} \cap E$ for any set $A \subseteq E$.

Let x and u be any distinct points of E. Since X is L-separated, there exists an L-neighborhood $(U_\lambda)_{\lambda\in L}$ of x in X such that $u \notin U_\lambda$ for every $\lambda \in L$. We set $V_\lambda = U_\lambda \cap E$. It is clear that V_λ is open in E, $x \in V_\lambda$, $u \notin V_\lambda$ for every $\lambda \in L$, and $[V_\lambda]_E = \overline{U}_\lambda \cap E \subseteq U_\mu \cap E = V_\mu$ for any elements $\lambda < \mu$ of L. Therefore $(V_\lambda)_{\lambda\in L}$ is an L-neighborhood of x in E, which separates points x and u. Thus E is an L-separated space. □

Proposition 3. *Any subspace E of an L-regular space X is L-regular.*

Proof. Fix $x \in E$ and let V be a neighborhood of x in E. There exists a neighborhood U of x in X, such that $V = U \cap E$. Since X is an L-regular space, there exists an L-neighborhood $(U_\lambda)_{\lambda\in L}$ of x in X such that $(U_\lambda)_{\lambda\in L} \prec U$. We put $V_\lambda = U_\lambda \cap E$ for every $\lambda \in L$. Then $(V_\lambda)_{\lambda\in L}$ is an L-neighborhood of x in E such that $(V_\lambda)_{\lambda\in L} \prec V$. Therefore E is L-regular. □

4. The axioms T_0 and T_1 in the Ceder product. We start with finding conditions on spaces X and Y guaranteeing that the Ceder product satisfies the separation axioms T_i for $i \in \{0, 1\}$.

Theorem 1. *Let X and Y be non-empty topological spaces, $b \in Y$ and $P = X \times_b Y$ be the Ceder product. Then P is a T_0-space if and only if X and $\dot{Y}$ are T_0-spaces.*

Proof. Necessity. Since X and X_b are homeomorphic, X is a T_0-space. Similarly, $\dot{Y}$ and $\dot{Y}_a$ are homeomorphic, where $a \in X$. So $\dot{Y}$ is a T_0-space.
Sufficiency. Let X and $\dot{Y}$ be T_0-spaces. Fix two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of P. If $x_1 \neq x_2$, then there is an open set U in X containing exactly one of the points $x_1,\ x_2$. Then $W = U \times Y$ is open in P and contains exactly one of the points p_1, p_2.

If $x_1 = x_2 = x$, then $y_1 \neq y_2$. Assume first that $y_1, y_2 \in \dot{Y}$. Since $\dot{Y}$ is a T_0-space, there is an open set V in $\dot{Y}$, which contains exactly one of the points y_1, y_2. Thus the set $W = \{x\} \times V$ is a neighborhood of one of the points p_1, p_2 and does not contain the other one.

Finally, consider the case where $y_1 = b$ or $y_2 = b$. Suppose, for example, that $y_1 = b \neq y_2$. Then the set $W = X \overset{x}{\times} Y$ is a neighborhood of $p_1 = (x, b)$ which does not contain the point $p_2 = (x, y_2)$. □

Theorem 2. *Let X and Y be non-empty topological spaces, $b \in Y$ and $P = X \times_b Y$ be the Ceder product. Then P is a T_1-space if and only if X and $\dot{Y}$ are T_1-spaces.*

Proof. Necessity. Let P be a T_1-space. As in the proof of Theorem 1 we obtain that X and $\dot{Y}$ are T_1-spaces.
Sufficiency. Let X and $\dot{Y}$ be T_1-spaces. Fix two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of P. If $x_1 \neq x_2$, then there exists a neighborhood U of x_1, such that $x_2 \notin U$. Then $W = U \times Y$ is a neighborhood of p_1 in P, and $p_2 \notin W$.

Consider the case, where $x_1 = x_2 = x$. Then $y_1 \neq y_2$. We first consider the subcase $y_1, y_2 \in \dot{Y}$. Since $\dot{Y}$ is a T_1-space, there exists a neighborhood V of y_1 in $\dot{Y}$, such that $y_2 \notin V$. Thus, the set $W = \{x\} \times V$ is a neighborhood of p_1 in P and $p_2 \notin W$.

