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MULTI-TERM POWER ASYMPTOTICS OF ENTIRE DIRICHLET SERIES AND CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS

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For entire Dirichlet series and entire characteristic functions of a probability law in terms of multi-term power asymptotics it is investigated the relation between the growth of the maximum modulus and the behavior of coefficients and the function of the distribution respectively.

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Для целых рядов Дирихле и целых характеристических функций вероятностных законов в терминах многочленной степенной асимптотики иследована связь между ростом максимума модуля и поведением соответственно коэффициентов и функции распределения.

1. Introduction. Let $\Lambda = (\lambda_n)$ be a sequence of nonnegative integers increasing to $+\infty$ ($\lambda_0 = 0$), and $S(\Lambda)$ be a class of entire Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it. \tag{1}$$

For $\sigma \in \mathbb{R}$ and $F \in S(\Lambda)$ we put $M(\sigma, F) = \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}$, and $\mu(\sigma) = \max\{|a_n| \exp(\sigma\lambda_n)\colon n \geq 0\}$ be the maximal term of (1). For entire Dirichlet series of the R-order $\rho_R \in (0, +\infty)$ and the R-type $T_R \in (0, +\infty)$ in [1–2] it is obtained the conditions on a_n and λ_n, under which

$$\ln M(\sigma, F) = T_R \exp\{\rho_R \sigma\} + (T + o(1)) \exp\{\rho\sigma\}, \quad \sigma \to +\infty,$$

where $0 < \rho < \rho_R$ and $T \in \mathbb{R}\backslash\{0\}$. Multi-term exponential asymptotics of $\ln M(\sigma, F)$ is investigated in [3–4].

Two-term power asymptotics for the maximal term of entire Dirichlet series of the form

$$\ln \mu(\sigma, F) = T_1 \sigma^{p_1} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty,$$

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where $p_1 > 1$, $0 < p < p_1$, $T_1 > 0$ and $\tau \in \mathbb{R}$, is indicated in [6]. In [6] it is established also conditions on a_n and λ_n, under which

$$\ln \mu(\sigma, F) = T_1\sigma^{p_1} + T_2\sigma^{p_2} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty, \tag{2}$$

where $p_1 > 1$, $0 < p < p_2 < p_1$, $T_1 > 0$, $T_2 \in \mathbb{R}\backslash\{0\}$ and $\tau \in \mathbb{R}\backslash\{0\}$. We put

$$\tau^* = \tau I_{\{p\colon p\ge 2p_2-p_1\}}(p) - \frac{T_2p_2}{2T_1p_1(p_1-1)} I_{\{p\colon p\le 2p_2-p_1\}}(p),$$

where $I_E(p) = 1$ for $p \in E$ and $I_E(p) = 0$ for $p \notin E$, in [6] the following theorem is proved.

Theorem A. *In order that the relation (2) hold, it is necessary and in the case $p+p_1 \ge 2p_2$ sufficient that for every $\varepsilon > 0$ the inequality*

$$\ln|a_n| \le -T_1(p_1-1)\Big(\frac{\lambda_n}{T_1p_1}\Big)^{p_1/(p_1-1)} + T_2\Big(\frac{\lambda_n}{T_1p_1}\Big)^{p_2/(p_1-1)} + (\tau^*+\varepsilon)\Big(\frac{\lambda_n}{T_1p_1}\Big)^{\frac{\max\{p,\,2p_2-p_1\}}{p_1-1}}$$

is valid and there exists a sequence (n_k) of positive integers such that

$$\lambda_{n_{k+1}} - \lambda_{n_k} = o\Big(\lambda_{n_k}^{\frac{p_1+\max\{p,\,2p_2-p_1\}-2}{2(p_1-1)}}\Big), \quad k\to\infty, \quad \ln|a_{n_k}| \ge -T_1(p_1-1)\Big(\frac{\lambda_{n_k}}{T_1p_1}\Big)^{p_1/(p_1-1)} +$$

$$+T_2\Big(\frac{\lambda_{n_k}}{T_1p_1}\Big)^{p_2/(p_1-1)} + (\tau^*-\varepsilon)\Big(\frac{\lambda_{n_k}}{T_1p_1}\Big)^{\frac{\max\{p,\,2p_2-p_1\}}{p_1-1}}.$$

Here we are going to find conditions under which $\ln M(\sigma, F)$ has the following asymptotics

$$\ln M(\sigma, F) = \sum_{j=1}^m T_j\sigma^{p_j} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty, \tag{3}$$

where $p_1 > 1$, $0 < p < p_m < \cdots < p_2 < p_1$ for $m \ge 2$, $T_1 > 0$, $T_j \in \mathbb{R}\backslash\{0\}$ for $2 \le j \le m$ and $\tau \in \mathbb{R}\backslash\{0\}$.

