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ENTIRE FUNCTIONS THAT SHARE A POLYNOMIAL WITH FINITE WEIGHT

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In this paper, with the aid of weighted sharing method we study the uniqueness problems of entire functions that share a nonconstant polynomial with weight two. The results of the paper improve and generalize some results due to [10] and [11].

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Методом весовых разделяемых значений изучается проблема единственности целых функций, которые разделяют полином отличний от константы с кратностью 2. Результаты статьи улучшают и обобщают некоторые результаты из [10] и [11].

1. Introduction, denitions and results. In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with standard notation and fundamental results of the Nevanlinna Theory as described in [6, 13, 14]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of a possible exceptional set E of finite linear measure. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$, by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$. The meromorphic function a is called a *small function* of f if $T(r, a) = S(r, f)$.

Two nonconstant meromorphic functions f and g share a small function a CM (counting multiplicities) provided that $f - a$ and $g - a$ have the same set of zeros with the same multiplicities; f and g share a IM (ignoring multiplicities) if we do not consider the multiplicities. A finite value z_0 is called a *fixed point* of $f(z)$ if $f(z_0) = z_0$. We define

$$E_f = \{z \in \mathbb{C} \colon f(z) = z, \text{counting multiplicities}\}.$$

Regarding a familiar question raised to W. K. Hayman ([5]), the following result was proved by M. L. Fang, X. H. Hua ([3]) in 1996.

Theorem A. *Let f and g be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

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In 2002 M. L. Fang ([2]) proved the following results extending Theorem A in which k-th derivative of f^n and g^n is taken into consideration.

Theorem B. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.*

Theorem C. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f = g$.*

Natural question arises: What can be said if the value sharing in the above theorems is replaced by sharing a fixed point? Afterwards research works concerning the above question have been done by many mathematicians such as M. L. Fang, H. L. Qiu ([4]), W. C. Lin, H. X. Yi ([9]), X. G. Qi, L. Z. Yang ([10]), P. Sahoo ([11]), J. L. Zhang ([15]). In this direction, we recall the following results due to J. L. Zhang ([15]) proved in 2008.

Theorem D. *Let f and g be two nonconstant entire functions, and n, k be two positive integers with $n > 2k + 4$. If $E_{(f^n)^{(k)}} = E_{(g^n)^{(k)}}$, then either*

(i) $k = 1$, $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$ or

(ii) $f = tg$ for a constant t such that $t^n = 1$.

Theorem E. *Let f and g be two nonconstant entire functions, and n, k be two positive integers with $n \geq 2k + 6$. If $E_{(f^n(f-1))^{(k)}} = E_{(g^n(g-1))^{(k)}}$, then $f = g$.*

In 2010 X. G. Qi and L. Z. Yang ([10]) and in 2011 J. Dou, X. G. Qi and L. Z. Yang ([1]) studied the uniqueness problem of entire functions concerning some general differential polynomials and proved the following results extending Theorems D and E, respectively.

Theorem F. *Let f and g be two transcendental entire functions, n, m and k be three positive integers with $n > 2k + m^* + 4$, λ and μ be constants that satisfy $|\lambda| + |\mu| \neq 0$. If $[f^n(\lambda f^m + \mu)]^{(k)}$ and $[g^n(\lambda g^m + \mu)]^{(k)}$ share z CM, then the following conclusions hold:*

(i) if $\lambda\mu \neq 0$, then $f^d(z) = g^d(z)$, where $d = \gcd(n, m)$; especially when $d = 1$, $f = g$;

(ii) if $\lambda\mu = 0$, then either $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$ or $k = 1$ and $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$ for three constants c_1, c_2 and c that satisfy

$$4(\lambda + \mu)^2(c_1c_2)^{n+m^*}[(n + m^*)c]^2 = -1,$$

where

$$m^* = \begin{cases} m & \text{if } \lambda \neq 0; \\ 0 & \text{if } \lambda = 0. \end{cases}$$

Theorem G. *Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \ldots + a_1z + a_0$ or $P(z) = C$, where a_0, $a_1, \ldots, a_{m-1}$, $a_m(\neq 0)$, $C(\neq 0)$ are complex constants. Suppose that f and g are two transcendental entire functions, and let n, k and m be three positive integers with $n > 2k + m^{**} + 4$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share z CM, then the following conclusions hold:*

