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NEW CHARACTERIZATIONS OF COMMUTATIVE CLEAN RINGS

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A ring is called clean if every its element is the sum of a unit and an idempotent. We introduce the notion of an avoidable element and describe class of the commutative clean rings as the rings in which zero is an avoidable element.

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Кольцо называется чистым, если каждый его элемент является сумой обратимого элемента и идемпотента. Мы вводим понятие раздельного элемента и описываем класс коммутативных чистых колец как колец, где ноль является раздельным элементом.

All rings considered will be commutative and have an identity. The adequate domains were introduced by O. Helmer in [1]. It had been known that principal ideal domains are elementary divisor rings and O. Helmer showed that the less restrictive hypothesis that an integral domain be adequate is sufficient. Following I. Kaplansky ([2]) a ring is said to be an *elementary divisor ring* if every matrix over R is equivalent to a diagonal matrix. Any elementary divisor ring is a *Bezout ring,* i. e. a ring in which every finitely generated ideal is principal. A ring R is an *Hermite ring* if every rectangular matrix A over R is equivalent to an upper or a lower triangular matrix ([2]).

In this paper we introduce the class of rings, called avoidable rings, which are generalizations of adequate rings. These rings are closely related to the neat rings. We say that a ring is a neat ring if every its non-trivial finite homomorphic image is clean ([4]).

In this paper, the commutative clean rings are described as the ring in which zero is an avoidable element.

Definition 1. An element a of a commutative ring R is said to be *avoidable* if for any elements $b, c \in R$ such that $aR + bR + cR = R$ there exist elements $r, s \in R$ such that

$$a = rs,\ rR + bR = R,\ sR + cR = R,\ rR + sR = R.$$

A ring R is called avoidable if every its nonzero element is avoidable. A ring R is an everywhere avoidable ring if every element of R is avoidable.

Definition 2. A ring R is said to be of *neat range 1* if for any elements $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $a + bt$ is an avoidable element.

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Definition 3. An element a of a commutative ring R is said to be *an adequate* if for any element $b \in R$ there exist elements $r, s \in R$ such that $a = rs$, where $rR + bR = R$ and $s'R + bR \neq R$ for every element s' such that $sR \subset s'R \neq R$. By [1], any ring R is an adequate ring if every nonzero element of R is adequate. A ring R is an everywhere adequate ring if every its element is adequate ([8]).

Theorem 1. *Any adequate element of a Bezout ring is avoidable.*

Proof. Let R be a commutative Bezout ring and a be an adequate element of R. Let $aR + bR + cR = R$ and $a = rs$ where $rR + bR = R$ and $s'R + bR \neq R$ for every element s' such that $sR \subset s'R \neq R$. Obviously $rR + sR = R$. Let $sR + cR = dR \neq R$. Since d is a noninvertible divisor of an element s, we see that $dR + bR = hR \neq R$. Due to such inclusions $cR \subset dR \subset hR$, $bR \subset hR$ and $aR \subset sR \subset hR$, we have $aR + bR + cR \subset hR \neq R$. It is impossible, because $aR + bR + cR = R$. □

As an obvious consequence we obtain the following result.

Corollary 1. *Any adequate Bezout commutative ring is an avoidable ring.*

Corollary 2. *Any everywhere adequate Bezout commutative ring is an everywhere avoidable ring.*

Consider the rings in which zero is an avoidable element. By [3], any regular ring is a ring in which zero is an adequate element. By Theorem 1, we have the following result.

Corollary 3. *Any regular ring is a ring in which zero is avoidable.*

Moreover, we have the following result.

Theorem 2. *Let a be an avoidable element of a commutative ring R. Then zero is an avoidable element of the factor-ring R/aR.*

Proof. Let $\overline{R} = R/aR$, and $\overline{bR} + \overline{cR} = \overline{R}$, where $\overline{b} = b + aR$, $\overline{c} = c + aR$. By the assumption we have $a = rs$, where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$. Now we have $\overline{0} = \overline{r} \cdot \overline{s}$, where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ and $\overline{r}\overline{R} + \overline{s}\overline{R} = \overline{R}$. □

Theorem 3. *Any commutative ring R is clean if and only if the zero element is avoidable.*

Proof. Let $bR + cR = R$ and $0 = rs$, where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$. Since $rR + sR = R$, we see that $ru + sv = 1$ for some elements $u, v \in R$. From the equality $0 = rs$ we have $r^2u = r$, $s^2v = s$. Denote $ru = e$, then $e^2 = e$ and $1 - e = sv$. Since $rR + bR = R$, we obtain $r\alpha + b\beta = 1$ for some elements $\alpha, \beta \in R$. Here $svb\beta = sv$, i. e. $1 - e \in bR$.