A non-decreasing function F continuous on the left on $(-\infty, +\infty)$ is said ([7, p. 10]) to be a probability law if $\lim_{x\to+\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$, and the function $\varphi(z) = \int_{-\infty}^{+\infty} e^{izx}dF(x)$ defined for real z is called ([7, p. 12]) a characteristic function of this law. If φ has an analytic continuation on $\mathbb{C}$, then we call φ an entire characteristic function of the law F. It is known ([7, p. 37–38]) that φ is an entire characteristic function of the law F if and only if for every $r \ge 0$

$$W_F(x) =: 1 - F(x) + F(-x) = O(e^{-rx}), \quad x \to +\infty. \tag{4}$$

Hence it follows that

$$\varliminf_{x\to+\infty} \frac{1}{x} \ln \frac{1}{W_F(x)} = +\infty. \tag{5}$$

For $0 \le r < +\infty$ we put $M_\varphi(r) = \max\{|\varphi(z)|\colon |z| = r\}$. Then [7, p. 45] there exists $\lim_{r\to+\infty} r^{-1}\ln M_\varphi(r) > 0$, that is φ has the growth not below of normal type of the order $\varrho = 1$. Therefore, we can investigate conditions, under which

$$\ln M_\varphi(r) = \sum_{j=1}^m T_jr^{\varrho_j} + (\tau + o(1))r^\varrho, \quad r \to +\infty, \tag{6}$$

where $\varrho_1 > 1$, $0 < \varrho < \varrho_m < \dots < \varrho_2 < \varrho_1$ for $m \ge 2$, $T_1 > 0$, $T_j \in \mathbb{R}\backslash\{0\}$ for $2 \le j \le m$ and $\tau \in \mathbb{R}\backslash\{0\}$.

2. Preliminary results. In [8] (see also [9]) the following result is obtained.

Lemma 1. *In order that relations*

$$\ln M(\sigma, F) \le \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty,$$

$$\ln \mu(\sigma, F) \le \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty,$$

be equivalent for each $F \in S(\Lambda)$, it is necessary and sufficient that $\ln n = o(\lambda_n^{p/(p_1-1)})$ as $n \to \infty$. The condition is sufficient for the equivalence of the asymptotic equalities (3) and

$$\ln \mu(\sigma, F) = \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty, \tag{7}$$

For an entire characteristic function φ of a law F we put $\mu_\varphi(r) = \sup\{W_F(x)e^{rx} \colon x \ge 0\}$. Then ([7, p. 55]) $\mu_\varphi(r) \le 2M_\varphi(r)$. On the other hand ([7, p. 52]),

$$M_\varphi(r) \le \int_0^\infty W_F(x)e^{rx}dx + 1 + W_F(0)$$

for all $r \ge 0$. Using this inequality we prove the following theorem.

Lemma 2. *The relations*

$$\ln M_\varphi(r) = \sum_{j=1}^{m} T_j r^{\varrho_j} + (\tau + o(1))r^{\varrho}, \quad r \to +\infty, \tag{8}$$

$$\ln \mu_\varphi(r) = \sum_{j=1}^{m} T_j r^{\varrho_j} + (\tau + o(1))r^{\varrho}, \quad r \to +\infty, \tag{9}$$

are equivalent.

Proof. At first we prove that if

$$\ln \mu_\varphi(r) \le \sum_{j=1}^{m} T_j r^{\varrho_j} + (\tau + o(1))r^{\varrho}, \quad r \to +\infty, \tag{10}$$

then

$$\ln M_\varphi(r) \le \sum_{j=1}^{m} T_j r^{\varrho_j} + (\tau + o(1))r^{\varrho}, \quad r \to +\infty. \tag{11}$$

Indeed,

$$M_\varphi(r - r^{\varrho-\varrho_1}) \le \int_0^\infty W_F(x)e^{rx}\exp\{-r^{\varrho-\varrho_1}x\}dx + 1 + W_F(0) \le \mu_\varphi(r)r^{\varrho_1-\varrho} + 1 + W_F(0),$$

whence in view of (10)

$$\ln M_\varphi(r-r^{\varrho-\varrho_1})\le \ln\mu_\varphi(r)+(1+o(1))(\varrho_1-\varrho)\ln r\le \sum_{j=1}^m T_j r^{\varrho_j}+(\tau+o(1))r^{\varrho},\quad r\to+\infty.$$

If we put $r-r^{\varrho-\varrho_1}=t$ then $r=t+(1+o(1))t^{\varrho-\varrho_1}$ as $t\to+\infty$. Therefore,

$$\ln M_\varphi(t)\le \sum_{j=1}^m T_j(t+(1+o(1))t^{\varrho-\varrho_1})^{\varrho_j}+(\tau+o(1))t^{\varrho}=\sum_{j=1}^m T_j t^{\varrho_j}(1+(1+o(1))t^{\varrho-\varrho_1-1})^{\varrho_j}+$$
$$+(\tau+o(1))t^{\varrho}=\sum_{j=1}^m T_j t^{\varrho_j}(1+(1+o(1))\varrho_j t^{\varrho-\varrho_1-1})+$$
$$+(\tau+o(1))t^{\varrho}=\sum_{j=1}^m T_j t^{\varrho_j}+(\tau+o(1))t^{\varrho},\quad t\to+\infty. \tag{12}$$

Thus, (10) implies (11). In view of the inequality $\ln\mu_\varphi(r)\le\ln M_\varphi(r)+\ln 2$ (11) implies (10). Hence it follows that (9) implies (8). We remark also that if (10) holds for some sequence (r_k) increasing to $+\infty$ then there exists an increasing to $+\infty$ sequence (t_k), for which (12) holds. Hence we obtain that (8) implies (9). Theorem 1 is prowed.