Now let $y_1 = b \neq y_2$. Then the set $W = X \overset{x}{\times} Y$ is a neighborhood of $p_1 = (x, b)$, which does not contain $p_2 = (x, y_2)$.

Finally, consider the case where $y_1 \neq b = y_2$. The set $W = \{x\} \times \dot{Y}$ is a neighborhood of $p_1 = (x, y_1)$, which does not contain $p_2 = (x, b)$. □

5. L-separatedness of the Ceder product.

Theorem 3. *Let L be a non-empty linearly ordered set, X and Y be non-empty topological spaces, $b \in Y$, $\dot{Y} = Y \setminus \{b\}$ and $P = X \times_b Y$ be the Ceder product. Then the following conditions are equivalent:*

(i) P is an L-separated space;

(ii) X and $\dot{Y}$ are L-separated spaces.

Proof. $(i) \Rightarrow (ii)$ Let P be an L-separated space. By Proposition 2, the subspaces $X_b = X \times \{b\}$ and $\dot{Y}_a = \{a\} \times \dot{Y}$ of the Ceder product P are L-separated, where $a \in X$. As in the proof of Theorem 1, we obtain that X and $\dot{Y}$ are L-separated.

$(ii) \Rightarrow (i)$ Consider two distinct points $p = (x, y)$ and $q = (u, v)$ of P.

Let $x \neq u$. There is an L-neighborhood $(U_\lambda)_{\lambda \in L}$ of x in X, such that $u \notin U_\lambda$ for every $\lambda \in L$, by the L-separatedness of X. The sets $W_\lambda = U_\lambda \times Y$ are open neighborhoods of p in P for all $\lambda \in L$, and

$$\overline{W}_\lambda \subseteq \overline{U}_\lambda \times Y \subseteq U_\mu \times Y = W_\mu$$

for any elements $\lambda < \mu$ of L. So, the indexed family $(W_\lambda)_{\lambda \in L}$ is an L-neighborhood of p in P. Moreover, $q \notin W_\lambda$ for every $\lambda \in L$, thus, this L-neighborhood separates the points p and q.

Now let $x = u$. Then $y \neq v$. Consider the following three logically possible cases.

Let $\{y, v\} \subseteq \dot{Y}$. Since $\dot{Y}$ is an L-separated space, there is an L-neighborhood $(V_\lambda)_{\lambda \in L}$ of y in $\dot{Y}$, which separates the points y and v. The sets $W_\lambda = \{x\} \times V_\lambda$ are open neighborhoods of p in P for every $\lambda \in L$, and $\overline{W}_\lambda = \{x\} \times [V_\lambda]_{\dot{Y}} \subseteq \{x\} \times V_\mu = W_\mu$ for any elements $\lambda < \mu$ of L. Moreover $q \notin W_\lambda$ for every $\lambda \in L$. Therefore $(W_\lambda)_{\lambda \in L}$ is an L-neighborhood of p in P, which separates the points p and q.

Let $y = b$. The set $W = X \overset{x}{\times} Y$ is clopen in P, $p \in W$ and $q \notin W$. Put $W_\lambda = W$ for every $\lambda \in L$. Then the indexed family $(W_\lambda)_{\lambda \in L}$ is an L-neighborhood of p in P, which separates the points p and q.

Finally, consider the case where $v = b$. Then the set $W = \{x\} \times \dot{Y}$ is clopen in P, $p \in W$ and $q \notin W$. Put $W_\lambda = W$ for every $\lambda \in L$. The indexed family $(W_\lambda)_{\lambda \in L}$ is the desired L-neighborhood.

Therefore the space P is L-separated. □

From Proposition 1 and Theorem 3 we immediately obtain the following consequence.

Corollary 1. *Let X and Y be non-empty topological spaces and $b \in Y$. The Ceder product $P = X \times_b Y$ is a Hausdorff (respectively Urysohn, functionally Hausdorff) space if and only if X and $\dot{Y}$ are Hausdorff (respectively Urysohn, functionally Hausdorff) spaces.*

6. L-regularity of the Ceder product. Recall that a point $x \in X$ of a topological space X is called an *accumulation point* if every neighborhood U of x in X is infinite.

A point $x \in X$ of a T_1-space X is accumulating if and only if x is not isolated.