(i) *if* $P(z) = a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0$ *is not a monomial, then either* $f = tg$ *for a constant* t *that satisfies* $t^d = 1$, *where* $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ *for some* $i \in \{0, 1, 2, \ldots, m\}$; *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$, *where*

$$R(w_1, w_2) = w_1^n(a_m w_1^m + \ldots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + \ldots + a_1 w_2 + a_0);$$

(ii) *when* $P(z) = C$ *or* $P(z) = a_m z^m$, *then either* $f = tg$ *for a constant* t *that satisfies* $t^{n+m^{**}} = 1$, *or* $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ *for three constants* b_1, b_2 *and* b *that satisfies* $4a_m^2(b_1b_2)^{n+m}((n+m)b)^2 = -1$, *or* $4C^2(b_1b_2)^n(nb)^2 = -1$, *where* m^{**} *is defined by*

$$m^{**} = \begin{cases} m & \text{if } P(z) \neq C; \\ 0 & \text{if } P(z) = C. \end{cases}$$

Observing the above results the following questions are natural.

Question 1. *What can be said if the fixed point sharing in Theorems F and G is replaced with sharing a nonconstant polynomial?*

Question 2. *Is it possible to relax the nature of sharing in Theorems F and G keeping the lower bound of* n *fixed?*

In the paper we will concentrate our attention to the above questions and provide an affirmative answer of Question 2. To state the main results we need the following definition known as weighted sharing of values introduced by I. Lahiri ([7, 8]) which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

In the paper, we prove the following two theorems which improve and generalize Theorems F and G, respectively, as well as deal with Question 1 and Question 2. We now state the main results of the paper.

Theorem 1. *Let* f *and* g *be two transcendental entire functions,* $P_1(z)$ *be a nonconstant polynomial of degree* p, *and let* n, k *and* m *be three positive integers with* $n > 2k+2p+m^*+2$. *Suppose further that* $k > p$ *when* $p \geq 2$. *If* $[f^n(\lambda f^m + \mu)]^{(k)} - P_1$ *and* $[g^n(\lambda g^m + \mu)]^{(k)} - P_1$ *share* $(0, 2)$ *where* λ, μ *are constants satisfying* $|\lambda| + |\mu| \neq 0$, *then the following conclusions hold:*

(i) *if* $\lambda\mu \neq 0$, *then* $f^d(z) = g^d(z)$, *where* $d = \gcd(n, m)$; *especially when* $d = 1$, $f = g$;

(ii) *if* $\lambda\mu = 0$, *then either* $f = tg$ *for a constant* t *that satisfies* $t^{n+m^*} = 1$ *or* $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, *where* $Q(z)$ *is a polynomial without constant such that* $Q'(z) = P_1(z)$, b_1, b_2 *and* b *are three constants satisfying* $\mu^2(nb)^2(b_1b_2)^n = -1$ *or* $\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1$.

Theorem 2. *Let* f *and* g *be two transcendental entire functions,* $P_1(z)$ *be a nonconstant polynomial of degree* p, *and let* n, k *and* m *be three positive integers with* $n > 2k+2p+m^{**}+2$. *Let* $P(z)$ *be defined as in Theorem G. If* $[f^nP(f)]^{(k)} - P_1$ *and* $[g^nP(g)]^{(k)} - P_1$ *share* $(0,2)$ *then the following conclusions hold:*

(i) *if* $P(z) = a_mz^m + a_{m-1}z^{m-1} + \ldots + a_1z + a_0$ *is not a monomial, then either* $f = tg$ *for a constant* t *that satisfies* $t^d = 1$, *where* $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ *for some* $i \in \{0, 1, 2, \ldots, m\}$; *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$, *where*

$$R(w_1, w_2) = w_1^n(a_mw_1^m + \ldots + a_1w_1 + a_0) - w_2^n(a_mw_2^m + \ldots + a_1w_2 + a_0);$$

(ii) *when* $P(z) = C$ *or* $P(z) = a_mz^m$, *then either* $f = tg$ *for a constant* t *that satisfies* $t^{n+m^{**}} = 1$, *or* $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, *where* $Q(z)$ *is a polynomial without constant such that* $Q'(z) = P_1(z)$, b_1, b_2 *and* b *are three constants satisfying* $C^2(nb)^2(b_1b_2)^n = -1$ *or* $a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1$.