Since $\ln\mu(\sigma)=\max\{\ln|a_n|+\sigma\lambda_n\colon n\ge 0\}$ for Dirichlet series and $\ln\mu_\varphi(r)=$ $=\sup\{\ln W_F(x)+rx\colon x\ge 0\}$ for characteristic functions, we need to investigate the connection between the growth of Young conjugated functions.

Thus, let $Q(\sigma)=\sup\{P(t)+\sigma t\colon t\ge 0\}$, where P is an arbitrary function defined on $[0,+\infty)$ and $\ne+\infty$ (can take on the value $-\infty$, but $P\not\equiv-\infty$). The functions Q and P are said to be Young conjugated.

As in [10–11] by Ω we denote the class of positive unbounded functions Φ on $(-\infty,+\infty)$ such that the derivative Φ' is positive, continuous and increasing to $+\infty$ on $(-\infty,+\infty)$. For $\Phi\in\Omega$ let $\Psi(\sigma)=\sigma-\Phi(\sigma)/\Phi'(\sigma)$ be the function associated with Φ in the sense of Newton and ϕ be the inverse function to Φ'. It is known ([10–11]) that the function Ψ is continuous and increasing to $+\infty$ on $(-\infty,+\infty)$, and the function ϕ is continuous and increasing to $+\infty$ on $(0,+\infty)$.

For $\Phi\in\Omega$ and $0<a<b<+\infty$ we put

$$G_1(a,b,\Phi)=\frac{ab}{b-a}\int_a^b\frac{\Phi(\phi(t))}{t^2}dt,\quad G_2(a,b,\Phi)=\Phi\left(\frac{1}{b-a}\int_a^b\phi(t)dt\right).$$

It is known [10] that $G_1(a,b,\Phi)<G_2(a,b,\Phi)$, and in [11] the following lemmas are proved.

Lemma 3. *Let $\Phi\in\Omega$. In order that $Q(\sigma)\le\Phi(\sigma)$ for all $\sigma\ge\sigma_0$ it is necessary and sufficient that $P(t)\le -t\Psi(\phi(t))$ for all $t\ge t_0$.*

Lemma 4. *Let $\Phi\in\Omega$ and $P(t_k)\ge -t_k\Psi(\phi(t_k))$ for some sequence (t_k) of positive integers increasing to $+\infty$. Then for all $k\ge k_0$ and all $\sigma\in[\phi(t_k),\,\phi(t_{k+1})]$ the inequality*

$$Q(\sigma)\ge\Phi(\sigma)-G_1(t_k,t_{k+1},\Phi)+G_2(t_k,t_{k+1},\Phi) \tag{13}$$

is valid.

Lemma 5. *Let $\Phi_j \in \Omega$ $(j \in \{1,2\})$, $\Phi_1(\sigma) \le Q(\sigma) \le \Phi_2(\sigma)$ for all $\sigma \ge \sigma_0$ i $P(t) \le -t\Psi_2(\phi_2(t))$ for all $t \ge t_0$. Then there exists a sequence (t_k) of positive integers increasing to $+\infty$ such that $P(t) \ge -t_k\Psi_1(\phi_1(t_k))$ and*

$$G_1(t_k, t_{k+1}, \Phi_2) \ge \Phi_1\left(\frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} \phi_2(t)dt\right). \tag{14}$$

Suppose $\Phi \in \Omega$ and Φ is a function of the form

$$\Phi(\sigma) = \sum_{j=1}^{m} T_j\sigma^{p_j} + \tau\sigma^p, \sigma \ge \sigma_0, \tag{15}$$

where $p_1 > 1$, $0 < p < p_m < \dots < p_2 < p_1$ for $m \ge 2$, $T_1 > 0$, $T_j \in \mathbb{R}\backslash\{0\}$ for $2 \le j \le m$ and $\tau \in \mathbb{R}\backslash\{0\}$.

3. Asymptotic behavior of ϕ and $G_j(t_k, t_{k+1}, \Phi)$. The following lemma is true.

Lemma 6. *Suppose that function $\Phi \in \Omega$ is of the form (15) and $2p_2 < p_1 + p$. Then*

$$\phi(x) =$$

$$= \left(\frac{x}{T_1p_1}\right)^{\frac{1}{p_1-1}} - \sum_{j=2}^{m}\frac{T_jp_j}{T_1p_1(p_1-1)}\left(\frac{x}{T_1p_1}\right)^{\frac{p_j-p_1+1}{p_1-1}} - \frac{(\tau+o(1))p}{T_1p_1(p_1-1)}\left(\frac{x}{T_1p_1}\right)^{\frac{p-p_1+1}{p_1-1}}, \quad x \to +\infty.$$

For $m = 2$ Lemma 6 is proved in [6]. In the general case the proof is analogous.