The following theorem provides sufficient conditions of L-regularity of the Ceder product.

Theorem 4. *Let L be a non-empty linearly ordered set, X and Y be non-empty topological spaces, $b \in Y$ and $P = X \times_b Y$ be the Ceder product. Assume that*

(i) X and $\dot{Y}$ are L-regular spaces and

(ii) if X has an accumulation point, then Y is L-regular at the point b.

Then the Ceder product $P = X \times_b Y$ is L-regular.

Proof. Let $p = (x, y) \in P$ and W be a neighborhood of p in P. We are going to find an L-neighborhood $(W_\lambda)_{\lambda\in L}$ of p in P, such that $(W_\lambda)_{\lambda\in L} \prec W$. First consider the case, where $y \neq b$. There exists a neighborhood V of y in Y, such that $\{x\} \times \dot{V} \subseteq W$. The set $\dot{V} = \dot{Y} \cap V$ is a neighborhood of y in $\dot{Y}$. Since $\dot{Y}$ is L-regular, there exists an L-neighborhood $(V_\lambda)_{\lambda\in L}$ of y in $\dot{Y}$, such that $(V_\lambda)_{\lambda\in L} \prec \dot{V}$. Then the sets $W_\lambda = \{x\} \times V_\lambda$ are open neighborhoods of p in P for all $\lambda \in L$. Moreover,

$$\overline{W}_\lambda = \{x\} \times [V_\lambda]_{\dot{Y}} \subseteq \{x\} \times V_\mu = W_\mu$$

for any elements $\lambda < \mu$ of L. Therefore $(W_\lambda)_{\lambda\in L}$ is an L-neighborhood of p in P. Since

$$W_\lambda = \{x\} \times V_\lambda \subseteq \{x\} \times \dot{V} \subseteq \{x\} \times V \subseteq W,$$

we have $(W_\lambda)_{\lambda\in L} \prec W$.

If $y = b$, then we can find a neighborhood $U \subseteq X$ of x, a neighborhood $V \subseteq Y$ of b and a finite subset $F \subseteq U$ such that $U \overset{F}{\times} V \subseteq W$.

If x is a non-accumulation point of X, then there exists a finite neighborhood $\widetilde{U}$ of x in X such that $\widetilde{U} \subseteq U$. The set $\widetilde{U} \overset{\widetilde{U}}{\times} V = \widetilde{U} \times \{b\}$ is open in P. Let $(\widetilde{U}_\lambda)_{\lambda\in L}$ be an L-neighborhood of x such that $(\widetilde{U}_\lambda)_{\lambda\in L} \prec \widetilde{U}$. Put $W_\lambda = \widetilde{U}_\lambda \times \{b\}$ for every $\lambda \in L$. The indexed family $(W_\lambda)_{\lambda\in L}$ is the desired L-neighborhood of p in P.

Suppose that X is L-regular, x is an accumulation point of X, then Y is L-regular at the point b. Then there exist L-neighborhoods $(U_\lambda)_{\lambda\in L}$ and $(V_\lambda)_{\lambda\in L}$ of x and b in X and Y respectively, such that $(U_\lambda)_{\lambda\in L} \prec U$ and $(V_\lambda)_{\lambda\in L} \prec V$. The set $W_\lambda = U_\lambda \overset{F}{\times} V_\lambda$ is an open neighborhood of p in P, $W_\lambda \subseteq W$ for every $\lambda \in L$, and

$$\overline{W}_\lambda \subseteq \overline{U}_\lambda \overset{F}{\times} \overline{V}_\lambda \subseteq U_\mu \overset{F}{\times} V_\mu = W_\mu$$

for any elements $\lambda < \mu$ of L. Therefore $(W_\lambda)_{\lambda\in L}$ is an L-neighborhood of p in P, such that $(W_\lambda)_{\lambda\in L} \prec W$. □