Similarly, $e \in cR$. By [5] R is an exchange ring. Since R is a commutative ring, by [5] R is a clean ring. The necessity is proved.

We will prove that any clean ring is a ring in which zero is avoidable. Let $bR + cR = R$. Since R is a commutative clean ring we have that R is an exchange ring. Therefore, R is an exchange ring. Then there exists an idempotent $e \in R$ such that $e \in bR$ and $1 - e \in cR$. Since $0 = e(1 - e)$, we obtain

$$(1 - e)R + bR = R, \; eR + cR = R, \; eR + (1 - e)R = R.$$

Putting $1 - e = r$, $e = s$ we obtain an appropriate representation of the zero element. □

Theorem 4. *Let R be a commutative Bezout domain. If $\overline{0}$ is an avoidable element of R/aR then a is an avoidable element of R.*

Proof. Denote $\overline{R} = R/aR$. Since zero is an avoidable element, for any elements $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{bR} + \overline{cR} = \overline{R}$ there exists $\overline{r}, \overline{s} \in \overline{R}$ such that $\overline{0} = \overline{r} \cdot \overline{s}$, $\overline{rR} + \overline{bR} = \overline{R}$, $\overline{sR} + \overline{cR} = \overline{R}$, $\overline{rR} + \overline{sR} = \overline{R}$. Denote

$$\overline{r} = r + aR,\ \overline{s} = s + aR,\ \overline{b} = b + aR,\ \overline{c} = c + aR.$$

Let b and c be arbitrary elements of R such that $aR + bR + cR = R$. Since $\overline{rR} + \overline{bR} = \overline{R}$, there are elements $u, v, t \in R$ such that $ru + bv = 1 + at$. Let $aR + rR = \delta R$. Then $a = \delta a_0$, $r = \delta r_0$ for some elements $a_0, r_0 \in R$. Moreover, $a_0R + r_0R = R$. It follows that $\delta R + bR = R$.

Since $\overline{0} = \overline{r} \cdot \overline{s}$, we obtain $rs = a\alpha$ for some element $\alpha \in R$. Then $\delta r_0 s = \delta a_0 \alpha$. Using that R is a domain we have $r_0 s = a_0 \alpha$. Since $r_0R + a_0R = R$, there exist elements $t, k \in R$ such that $r_0 t + a_0 k = 1$. This means that $a_0 \beta = s$ for some element $\beta \in R$. Therefore $a = \delta a_0$, where $\delta R + bR = R$ and $sR \subset a_0R$. Since $\overline{sR} + \overline{cR} = \overline{R}$, it is obvious that $a_0R + cR = R$. Since $\overline{rR} + \overline{sR} = \overline{R}$ obviously, $\delta R + a_0R = R$. Thus we have shown that a is an avoidable element. □

As an obvious consequence we obtain the following result.

Theorem 5. *Let R be a commutative Bezout domain. The following properties are equivalent:*

1) *R is an avoidable domain;*
2) *R is a neat domain.*

Based on the results of [6, 7], we obtain the following theorems.

Theorem 6. *Any commutative Hermite ring R is an elementary divisor ring if and only if R is a ring of neat range 1.*

Consequently, we have following results.

Theorem 7. *Any avoidable Hermite ring is an elementary divisor ring.*

Theorem 8. *Let R be a commutative Bezout domain in which for any nonzero and noninvertible element a the factor-ring R/aR is a ring in which zero is an avoidable element. Then R is an avoidable domain.*

REFERENCES

1. O. Helmer, *The elementary divisor for rings without chain condition*, Bull. Amer. Math. Soc., **49** (1943), №2, 235–236.
2. I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc., **66** (1949), 464–491.
3. M. Larsen, W. Lewis, T. Shores, *Elemetary divisor rings and finitely presented modules*, Trans. Amer. Mat. Soc., **187** (1974), 231–248.
4. W. McGovern, *Neat rings*, J. Pure Appl. Alg., **205** (2006), №2, 243–265.
5. W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Mat. Soc., **229** (1977), 269–279.

6. M. Roitman, *The Kaplansky condition and rings of almost stable range 1*, Trans. Amer. Math. Soc., **191** (2013), 3013–1319.
7. B.V. Zabavsky, *Diagonal reduction of matrices over finite stable range*, Mat. Stud., **41** (2014), №1, 101–108.
8. B.V. Zabavsky, Diagonal reduction of matrices over rings, Mathematical Studies, Monograph Series, XVI, 2012.

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