Since $(x\Psi(\phi(x)))' = \phi(x)$, from (16) it follows that

$$x\Psi(\phi(x)) = T_1(p_1-1)\left(\frac{x}{T_1p_1}\right)^{\frac{p_1}{p_1-1}} - \sum_{j=2}^{m} T_j\left(\frac{x}{T_1p_1}\right)^{\frac{p_j}{p_1-1}} -$$
$$-(\tau+o(1))\left(\frac{x}{T_1p_1}\right)^{\frac{p}{p_1-1}}, \ x \to +\infty. \tag{16}$$

Let (t_k) be an increasing to $+\infty$ sequence of positive integers and $t_{k+1} = (1+\theta_k)t_k$. Since

$$\int_a^b \frac{\Phi(\phi(t))}{t^2}dt = \int_a^b \Phi(\phi(t))d\left(-\frac{1}{t}\right) = \Psi(\phi(b)) - \Psi(\phi(a)),$$

from (17) we can obtain that

$$G_1(t_k, t_{k+1}, \Phi) = T_1(p_1-1)\left(\frac{t_k}{T_1p_1}\right)^{p_1/(p_1-1)}\frac{1+\theta_k}{\theta_k}\left((1+\theta_k)^{1/(p_1-1)}-1\right) -$$
$$-\sum_{j=2}^{m} T_j\left(\frac{t_k}{T_1p_1}\right)^{p_j/(p_1-1)}\frac{1+\theta_k}{\theta_k}\left((1+\theta_k)^{(p_j-p_1+1)/(p_1-1)}-1\right) -$$
$$-\tau(1+o(1))\left(\frac{t_k}{T_1p_1}\right)^{p/(p_1-1)}\frac{1+\theta_k}{\theta_k}\left((1+\theta_k)^{(p-p_1+1)/(p_1-1)}-1\right), \quad k \to \infty. \tag{17}$$

Hence it follows that if there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j} \to +\infty$ as $j \to \infty$, then

$$G_1(t_{k_j}, (1+\theta_{k_j})k_j, \Phi) = T_1(p_1-1)\left(\frac{t_{k_j}}{T_1 p_1}\right)^{p_1/(p_1-1)} \theta_{k_j}^{1/(p_1-1)}(1+o(1)), \quad j \to \infty. \tag{18}$$

If there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j} \to \theta \in (0, +\infty)$ as $j \to \infty$, then

$$G_1(t_{k_j}, (1+\theta_{k_j})k_j, \Phi) = \\ = T_1(p_1-1)\left(\frac{t_{k_j}}{T_1 p_1}\right)^{p_1/(p_1-1)} \frac{1+\theta}{\theta}\left((1+\theta)^{1/(p_1-1)}-1\right)(1+o(1)), \quad j \to \infty. \tag{19}$$

Finally, we get $\theta_k \to 0$ as $k \to \infty$. Then from (18) it follows that

$$G_1(t_k, (1+\theta_k)t_k, \Phi) = \\ = T_1\left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} + \frac{T_1 p_1}{2(p_1-1)}\theta_k\left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} + \frac{T_1(2-p_1)p_1}{6(p_1-1)^2}\theta_k^2\left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} - \\ - \sum_{j=2}^{m}\frac{T_j(p_j-p_1+1)}{p_1-1}\left(\frac{t_k}{T_1 p_1}\right)^{p_j/(p_1-1)} - \sum_{j=2}^{m}\frac{T_j p_j(p_j-p_1+1)}{2(p_1-1)^2}\theta_k\left(\frac{t_k}{T_1 p_1}\right)^{p_j/(p_1-1)} - \\ - \frac{\tau(p-p_1+1)(1+o(1))}{p_1-1}\left(\frac{t_k}{T_1 p_1}\right)^{p/(p_1-1)} + O(t_k^{p_1/(p_1-1)}\theta_k^3) + O(t_k^{p_2/(p_1-1)}\theta_k^2)$$

as $k \to \infty$.

Now we consider an asymptotic behaviour of $G_2(t_k, t_{k+1}, \Phi)$. At first we put $\varkappa(t_k, t_{k+1}, \Phi) = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} \varphi(x)dx$. Then in view of (16) (or (17))

$$\varkappa(t_k, (1+\theta_k)t_k, \Phi) = \frac{1}{\theta_k t_k}\Bigg\{T_1(p_1-1)\left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)}\left((1+\theta_k)^{p_1/(p_1-1)}-1\right) - \\ - \sum_{j=2}^{m} T_j\left(\frac{t_k}{T_1 p_1}\right)^{p_j/(p_1-1)}\left((1+\theta_k)^{p_j/(p_1-1)}-1\right) - \\ - \tau(1+o(1))\left(\frac{t_k}{T_1 p_1}\right)^{p/(p_1-1)}\left((1+\theta_k)^{p/(p_1-1)}-1\right)\Bigg\}, \quad k \to \infty. \tag{20}$$