A subset M of a linearly ordered set L is called *coinitial* if for every $\lambda \in L$ there is $\mu \in M$ such that $\mu \leq \lambda$. The smallest cardinality of a coinitial subset in L is called the *coinitiality* of L and is denoted by $\mathrm{ci}(L)$. A linearly ordered set L is called *a set of countable coinitiality* if $\mathrm{ci}(L) \leq \aleph_0$. It is clear that for a non-empty linearly ordered set L of countable coinitiality we have $\mathrm{ci}(L) = 1$ or $\mathrm{ci}(L) = \aleph_0$. Indeed, if $M = \{\mu_0, \mu_1, \dots, \mu_n\}$ is a finite coinitial subset of a linearly ordered set L, where $\mu_0 < \mu_1 < \cdots < \mu_n$, then the subset $M_0 = \{\mu_0\}$ is coinitial in L. So, in this case we have $\mathrm{ci}(L) = 1$. Note that μ_0 is the smallest element of the set L. If there is no subset of finite coinitiality of a linearly ordered set L and $\mathrm{ci}(L) \leq \aleph_0$, then $\mathrm{ci}(L) = \aleph_0$. In this case there is a sequence of elements l_n in L, such that $l_{n+1} < l_n$ for every $n \in \mathbb{N}$ and the subset $L_0 = \{l_n : n \in \mathbb{N}\}$ is coinitial in L. For example, $\mathrm{ci}(\mathbb{Q}) = \mathrm{ci}(\mathbb{R}) = \aleph_0$.

In the case where L is an arbitrary linearly ordered set, it is easily seen that if the Ceder product $P = X \times_b Y$ is L-regular, then X and $\dot{Y}$ are L-regular.

For linearly ordered sets L of countable coinitiality the following theorem provides necessary and sufficient conditions of L-regularity of the Ceder product.

Let $E \subseteq X \times Y$ and $x \in X$. We denote $E^x = \{y \in Y : (x, y) \in E\}$.

Theorem 5. *Let L be a non-empty linearly ordered set of countable coinitiality and let $P = X \times_b Y$ be the Ceder product of non-empty topological spaces X and Y, where $b \in Y$ and $\overline{\{b\}} = \{b\}$. Then P is an L-regular space if and only if the following conditions hold:*

(*i*) *X and $\dot Y$ are L-regular;*

(*ii*) *if X has an accumulation point, then Y is L-regular at b.*

Proof. The “if” part follows from Theorem 4. To prove the “only if” part let P be an L-regular space. By Proposition 3, the subspaces $X_b = X \times \{b\}$ and $\dot Y_a = \{a\} \times \dot Y$ of the Ceder product P are L-regular, where $a \in X$. Since X and X_b are homeomorphic, X is L-regular. Analogously, $\dot Y$ and $\dot Y_a$ are homeomorphic, so $\dot Y$ is L-regular.

Assuming that X contains an accumulation point $a \in X$, we shall prove that Y is L-regular at b. Given any neighborhood $V \subseteq Y$ of b, we shall construct an L-neighborhood $(V_\lambda)_{\lambda\in L}$ of b in Y with $(V_\lambda)_{\lambda\in L} \prec V$.

If b is isolated in Y, then we put $V_\lambda = \{b\}$ for every $\lambda \in L$.

We suppose that the point b is non-isolated in Y.

If $\mathrm{ci}(L) = 1$, then the set L has the smallest element $l \in L$. Since the Ceder product P is L-regular, for the neighborhood $W = X \times V$ of the point $c = (a, b)$ in P there is an L-neighborhood $(W_\lambda)_{\lambda\in L}$ of c in P such that $(W_\lambda)_{\lambda\in L} \prec W$. For the neighborhood W_l of c in P there are neighborhoods $\widetilde U$ of a in X, $\widetilde V$ of b in Y and a finite subset $F \subseteq U$ such that $\widetilde U \overset{F}{\times} \widetilde V \subseteq W_l$. Since a is an accumulation point of X, there is a point $x \in \widetilde U \setminus F$. For every $\lambda \in L$ consider the set $V_\lambda = W^x_\lambda$. We claim that $(V_\lambda)_{\lambda\in L}$ is the desired L-neighborhood of b in Y.

Since $x \in \widetilde U \setminus F$, for any $\lambda \in L$ we have

$$\widetilde V = (\widetilde U \overset{F}{\times} \widetilde V)^x \subseteq W^x_l \subseteq W^x_\lambda = V_\lambda,$$

because $l \le \lambda$. Thus $\widetilde V \subseteq V_\lambda$, and so V_λ is a neighborhood of b in Y.