We now explain the following definitions and notations which are used in the paper.

Definition 2 ([6])**.** Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer p we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

Analogously we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 3. Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

2. Lemmas.

Lemma 1 ([12])**.** *Let* f *be a nonconstant meromorphic function and let* $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, $\ldots$, $a_0(z)$ *be small functions of* f. *Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + \ldots + a_1f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 ([16])**.** *Let* f *be a nonconstant meromorphic function, and* p, k *be positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{1}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2}$$

Lemma 3 ([8]). *Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. Then one of the following cases hold:*

(i) $T(r) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r)$,

(ii) $f = g$,

(iii) $fg = 1$.

Lemma 4 ([6]). *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, $i \in \{1,2\}$. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 5 ([6]). *Suppose that f is a nonconstant meromorphic function, $k \geq 2$ is an integer. If $N(r,\infty;f) + N(r,0;f) + N(r,0;f^{(k)}) = S(r,\frac{f'}{f})$, then $f = e^{az+b}$, where $a(\neq 0)$, b are constants.*

Lemma 6 ([11]). *Let f and g be two nonconstant entire functions and let n, k be two positive integers. Suppose that $F_1 = (f^nP(f))^{(k)}$ and $G_1 = (g^nP(g))^{(k)}$ where $P(z) = a_mz^m + a_{m-1}z^{m-1} + \ldots + a_1z + a_0$, $a_0(\neq 0)$, $a_1, \ldots, a_{m-1}$, $a_m(\neq 0)$ are complex constants. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F_1) = \overline{N}(r,0;G_1)$ and $\overline{N}(r,c_2;G_1) = \overline{N}(r,0;F_1)$, then $n \leq 2k + m + 2$.*

Lemma 7. *Let f and g be two nonconstant entire functions, n, m and k be three positive integers. Suppose that $F_1G_1 = P_1^2$, where F_1, G_1 are defined as in Lemma 6 and $P_1(z)$ is defined as in Theorem 1. Then $n \leq k + 2p$.*

Proof. If possible, we assume that $n > k + 2p$. From $F_1G_1 = P_1^2$, we have

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = P_1^2.$$

Let z_0 be a zero of f with multiplicity l. Then z_0 is a zero of $(f^nP(f))^{(k)}$ with multiplicity $nl - k$. Since g is an entire function and $n > k + 2p$, z_0 is a zero of P_1^2 with multiplicity at least $2p+1$, which is absurd. Thus f has no zeros. We put $f = e^{\alpha}$, where α is a nonconstant entire function. Now

$$(a_mf^{n+m})^{(k)} = t_m(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{(n+m)\alpha}, \tag{3}$$

$$\ldots$$

$$(a_0f^n)^{(k)} = t_0(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{n\alpha}, \tag{4}$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)})$ $(i \in \{0,1,\ldots,m\})$ are differential polynomials in α', α'', $\ldots$, $\alpha^{(k)}$. Obviously $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}) \not\equiv 0$ for $i \in \{0,1,2,\ldots,m\}$, and $(f^nP(f))^{(k)} \not\equiv 0$. Therefore from (3) and (4) we obtain

$$t_m(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{m\alpha} + \ldots + t_0(\alpha', \alpha'', \ldots, \alpha^{(k)}) \not\equiv 0. \tag{5}$$

Since α is an entire function, we have $T(r,\alpha^{(j)}) = S(r,f)$ for $j \in \{1,2,\ldots,k\}$, and hence $T(r,t_i) = S(r,f)$ for $i \in \{0,1,2,\ldots,m\}$. Therefore using (5), Lemmas 1 and 4 we deduce that

$$\begin{aligned} mT(r,f) &= T(r,t_me^{m\alpha} + \ldots + t_1e^{\alpha}) + S(r,f) \leq \\ &\leq \overline{N}(r,0;t_me^{m\alpha} + \ldots + t_1e^{\alpha}) + \overline{N}(r,0;t_me^{m\alpha} + \ldots + t_1e^{\alpha} + t_0) + S(r,f) \leq \\ &\leq \overline{N}(r,0;t_me^{(m-1)\alpha} + \ldots + t_1) + S(r,f) \leq (m-1)T(r,f) + S(r,f), \end{aligned}$$