Hence it follows that if there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j} \to +\infty$ as $j \to \infty$, then

$$\varkappa(t_{k_j}, (1+\theta_{k_j})t_{k_j}, \Phi) = \frac{p_1-1}{p_1}\left(\frac{t_{k_j}}{T_1 p_1}\right)^{1/(p_1-1)}\theta_{k_j}^{1/(p_1-1)}(1+o(1)), \quad j \to \infty,$$

and since $G_2(t_k, t_{k+1}, \Phi) = \Phi(\varkappa(t_k, t_{k+1}, \Phi)$ and $\Phi(\sigma) = (1+o(1))T_1\sigma^{p_1}$ as $\sigma \to +\infty$, we have

$$G_2(t_{k_j}, (1+\theta_{k_j})t_{k_j}, \Phi) = T_1\left(\frac{p_1-1}{p_1}\right)^{p_1}\left(\frac{t_{k_j}}{T_1 p_1}\right)^{p_1/(p_1-1)}\theta_{k_j}^{p_1/(p_1-1)}(1+o(1)), \quad j \to \infty. \tag{21}$$

If there exists an increasing sequence (k_j) of positive integers such that $\theta_{k_j} \to \theta \in (0, +\infty)$ as $j \to \infty$, then

$$\varkappa(t_{k_j}, (1+\theta_{k_j})t_{k_j}, \Phi) = \frac{p_1 - 1}{p_1} \left(\frac{t_{k_j}}{T_1 p_1}\right)^{1/(p_1-1)} \frac{(1+\theta)^{p_1/(p_1-1)} - 1}{\theta}(1+o(1)), \quad j \to \infty,$$

and, thus,

$$G_2(t_{k_j}, (1+\theta_{k_j})t_{k_j}, \Phi) = \\ = T_1 \left(\frac{p_1-1}{p_1}\right)^{p_1} \left(\frac{t_{k_j}}{T_1 p_1}\right)^{p_1/(p_1-1)} \left(\frac{(1+\theta)^{p_1/(p_1-1)} - 1}{\theta}\right)^{p_1} (1+o(1)), \quad j \to \infty.$$

Finally, if $\theta_k \to 0$ as $k \to \infty$ then from (22) we obtain

$$\varkappa(t_k, (1+\theta_k)t_k, \Phi) = \left(\frac{t_k}{T_1 p_1}\right)^{1/(p_1-1)} \left\{1 + \frac{\theta_k}{2(p_1-1)} + \frac{(2-p_1)\theta_k^2}{6(p_1-1)^2} + O(\theta_k^3)\right\} - \\ - \sum_{j=2}^{m} \frac{T_j p_j}{T_1 p_1 (p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{(p_j-p_1+1)/(p_1-1)} \left\{1 + \frac{(p_j - p_1 + 1)\theta_k}{2(p_1-1)} + O(\theta_k^2)\right\} - \\ - \frac{\tau p(1+o(1))}{T_1 p_1 (p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{(p-p_1+1)/(p_1-1)}, \quad k \to \infty. \tag{22}$$

Hence for $q > 0$ we have

$$\varkappa(t_k, (1+\theta_k)t_k, \Phi)^q = \left(\frac{t_k}{T_1 p_1}\right)^{q/(p_1-1)} \left\{1 + \frac{q\theta_k}{2(p_1-1)} + \frac{q(2-p_1)\theta_k^2}{6(p_1-1)^2} + O(\theta_k^3) - \right. \\ - \sum_{j=2}^{m} \frac{T_j p_j q}{T_1 p_1 (p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{(p_j-p_1)/(p_1-1)} - \sum_{j=2}^{m} \frac{T_j p_j q (p_j - p_1 + 1)\theta_k}{2T_1 p_1 (p_1-1)^2} \left(\frac{t_k}{T_1 p_1}\right)^{(p_j-p_1)/(p_1-1)} + \\ + O(\theta_k^2 t_k^{(p_j-p_1)/(p_1-1)}) - \frac{\tau q p (1+o(1))}{T_1 p_1 (p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{(p-p_1)/(p_1-1)} + \frac{q(q-1)\theta_k^2}{8(p_1-1)^2} - \\ \left. - \frac{q(q-1)\theta_k}{2(p_1-1)} \sum_{j=2}^{m} \frac{T_j p_j q}{T_1 p_1 (p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{(p_j-p_1)/(p_1-1)} + O(t_k^{2(p_2-p_1)/(p_1-1)}) + O(\theta_k^3)\right\},$$