For every $\lambda \in L$ we have $W_\lambda \subseteq W$. Therefore $V_\lambda = W^x_\lambda \subseteq W^x = V$. We prove that $\overline{V}_\lambda \subseteq V_\mu$ for any elements $\lambda < \mu$ of L. We have $\overline{W}_\lambda \subseteq W_\mu$. We show that $\{x\} \times \overline{V}_\lambda \subseteq \overline{W}_\lambda$. Given any point $y \in \overline{V}_\lambda$, consider the point $p = (x, y)$. Suppose that $y \ne b$. Consider a basic neighborhood $W^* = \{x\} \times \dot V^*$ of p in P, where V^* is an open neighborhood of y in Y. Since the singleton $\{b\}$ is closed in Y, the set $\dot V^*$ is open in Y and $y \in \dot V^*$, because $y \ne b$. Therefore $\dot V^*$ is a neighborhood of y. Then $\dot V^* \cap V_\lambda \ne \varnothing$. Thus there is $y^* \in \dot V^* \cap V_\lambda$. Since $y^* \in \dot V^*$, we have $p^* = (x, y^*) \in W^*$. On the other hand $y^* \in V_\lambda = W^x_\lambda$ and so $p^* \in W_\lambda$. Therefore $p^* \in W^* \cap W_\lambda$, hence $W^* \cap W_\lambda \ne \varnothing$. In this way $p \in \overline{W}_\lambda$. If $y = b$, then $p = (x, b) \in \widetilde U \overset{F}{\times} \widetilde V \subseteq W_l \subseteq W_\lambda \subseteq \overline{W}_\lambda$.

Consequently,

$$\overline{V}_\lambda \subseteq \overline{W}^x_\lambda \subseteq W^x_\mu = V_\mu,$$

and so $\overline{V}_\lambda \subseteq V_\mu$.

Therefore $(V_\lambda)_{\lambda\in L}$ is an L-neighborhood of b in Y, such that $(V_\lambda)_{\lambda\in L} \prec V$.

Now assume that $\mathrm{ci}(L) = \aleph_0$ and fix a strictly decreasing sequence $(l_n)_{n\in\mathbb{N}}$, such that the subset $L_0 = \{l_n : n \in \mathbb{N}\}$ is coinitial in L.

Put $U_0 = X$, $V_0 = V$ and $W_0 = U_0 \times V_0$. The set W_0 is a neighborhood of the point $c = (a, b)$ in P. By L-regularity of P, there is an L-neighborhood $(W_{1,\lambda})_{\lambda\in L}$ of c in P, such that $(W_{1,\lambda})_{\lambda\in L} \prec W_0$. The set W_{1,l_1} is a neighborhood of c in P. Then there are neighborhoods U_1 of a in X, V_1 of b in Y and a finite subset $F_1 \subseteq U_1$ such that $W_1 = U_1 \overset{F_1}{\times} V_1 \subseteq W_{1,l_1}$. The set W_1 is also a neighborhood of c in P. Then there is an L-neighborhood $(W_{2,\lambda})_{\lambda\in L}$ of c in P, such that $(W_{2,\lambda})_{\lambda\in L} \prec W_1$. Next, we consider the neighborhood W_{2,l_2} of c in P

and choose neighborhoods U_2 of a in X, V_2 of b in Y and a finite subset $F_2 \subseteq U_2$ such that $W_2 = U_2 \overset{F_2}{\times} V_2 \subseteq W_{2,l_2}$. For every $n \in \mathbb{N}$ we can construct recursively neighborhoods U_n of a in X, V_n of b in Y, a finite subset $F_n \subseteq U_n$ and an L-neighborhood $(W_{n,\lambda})_{\lambda \in L}$ of c in P such that the following conditions are satisfied:

(a) $W_{n+1,\lambda} \subseteq W_n = U_n \overset{F_n}{\times} V_n$ for every $\lambda \in L$ and $n \in \mathbb{N}$;

(b) $W_n \subseteq W_{n,l_n}$ for every $n \in \mathbb{N}$.