a contradiction. Hence $n \leq k + 2p$ and the lemma follows. □

Lemma 8 ([10]). *Let f and g be two nonconstant entire functions, n, m and k be three positive integers, and let $F_2 = [f^n(\lambda f^m + \mu)]^{(k)}$ and $G_2 = [g^n(\lambda g^m + \mu)]^{(k)}$ where $|\lambda| + |\mu| \neq 0$, and $\lambda\mu = 0$. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F_2) = \overline{N}(r, 0; G_2)$ and $\overline{N}(r, c_2; G_2) = \overline{N}(r, 0; F_2)$, then $n \leq 2k + m^* + 2$.*

Lemma 9. *Let f and g be two nonconstant entire functions, n, m and k be three positive integers with $n > 2k + 2p + m^* + 2$. Further assume that $k > p$ when $p \geq 2$. Suppose that $F_2G_2 = P_1^2$, where F_2, G_2 are defined as in Lemma 8, $|\lambda| + |\mu| \neq 0$ and $P_1(z)$ is defined as in Theorem 1. Then $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1$ or $\mu^2(nb)^2(b_1b_2)^n = -1$ and $Q(z)$ is same as in Theorem 1.*

Proof. We discuss the following two cases separately.

Case I. Let $\lambda\mu = 0$. Since $|\lambda| + |\mu| \neq 0$, we may take $\mu = 0$, $\lambda \neq 0$ and therefore $m^* = m$. The case $\mu \neq 0$, $\lambda = 0$ can be proved similarly. First we assume that $k = 1$. Then $F_2G_2 = P_1^2$ gives

$$(\lambda f^{n+m})'(\lambda g^{n+m})' = P_1^2. \tag{6}$$

Since f and g are entire functions and $n > 2k + 2p + m + 2$, we deduce from (6) that f and g have no zeros. We put

$$f = e^{\alpha}, \quad g = e^{\beta}, \tag{7}$$

where α and β are two nonconstant entire functions. Therefore

$$\lambda^2(n+m)^2\alpha'\beta' e^{(n+m)(\alpha+\beta)} = P_1^2. \tag{8}$$

From (8) it follows that α, β must be polynomials and $\alpha+\beta \equiv C$, where C is a constant. Thus $\deg(\alpha) = \deg(\beta)$. Therefore $\alpha' + \beta' \equiv 0$ and $\lambda^2(n+m)^2\alpha'\beta' e^{(n+m)C} = P_1^2$. Simplifying we obtain $\alpha' = bP_1(z)$ and $\beta' = -bP_1(z)$, where $b(\neq 0)$ is a constant. This gives $\alpha = bQ(z) + d_1$ and $\beta = -bQ(z) + d_2$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = P_1(z)$ and d_1, d_2 are constants. Therefore $f = b_1 e^{bQ(z)}$, $g = b_2 e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1$. Next we assume that $k \geq 2$. Then $F_2G_2 = P_1^2$ gives

$$(\lambda f^{n+m})^{(k)}(\lambda g^{n+m})^{(k)} = P_1^2. \tag{9}$$

Since f and g are transcendental entire function, from (9) we obtain $N(r, 0; (\lambda f^{n+m})^{(k)}) = O\{\log r\}$. From this and (7) we get

$$N(r, \infty; \lambda f^{n+m}) + N(r, 0; \lambda f^{n+m}) + N(r, 0; (\lambda f^{n+m})^{(k)}) = O\{\log r\}.$$