as $k \to \infty$

Therefore,

$$G_2(t_{k_j}, (1+\theta_{k_j})t_{k_j}, \Phi) = T_1 \left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} + \frac{T_1 p_1 \theta_k}{2(p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} + \\ + \frac{T_1 p_1 (3p_1+1)\theta_k^2}{24(p_1-1)^2} \left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} - \sum_{j=2}^{m} \frac{T_j (p_j - p_1 + 1)}{p_1 - 1} \left(\frac{t_k}{T_1 p_1}\right)^{p_j/(p_1-1)} - \\ - \sum_{j=2}^{m} \frac{T_j (p_j - p_1 + 1)\theta_k}{2(p_1-1)^2} \left(\frac{t_k}{T_1 p_1}\right)^{p_j/(p_1-1)} - \frac{\tau(p - p_1 + 1)(1+o(1))}{p_1 - 1} \left(\frac{t_k}{T_1 p_1}\right)^{(p-p_1)/(p_1-1)} + \\ + O(\theta_k^3 t_k^{p_1/(p_1-1)}) + O(\theta_k^2 t_k^{p_2/(p_1-1)}) + O(t_k^{(2p_2-p_1)/(p_1-1)}), \quad k \to \infty. \tag{23}$$

The following lemma is a consequence of (21) and (26).

Lemma 7. *Suppose that $\Phi \in \Omega$ and Φ has a form (15) and $2p_2 < p_1 + p$. If $\theta_k \to 0$ $(k \to \infty)$ then*

$$G_2(t_k, t_k(1+\theta_k), \Phi) - G_1(t_k, t_k(1+\theta_k), \Phi) = \frac{T_1 p_1 \theta_k^2}{8(p_1-1)} \left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} + \\ + O\left(\theta_k^3 t_k^{p_1/(p_1-1)}\right) + O\left(\theta_k^2 t_k^{p_1/(p_1-1)}\right) + o\left(t_k^{p/(p_1-1)}\right), \quad k \to \infty.$$

We will also need the following statement.

Lemma 8. *Let $\Phi_1 \in \Omega$ and $\Phi_2 \in \Omega$ be such functions that*

$$\Phi_1(\sigma) = \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau - \delta)\sigma^p, \quad \Phi_2(\sigma) = \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau + \delta)\sigma^p$$

for $\sigma \geq \sigma_0$, where $\delta \in (0, |\tau|)$. We suppose that $p > 2p_2 - p_1$, $t_{k+1} = (1+\theta_k)t_k$ and for all $k \geq k_0$

$$G_1(t_k, t_{k+1}, \Phi_2) \geq \Phi_1(\varkappa(t_k, (1+\theta_k)t_k, \Phi_2)). \tag{24}$$

Then $\theta_k \to 0$ $(k \to \infty)$ and

$$\theta_k^2 \leq \frac{16(p_1-1)}{T_1 p_1}(\delta + o(1)) \left(\frac{t_k}{T_1 p_1}\right)^{(p-p_1)/(p_1-1)} + o\left(t_k^{(p-p_1)/(p_1-1)}\right), \quad k \to \infty. \tag{25}$$

Proof. Since $\Phi_1(\sigma) = \Phi_2(\sigma) - 2\delta\sigma^p$ and $\Phi_2(\varkappa(t_k, (1+\theta_k)t_k, \Phi_2)) = G_2(t_k, t_k(1+\theta_k), \Phi_2)$, from (33) we have

$$G_1(t_k, t_{k+1}, \Phi_2) \geq G_2(t_k, t_k(1+\theta_k), \Phi_2) - 2\delta\varkappa(t_k, (1+\theta_k)t_k, \Phi_2)^p. \tag{26}$$

Using (29) and (19), (23) and respectively (20), (24), as in [6], it is easy to show that $\theta_k \to 0$ $(k \to \infty)$. Therefore, from (25) we get the asymptotics

$$\varkappa(t_k, (1+\theta_k)t_k, \Phi_2)^p = (1 + o(1)) \left(\frac{t_k}{T_1 p_1}\right)^{p/(p_1-1)}, \quad k \to \infty,$$

and in view of Lemma 7 from (29) we have

$$\frac{T_1 p_1 \theta_k^2}{8(p_1-1)} \leq 2(1+o(1))\delta \left(\frac{t_k}{T_1 p_1}\right)^{(p-p_1)/(p_1-1)} + \\ + O\left(\theta_k^3\right) + O\left(\theta_k^2 t_k^{(p_2-p_1)/(p_1-1)}\right) + o\left(t_k^{(p-p_1)/(p_1-1)}\right), \quad k \to \infty,$$

whence we obtain (28). □

4. Main results. At first we prove the following main theorem for Young conjugated functions.

Theorem 1. *Let $p_1 > 1$, $0 < p < p_m < \cdots < p_2 < p_1$ for $m \geq 2$, $T_1 > 0$, $T_j \in \mathbb{R}$ for $2 \leq j \leq m$, $\tau \in \mathbb{R}$ and $p + p_1 > 2p_2$ for $m \geq 3$. Then in order that*

$$Q(\sigma) = \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau + o(1))\sigma^p, \quad \sigma \to +\infty, \tag{27}$$