Note that $U_{n+1} \subseteq U_n$ and $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Indeed, for a fixed n we have

$$W_{n+1} = U_{n+1} \overset{F_{n+1}}{\times} V_{n+1} \subseteq W_{n+1,l_{n+1}} \subseteq W_n = U_n \overset{F_n}{\times} V_n.$$

Then $U_{n+1} \times \{b\} = W_{n+1} \cap (X \times \{b\}) \subseteq W_n \cap (X \times \{b\}) = U_n \times \{b\}$. It follow that $U_{n+1} \subseteq U_n$. Since the point a is an accumulation point of X, there is $x \in U_{n+1} \setminus F_{n+1}$. By the above $x \in U_n$. Since $W_{n+1} \subseteq W_n$ and $x \notin F_{n+1}$, we have $x \notin F_n$ because the point b is not isolated. Hence

$$\{x\} \times V_{n+1} = W_{n+1} \cap (\{x\} \times Y) \subseteq W_n \cap (\{x\} \times Y) = \{x\} \times V_n.$$

In this way $V_{n+1} \subseteq V_n$. In particular, $V_n \subseteq V_0 = V$ for every $n \in \mathbb{N}$.

Since U_n is a neighborhood of the accumulation point a in X, we can choose a point $x_n \in U_n \setminus F_n$ for every $n \in \mathbb{N}$. Note that from $W_n \subseteq W_{n-1}$ and $x_n \notin F_n$ it follows that $x_n \notin F_{n-1}$ for every $n \in \mathbb{N}$.

For every $\lambda \in L$ there is the smallest number $n = n(\lambda)$ such that $l_n \leq \lambda$. For this number n the inequalities $l_n \leq \lambda < l_{n-1}$ where $n \geq 2$ and $l_1 \leq \lambda$ where $n = 1$ are true. We put $V(\lambda) = W_{n,\lambda}^{x_n}$ for every $\lambda \in L$, where $n = n(\lambda)$. We show that $(V(\lambda))_{\lambda \in L}$ is an L-neighborhood of b in Y, such that $(V(\lambda))_{\lambda \in L} \prec V$.

Let us verify that $V(\lambda)$ is a neighborhood of b in Y. Since $l_n \leq \lambda$, we have $W_n = U_n \overset{F_n}{\times} V_n \subseteq W_{n,l_n} \subseteq W_{n,\lambda}$. Then

$$V_n = W_n^{x_n} \subseteq W_{n,\lambda}^{x_n} = V(\lambda).$$

Therefore, $V_n \subseteq V(\lambda)$, and hence the set $V(\lambda)$ is a neighborhood of b in Y, because V_n is a neighborhood of b in Y.

We prove that $V(\lambda) \subseteq V$ for every $\lambda \in L$. Suppose that $n = n(\lambda) \geq 2$ for $\lambda \in L$. Then $l_n \leq \lambda < l_{n-1}$ and

$$W_{n,\lambda} \subseteq W_{n,l_{n-1}} \subseteq W_{n-1} = U_{n-1} \overset{F_{n-1}}{\times} V_{n-1}.$$

Since $U_n \subseteq U_{n-1}$ and $x_n \in U_n$, we have $x_n \in U_{n-1}$. Note that $x_n \notin F_{n-1}$. Therefore

$$V(\lambda) = W_{n,\lambda}^{x_n} \subseteq W_{n-1}^{x_n} = V_{n-1}.$$

But $V_{n-1} \subseteq V$ then $V(\lambda) \subseteq V$. If $n(\lambda) = 1$, then

$$W_{1,\lambda} \subseteq W_0 = U_0 \times V_0 \text{ and } V(\lambda) = W_{1,\lambda}^{x_1} \subseteq W_0^{x_1} = V_0 = V.$$

Finally, let us prove that $\overline{V(\lambda)} \subseteq V(\mu)$ for any elements $\lambda < \mu$ of L. Fix $\lambda \in L$ and verify that $\{x_n\} \times \overline{V(\lambda)} \subseteq \overline{W}_{n,\lambda}$, where $n = n(\lambda)$. Let $p_n = (x_n, y)$ be an arbitrary point of $\{x_n\} \times \overline{V(\lambda)}$. We shall prove that $p_n \in \overline{W}_{n,\lambda}$. Let $y \neq b$. Consider any neighborhood V^* of y in Y. The set $W^* = \{x_n\} \times \dot{V}^*$ is a basic neighborhood of p_n in P. Note that $\dot{V}^*$ is an open neighborhood of y in Y. Since $y \in \overline{V(\lambda)}$, there is a point $y^* \in \dot{V}^* \cap V(\lambda)$. It is clear that $p^* = (x_n, y^*) \in \{x_n\} \times \dot{V}^* = W^*$ and $p^* \in \{x_n\} \times V(\lambda) \subseteq W_{n,\lambda}$. Thus $p^* \in W^* \cap W_{n,\lambda}$ and hence $p_n \in \overline{W}_{n,\lambda}$. If $y = b$, then $p_n = (x_n, b) \in U_n \overset{F_n}{\times} V_n \subseteq W_{n,l_n} \subseteq W_{n,\lambda} \subseteq \overline{W}_{n,\lambda}$.