Suppose that α is a transcendental entire function. Then by Lemma 5 we deduce that α is a polynomial, a contradiction. Next we assume that α, β are polynomials of degree p_1 and p_2 respectively. If $p_1 = p_2 = 1$, then $f = e^{Az+B}$, $g = e^{Cz+D}$, where $A(\neq 0)$, B, $C(\neq 0)$ and D are constants. So from (9) we obtain

$$\lambda^2(AC)^k(n+m)^{2k} e^{(n+m)\{(A+C)z+(B+D)\}} = P_1^2,$$

which is not possible. Hence $\max\{p_1, p_2\} > 1$. We assume that $p_1 > 1$. Then $(\lambda f^{n+m})^{(k)} = Q_1 e^{(n+m)\alpha}$ and $(\lambda g^{n+m})^{(k)} = Q_2 e^{(n+m)\beta}$, where Q_1, Q_2 are polynomials of degree $k(p_1 - 1)$ and $k(p_2 - 1)$, respectively. So from (9) we obtain $\alpha + \beta \equiv k_1$, a constant, and hence $p_1 = p_2$ and $k(p_1 - 1) = p$. This shows that $p \geq k \geq 2$, contradicting with the assumption that $k > p$ when $p \geq 2$.

Case II. Let $\lambda\mu \neq 0$. Since $n > 2k + 2p + m + 2 > k + 2p$, using the argument similar as in Lemma 7 we obtain a contradiction. □

Lemma 10 ([10]). *Suppose that F_2 and G_2 are given as in Lemma 8 where $\lambda\mu \neq 0$. If $n > 2k + m$ and $F_2 = G_2$, then $f^d(z) = g^d(z)$ where $d = \gcd(n, m)$.*

Lemma 11 ([10]). *Suppose that F_2 and G_2 are given as in Lemma 8 where $\lambda\mu = 0$. If $n > 2k + m^*$ and $F_2 = G_2$, then $f = tg$ for a constant t satisfying $t^{n+m^*} = 1$.*

3. Proof of the Theorems 1 and 2.

Proof of Theorem 2. We discuss the following three cases separately.

Case (i) Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_2 z^2 + a_1 z + a_0$, where $a_0 (\neq 0)$, a_1, …,a_{m-1}, $a_m (\neq 0)$ are complex constants. We consider $F = \frac{(f^n P(f))^{(k)}}{P_1(z)}$ and $G = \frac{(g^n P(g))^{(k)}}{P_1(z)}$. Then F and G are transcendental meromorphic functions that share $(1, 2)$. Now from Lemma 1 and (1) we obtain

$$N_2(r, 0; F) \leq N_2\left(r, 0; (f^n P(f))^{(k)}\right) + S(r, f) \leq$$
$$\leq T\left(r, (f^n P(f))^{(k)}\right) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) + S(r, f) \leq$$
$$\leq T(r, F) - (n + m)T(r, f) + N_{k+2}(r, 0; f^n P(f)) + S(r, f). \tag{10}$$

Similarly

$$N_2(r, 0; G) \leq T(r, G) - (n + m)T(r, g) + N_{k+2}(r, 0; g^n P(g)) + S(r, g). \tag{11}$$

Again by (2) we have

$$N_2(r, 0; F) \leq N_{k+2}(r, 0; f^n P(f)) + S(r, f), \tag{12}$$
$$N_2(r, 0; G) \leq N_{k+2}(r, 0; g^n P(g)) + S(r, g). \tag{13}$$

From (10) and (11) we get

$$(n + m)\{T(r, f) + T(r, g)\} \leq T(r, F) + T(r, G) + N_{k+2}(r, 0; f^n P(f))+$$
$$+N_{k+2}(r, 0; g^n P(g)) - N_2(r, 0; F) - N_2(r, 0; G) + S(r, f) + S(r, g). \tag{14}$$

We assume that the conclusion (i) of Lemma 3 holds. Then using Lemma 1, (12) and (13) we obtain from (14)

$$(n + m)\{T(r, f) + T(r, g)\} \leq N_2(r, 0; F) + N_2(r, 0; G) + 2N_2(r, \infty; F)+$$
$$+2N_2(r, \infty; G) + N_{k+2}(r, 0; f^n P(f)) + N_{k+2}(r, 0; g^n P(g)) + S(r, f) + S(r, g) \leq$$
$$\leq 2N_{k+2}(r, 0; f^n P(f)) + 2N_{k+2}(r, 0; g^n P(g)) + S(r, f) + S(r, g) \leq$$
$$\leq 2(k + m + 2)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).$$