it is necessary and sufficient that for every $\varepsilon > 0$ the inequality

$$P(t) \le -T_1(p_1-1)\left(\frac{t}{T_1p_1}\right)^{p_1/(p_1-1)} + \sum_{j=2}^{m} T_j\left(\frac{t}{T_1p_1}\right)^{p_j/(p_1-1)} + (\tau+\varepsilon)\left(\frac{t}{T_1p_1}\right)^{p/(p_1-1)}, \quad (28)$$

for $t \ge t_0(\varepsilon)$ is valid and there exists an increasing to $+\infty$ sequence (t_k) of positive integers such that

$$P(t_k) \ge -T_1(p_1-1)\left(\frac{t_k}{T_1p_1}\right)^{p_1/(p_1-1)} + \sum_{j=2}^{m} T_j\left(\frac{t_k}{T_1p_1}\right)^{p_j/p_1-1)} + (\tau-\varepsilon)\left(\frac{t_k}{T_1p_1}\right)^{p/p_1-1}, \quad (29)$$

$$t_{k+1} - t_k = o(t_k^{(p_1+p-2)/2(p_1-1)}), \quad k \to \infty. \quad (30)$$

Proof. We begin with the necessity. Asymptotics (36) implies for every $\delta \in (0, |\tau|)$ and all $\sigma \ge \sigma_0(\delta)$ the condition of Lemma 5 is true with

$$\Phi_1(\sigma) = \sum_{j=1}^{m} T_j\sigma^{p_j} + (\tau-\varepsilon)\sigma^p, \quad \Phi_2(\sigma) = \sum_{j=1}^{m} T_j\sigma^{p_j} + (\tau+\varepsilon)\sigma^p.$$

Therefore, according to this lemma the inequalities $P(t) \le -t\Psi(\phi(t))$ for all $t \ge t_0$ and $P(t_k) \le -t_k\Psi(\phi(t_k))$ for an increasing to $+\infty$ sequence (t_k) of positive integers such that (14) and, thus, (33) holds. But by (17)

$$t\Psi_2(\phi_2(t)) = T_1(p_1-1)\left(\frac{t}{T_1p_1}\right)^{\frac{p_1}{p_1-1}} - \sum_{j=2}^{m} T_j\left(\frac{t}{T_1p_1}\right)^{\frac{p_j}{p_1-1}} - (\tau+\delta+o(1))\left(\frac{t}{T_1p_1}\right)^{\frac{p}{p_1-1}}$$

as $t \to +\infty$ and

$$t_k\Psi_2(\phi_2(t_k)) = T_1(p_1-1)\left(\frac{t_k}{T_1p_1}\right)^{\frac{p_1}{p_1-1}} - \sum_{j=2}^{m} T_j\left(\frac{t_k}{T_1p_1}\right)^{\frac{p_j}{p_1-1}} - (\tau-\delta+o(1))\left(\frac{t_k}{T_1p_1}\right)^{\frac{p}{p_1-1}}$$

as $k \to \infty$, and by Lemma 8 we have

$$\left(\frac{t_{k+1}-t_k}{t_k}\right)^2 = \theta_k^2 \le \frac{16(p_1-1)}{T_1p_1}(\delta+o(1))\left(\frac{t_k}{T_1p_1}\right)^{(p-p_1)/(p_1-1)}, \quad k \to \infty,$$

i.e.

$$t_{k+1} - t_k \le 4\sqrt{p_1-1}(\sqrt{\delta}+o(1))(T_1p_1)^{(p-p_1)/2(p_1-1)}t_k^{(p_1+p-2)/2(p_1-1)}, \quad k \to \infty.$$

Taking into account arbitrariness of δ these relation imply (31)–(33).

We will now prove the sufficiency of conditions (31)–(33). Using Lemma 4 and equality (17), it is easy to show that condition (31) implies the asymptotic inequality

$$Q(\sigma) \le \sum_{j=1}^{m} T_j\sigma^{p_j} + (\tau +)\sigma^p, \quad \sigma \ge \sigma(\delta), \quad (31)$$

for an arbitrary positive δ. Further by Lemmas 4 and 7 for $k \geq k_0$ and $\sigma \in [\phi_1(t_k), \phi_1(t_{k+1})]$ in view of condition (32) we obtain

$$Q(\sigma) \geq \Phi_1(\sigma) - (G_2(t_k, t_{k+1}, \Phi_1) - G_1(t_k, t_{k+1}, \Phi_1) = \Phi_1(\sigma) - \frac{T_1 p_1 \theta_k^2}{8(p_1 - 1)} \left(\frac{t_k}{T_1 p_1}\right)^{p_1/(p_1-1)} + \\ + O\left(\theta_k^3 t_k^{p_1/(p_1-1)}\right) + O\left(\theta_k^2 t_k^{p_1/(p_1-1)}\right) + o\left(t_k^{p/(p_1-1)}\right) = \Phi_1(\sigma) + o\left(t_k^{p/(p_1-1)}\right), \ k \to \infty, \quad (32)$$

because in view of (33)

$$\theta_k = \frac{t_{k+1} - t_k}{t_k} = o(t_k^{(p-p_1)/2(p_1-1)}), \quad k \to \infty.$$

