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DERIVATIONS AS HOMOMORPHISMS OR ANTI-HOMOMORPHISMS IN DIFFERENTIALLY SEMIPRIME RINGS

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Let R be a ring with an identity, U a nonzero right d-ideal and $d \in \operatorname{Der} R$. We prove that if R is d-semiprime and d is a homomorphism (respectively an anti-endomorphism) of R (respectively acts as a homomorphism on U), then $d = 0$. If R is d-prime and d acts as an anti-homomorphism on U, then $d = 0$.

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Пусть R — ассоциативное кольцо с единицей, U — его ненулевой правый d-идеал и $d \in \operatorname{Der} R$. Доказано, что если R — d-полупервичное кольцо и d — гомоморфизм (соответственно антигомоморфизм) на R (соответственно гомоморфизм на U), то $d = 0$. Если же R — d-первичное кольцо и d действует как антигомоморфизм на U, то $d = 0$.

1. Introduction. Let R be an associative ring with an identity. An additive mapping $d\colon R \to R$ is called a *derivation* of R if

$$d(ab) = d(a)b + ad(b)$$

for any $a, b \in R$. Let $\operatorname{Der} R$ be the set of all derivations of R. H. E. Bell and L. C. Kappe ([5]) proved that if d is a derivation of R which acts as either an endomorphism or an anti-endomorphism on a ring R, then $d = 0$ is trivial. M. Yenigul and N. Argaç ([14]), M. Ashraf, N. Rehman and M. A. Quadri ([1]) extended this result for (σ, τ)-derivations for a prime ring. A. Asma and K. Deepak ([2]), A. Asma, N. Rehman and A. Shakir ([4]) obtained the above result for a generalized (σ, τ)-derivations acting as a homomorphism or as an anti-homomorphism on a nonzero Lie ideal U of a prime ring of characteristic $\neq 2$. Recently A. Asma and K. Deepak ([3]), N. Rehman and M. A. Raza ([10, 11]), B. Dhara ([7]), Y. Wang and H. You ([13]) and G. Scudo ([12]) extended these results for the generalized derivation acting on an ideal (respectively Lie ideal) in prime and semiprime rings.

Recall that a ring R (with 1) is called

(*i*) *d-semiprime* (or *differentialy semiprime*) if, for a nonzero d-ideal I of R, the condition $aId^k(a) = 0$ (where $a \in R$) for all $k \geqslant 0$ implies that $a = 0$,

(*ii*) *d-prime* (or *differentialy prime*) if, for a nonzero d-ideal I of R, the condition $aId^k(b) = 0$ for any integer $k \geqslant 0$ (where $a, b \in R$) implies that $a = 0$ or $b = 0$.

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In this paper we prove the following two theorems.

Theorem 1. *Let R be a ring and $d \in \operatorname{Der} R$. If R is d-semiprime and d is a homomorphism (respectively an anti-endomorphism) of R, then $d = 0$.*

Theorem 2. *Let R be a ring, $d \in \operatorname{Der} R$ and U be a nonzero right d-ideal of R. Then the following hold:*

(1) *if R is d-semiprime and d acts as a homomorphism on U, then $d = 0$,*

(2) *if R is d-prime and d acts as an anti-homomorphism on U, then $d = 0$.*

Throughout the paper $Z(R)$ is the center of an associative ring $R, d \in \operatorname{Der} R$ and $\mathbb{P}(R)$ is the prime radical of R. More details about derivations can be found in [6] and [9].

2. Preliminaries. We extend Theorem 3 from [8] in the following

Proposition 1. *Let R be a ring and $d \in \operatorname{Der} R$. If R is a d-semiprime ring and $(d(x))^n = 0$ for any $x \in R$, where $n \geq 1$ is a fixed integer, then $d = 0$.*

Proof. Assume that P is a prime ideal of $R, a \in P$ and $x \in R$. Since

$$0 = (d(ax))^n = (d(a)x)^n \bmod P,$$

we have $(\overline{d(a)x})^n = \overline{0}$ in the quotient ring $\overline{R} = R/P$. This means that a prime ring $\overline{R}$ contains the nilpotent ideal $\overline{d(a)} \cdot \overline{R}$. Consequently, $d(a) \in P$ and $d(P) \subseteq P$. The rule

$$\overline{d}\colon \overline{R} \ni x + P \to d(x) + P \in \overline{R}$$

determines a derivation $\overline{d}$ of $\overline{R}$ such that $(\overline{d}(\overline{x}))^n = \overline{0}$, and so, by Theorem 1 of [8], $\overline{d} = \overline{0}$. Hence $d(R) \subseteq P$. From this it follows that $d(R) \subseteq \mathbb{P}(R)$. Since $d(\mathbb{P}(R)) \subseteq \mathbb{P}(R)$ and R is d-semiprime, we deduce that $d = 0$. □

Corollary 1. *Let R be a $\triangle$-semiprime ring, where $\triangle \subseteq \operatorname{Der} R$. If any inner derivation of R is its ring endomorphism, then R is commutative.*