From this we get $(n-m-2k-4)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$, which leads to a contradiction as $n > 2k+2p+m+2$. Hence by Lemma 3 we have either $FG=1$ or $F=G$. If $FG=1$, then $(f^nP(f))^{(k)}(g^nP(g))^{(k)} = P_1^2$, a contradiction by Lemma 7. Hence $F=G$. That is $[f^nP(f)]^{(k)} = [g^nP(g)]^{(k)}$. Integrating we get $[f^nP(f)]^{(k-1)} = [g^nP(g)]^{(k-1)} + c_{k-1}$, where c_{k-1} is a constant. If $c_{k-1} \neq 0$, from Lemma 6 we obtain $n \leq 2k+m$, a contradiction. Hence $c_{k-1}=0$. Repeating k-times, we obtain $f^nP(f) = g^nP(g)$. Then

$$f^n(a_mf^m + a_{m-1}f^{m-1} + \ldots + a_1f + a_0) = g^n(a_mg^m + a_{m-1}g^{m-1} + \ldots + a_1g + a_0). \quad (15)$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (15) we get

$$a_mg^{n+m}(h^{n+m}-1) + a_{m-1}g^{n+m-1}(h^{n+m-1}-1) + \ldots + a_0g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$. Thus $f = tg$ for a constant t such that $t^d = 1$, $d = (n+m, \ldots, n+m-i, \ldots, n+1, n)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$.

If h is not a constant, then from (15) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(w_1,w_2) = w_1^n(a_mw_1^m + \ldots + a_1w_1 + a_0) - w_2^n(a_mw_2^m + \ldots + a_1w_2 + a_0)$.

Case (ii) Now we assume that $P(z) = a_mz^m$, where a_m $(\neq 0)$ is a complex constant. Let $F = \frac{(a_mf^{n+m})^{(k)}}{P_1(z)}$ and $G = \frac{(a_mg^{n+m})^{(k)}}{P_1(z)}$. Then F and G are transcendental meromorphic functions that share the value 1 with weight two. Proceeding in the similar manner as in Case (i) above we obtain either $FG=1$ or $F=G$.

If $FG=1$, then $(a_mf^{n+m})^{(k)}(a_mg^{n+m})^{(k)} = P_1^2$. So by Lemma 9 we obtain $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $a_m^2((n+m)b)^2(b_1b_2)^{n+m} = -1$ and $Q(z)$ is same as in Theorem 1. If $F=G$, then using Lemmas 8 and 11 we obtain $f = tg$ for a constant t such that $t^{n+m} = 1$.

Case (iii) Let $P(z) = C$. Taking $F = \frac{(Cf^n)^{(k)}}{P_1(z)}$, $G = \frac{(Cg^n)^{(k)}}{P_1(z)}$ and arguing similarly as in Case (ii) we obtain either $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $C^2(nb)^2(b_1b_2)^n = -1$, $Q(z)$ is same as in Theorem 1 or $f = tg$ for a constant t satisfying $t^n = 1$. □

Proof of Theorem 1. Let $F = \frac{[f^n(\lambda f^m+\mu)]^{(k)}}{P_1(z)}$ and $G = \frac{[g^n(\lambda g^m+\mu)]^{(k)}}{P_1(z)}$. Then F and G are transcendental meromorphic functions that share the value 1 with weight two. Proceeding similarly as in Theorem 2 we obtain either $FG=1$ or $F=G$. First we assume that $\lambda\mu \neq 0$. Then $FG \not\equiv 1$, by Lemma 7. Hence $F=G$ and so by Lemmas 6 and 10 we obtain $f^d(z) = g^d(z)$ where $d = \gcd(n,m)$. Next we assume that $\lambda\mu = 0$. Let $\lambda \neq 0$ and $\mu = 0$. Then if $FG=1$, by Lemma 9 we have $f(z) = b_1e^{bQ(z)}$, $g(z) = b_2e^{-bQ(z)}$, where b_1, b_2 and b are three constants satisfying $\lambda^2((n+m)b)^2(b_1b_2)^{n+m} = -1$ and $Q(z)$ is defined as in Theorem 1. Similar result holds when $\mu \neq 0$ and $\lambda = 0$. If $F=G$, by Lemmas 8 and 11 we conclude that $f = tg$ for a constant t that satisfies $t^{n+m^*} = 1$. □

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