Since $\phi_1(t_k) \leq \sigma \leq \phi_1(t_{k+1})$, we have $t_k \leq \Phi'(\sigma) \leq t_{k+1}$ and from (40) we obtain

$$Q(\sigma) \geq \Phi_1(\sigma) + o\left(\Phi'(\sigma))^{p/(p_1-1)}\right) = \Phi_1(\sigma) + o\left((\sigma^{p_1-1})^{p/(p_1-1)}\right) = \\ = \Phi_1(\sigma) + o\left(\sigma^p\right) = \sum_{j=1}^{m} T_j \sigma^{p_j} + (\tau - \delta)\sigma^p, \quad \sigma \geq \sigma(\delta_1), \quad (33)$$

for arbitrary positive δ_1. According to arbitrariness of δ and δ_1 (34) and (35) imply (30). □

If we choose for an entire Dirichlet series (1)

$$P(t) = \begin{cases} \ln |a_n|, & t = \lambda_n \, (n \in \mathbb{Z}_+), \\ -\infty, & t =\in (0 \, +\infty) \setminus \{\lambda_n\} \end{cases},$$

then $Q(\sigma) = \ln \mu(\sigma, F)$, and from Theorem 1 we obtain the corresponding corollary. Uniting it with Lemma 1 we get the following statement.

Theorem 2. *Let $p_1 > 1$, $0 < p < p_m < \cdots < p_2 < p_1$ for $m \geq 2$, $T_1 > 0$, $T_j \in \mathbb{R}$ for $2 \leq j \leq m$, $\tau \in \mathbb{R}$ and $p + p_1 > 2p_2$ for $m \geq 3$. Suppose that for entire Dirichlet series (1) $\ln n = o(l_n^{p/(p_1-1)})$ as $n \to \infty$. The asymptotic equality (3) hold if and only if for every $\varepsilon > 0$ the inequality*

$$\ln |a_n| \leq -T_1(p_1 - 1) \left(\frac{\lambda_n}{T_1 p_1}\right)^{p_1/(p_1-1)} + \sum_{j=2}^{m} T_j \left(\frac{\lambda_n}{T_1 p_1}\right)^{p_j/(p_1-1)} + (\tau + \varepsilon) \left(\frac{\lambda_n}{T_1 p_1}\right)^{p/(p_1-1)},$$

for $n \geq n_0(\varepsilon)$, is valid and there exists a sequence (n_k) of positive integers such that

$$\ln |a_{n_k}| \geq -T_1(p_1 - 1) \left(\frac{\lambda_{n_k}}{T_1 p_1}\right)^{p_1/(p_1-1)} + \sum_{j=2}^{m} T_j \left(\frac{\lambda_{n_k}}{T_1 p_1}\right)^{p_j/p_1-1)} + (\tau - \varepsilon) \left(\frac{\lambda_{n_k}}{T_1 p_1}\right)^{p/p_1-1},$$

$$\lambda_{n_{k+1}} - \lambda_{n_k} = o(\lambda_{n_k}^{(p_1+p-2)/2(p_1-1)}), \quad k \to \infty.$$

Since $\ln \mu_\varphi(r) = \sup\{\ln W_F(t) + rt \colon x \geq 0\}$ for the entire characteristic function φ of a probability law F, choosing $P(t) = \ln W_F(t)$ we have $Q(r) = \ln \mu_\varphi(r)$, and thus, Lemma 2 and Theorem 1 imply the following statement.

Theorem 3. *Let $\varrho_1 > 1$, $0 < \varrho < \varrho_m < \dots < \varrho_2 < \varrho_1$ for $m \geq 2$, $T_1 > 0$, $T_j \in \mathbb{R}\backslash\{0\}$ for $2 \leq j \leq m$, $\tau \in \mathbb{R}\backslash\{0\}$ and φ be the entire characteristic function of a probability law F. The asymptotical equality (6) hold if and only if for every $\varepsilon > 0$ the inequality*

$$\ln W_F(t) \leq -T_1(p_1-1)\left(\frac{t}{T_1p_1}\right)^{p_1/(p_1-1)} + \sum_{j=2}^m T_j\left(\frac{t}{T_1p_1}\right)^{p_j/(p_1-1)} + (\tau+\varepsilon)\left(\frac{t}{T_1p_1}\right)^{p/(p_1-1)},$$

for $t \geq t_0(\varepsilon)$ is valid and there exists an increasing to $+\infty$ sequence (t_k) of positive integers such that

$$\ln W_F(t_k) \geq -T_1(p_1-1)\left(\frac{t_k}{T_1p_1}\right)^{p_1/(p_1-1)} + \sum_{j=2}^m T_j\left(\frac{t_k}{T_1p_1}\right)^{p_j/p_1-1)} + (\tau-\varepsilon)\left(\frac{t_k}{T_1p_1}\right)^{p/p_1-1},$$

$$t_{k+1}-t_k = o(t_k^{(p_1+p-2)/2(p_1-1)}), \quad k\to\infty.$$

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