Let $\lambda, \mu \in L$, $\lambda < \mu$, $n = n(\lambda)$ and $m = n(\mu)$. It is clear that $n \geq m$.

Consider the case where $n = m$. Then $V(\lambda) = W^{x_n}_{n,\lambda}$ and $V(\mu) = W^{x_n}_{n,\mu}$. Since $\overline{W}_{n,\lambda} \subseteq W_{n,\mu}$, we have

$$\{x_n\} \times \overline{V(\lambda)} \subseteq \overline{W}_{n,\lambda} \subseteq W_{n,\mu}.$$

Therefore, $\{x_n\} \times \overline{V(\lambda)} \subseteq W_{n,\mu}$, and hence $\overline{V(\lambda)} \subseteq W^{x_n}_{n,\mu} = V(\mu)$.

Now let $n > m$. In this case $n > 1$, $n - 1 \geq m$ and

$$\lambda < l_{n-1} \leq l_m \leq \mu.$$

Then $\overline{W}_{n,\lambda} \subseteq W_{n,l_{n-1}} \subseteq W_{n-1} \subseteq W_{n-1,l_{n-1}} \subseteq W_{n-2} \subseteq \cdots \subseteq W_m \subseteq W_{m,l_m} \subseteq W_{m,\mu}$. Thus, $\overline{W}_{n,\lambda} \subseteq W_m \subseteq W_{m,l_m} \subseteq W_{m,\mu}$. Therefore

$$\{x_n\} \times \overline{V(\lambda)} \subseteq \overline{W}_{n,\lambda} \subseteq W_m,$$

and we have $\overline{V(\lambda)} \subseteq W^{x_n}_m = V_m$, because $x_n \in U_n \subseteq U_m$ and $x_n \notin F_m$. Therefore $\overline{V(\lambda)} \subseteq V_m$. On the other hand, if $x_m \in U_m$ and $x_m \notin F_m$, then

$$V_m = W^{x_m}_m \subseteq W^{x_m}_{m,\mu} = V(\mu),$$

hence, $\overline{V(\lambda)} \subseteq V_m \subseteq V(\mu)$. Thus $\overline{V(\lambda)} \subseteq V(\mu)$. □

Corollary 2. *Let X be a topological space containing an accumulation point, Y be a topological space, $b \in Y$, $\overline{\{b\}} = \{b\}$ and L be a non-empty countable linearly ordered set. The Ceder product $P = X \times_b Y$ is L-regular if and only if X and Y are L-regular.*

Problem 1. *Let L be a non-empty linearly ordered set, X be a topological space containing an accumulation point, Y be a topological space, $b \in Y$, $\overline{\{b\}} = \{b\}$ and the Ceder product $P = X \times_b Y$ be an L-regular space. Is it true that Y is L-regular at the point b?*

A topological space X is called *regular at a point* $x \in X$ if for every neighborhood V of x there exists a neighborhood U of x in X such that $\overline{U} \subseteq V$.

A topological space X is called *completely regular at a point* $x \in X$ if for every neighborhood U of x there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(u) = 1$ for every $u \in X \setminus U$.

From Proposition 1 and Corollary 2 we immediately obtain the following consequence.

Corollary 3. *Let X, Y be topological spaces, $X \neq \varnothing$, $b \in Y$ and $\overline{\{b\}} = \{b\}$. The Ceder product $P = X \times_b Y$ is a regular (completely regular) space if and only if the following conditions are valid:*

(i) X and $\dot{Y}$ are regular (completely regular) spaces;

(ii) if X has an accumulation point, then Y is regular (completely regular) at b.

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