Proof. Let $a, x, y \in R$. Since $\partial_b \in \operatorname{End} R$ for any

$$b \in \{\delta_1^{m_1} \dots \delta_k^{m_k}(a) \mid \delta_i \in \triangle,\ k \geq 1 \text{ and } m_i \geq 0 \text{ are integers } (i \in \{1, \dots, k\})\},\ a \in C_R(R^2)$$

by Lemma 1 of [5], where $R^2 = \{ab \mid a, b \in R\}$. Then

$$[x,b][y,b] = xbxb - xb^2x - bxyb - bxby = 0 = [xy,b] = xyb - bxy,$$
$$[x,b]y[x,b] = [x,b](yxb - ybx) = [x,b]([yx,b] - [y,b]x) = 0.$$

In view of $\triangle$-primeness of R, we deduce that $[x,b] = 0$ for any $x \in R$. Hence $a \in Z(R)$. □

3. Proofs. Proof of Theorem 1. 1) Assume that d is an endomorphism of R. By Lemma 2(a) of [5], we have that

$$d(x)x(y - d(y)) = 0 \tag{1}$$

for any $x, y \in R$. The replacement of y with yt for an arbitrary $t \in R$ yields that

$$d(x)x(y - d(y))t - d(x)xyd(t) = 0.$$

From this, in view of (1), it follows that

$$d(x)xRd(R) = 0. \tag{2}$$

By the same argument, substituting y^2 for y in (1) we get

$$xd(y)y + xyd(y) + d(x)y^2 = d(x)d(y)y + d(x)yd(y).$$

Recalling (1), we obtain that

$$(d(x) - x)yd(y) = 0. \tag{3}$$

The replacement of x with rx in (3), where $r \in R$, gives $(d(r)x + rd(x) - rx)yd(y) = 0$ or equivalently

$$d(r)xyd(y) = 0. \tag{4}$$

Since

$$d(x)^5 = (d(x^2))^2 d(x) = (d(x)x + xd(x))^2 d(x) = (xd(x)d(x)x)d(x) = 0$$

in view of (2) and (4), we deduce that $d = 0$ by Proposition 1.

2) Now assume that d is an anti-homomorphism of R. By the same reason as in the proof of Theorem 2 from [5], $[r, d(x)]R[r, d(x)] = 0$ for any $x, r \in R$. Then $I = \sum_{x,r \in R} R[r, d(x)]R$ is a nil ideal. If $a, b \in R$, then

$$d(a[r, d(x)]b) = d(a)[r, d(x)]b + a[r, d(x)]d(b) + a[d(r), d(x)]b + a[r, d^2(x)]b \in I.$$

Hence I is a d-ideal. Then $I = 0$ and $d(R) \subseteq Z(R)$. Therefore $d \in \operatorname{End} R$. The rest follows from the part 1).

Lemma 1 (Lemma 1.1, [9])**.** *Let R be a ring and U a nonzero right ideal of R. Suppose that, given $a \in U$, $a^n = 0$ for a fixed integer n; then R has a nonzero nilpotent ideal.*

Proof of Theorem 2. 1) Assume that d acts as a homomorphism on U. By Lemma 2(a) of [5], we obtain condition (1) for any $x, y \in U$. Replacing x with vx in (1), where $v \in U$, we obtain

$$v(xd(y) + d(x)y) + d(v)xy = d(v)xd(y) + vd(x)d(y).$$

Since $d(xy) = d(x)d(y)$, from this it holds that $vd(x)d(y) + d(v)xy = d(v)xd(y) + vd(x)d(y)$ and so

$$d(v)x(y - d(y)) = 0. \tag{5}$$

Substituting yr for y in (5), we obtain $d(v)xyr - d(v)xd(y)r - d(v)xyd(r) = 0$ or equivalently

$$d(v)xyd(r) = 0. \tag{6}$$

Replacing r with rs in (6), where $s \in R$, we have $d(v)xyRd(R) = 0$ and therefore $(d(v)xyR)^2 = 0$. This means that $I = \sum_{x,v \in U} d(v)xU$ is a nil ideal. Since I is a d–ideal, we deduce that $I = 0$. Then $d(x)xy = 0$ and, in particular, $d(x)xd(y) = 0$. As in the proof of Theorem 3 from [5] (see its equations (10) and (11)), we can obtain that $x^2 d(y) = 0$.

The replacement of y with yt for an arbitrary $t \in R$ gives

$$0 = x^2 d(yt) = x^2 d(y)t + x^2 yd(t) = x^2 yd(t).$$

Hence $x^2 yRd^k(R) = 0$ for any integer $k \geq 0$. Then $x^2 y = 0$. As a consequence, $x^3 = 0$ for any $x \in U$. By Lemma 1, we obtain a contradiction. Hence $d = 0$.

2) Now assume that d acting on U as an anti-homomorphism. As in the proof of Theorem 3 from [5], we can prove that $xR[r, d(y)] = 0$ for any $x, y \in U$ and $r \in R$. Then $xRd^k([r, d(y)]) = 0$ for any integer $k \geq 0$. Since U is nonzero, we deduce that $[r, d(y)] = 0$. This means that $d(U) \subseteq Z(R)$ and d acts on U as a homomorphism. The rest follows from the part 1).

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