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SUFFICIENT CONDITIONS OF BOUNDEDNESS OF L-INDEX IN JOINT VARIABLES

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A concept of boundedness of **L**-index in joint variables (see in Bordulyak M.T. *The space of entire in* $\mathbb{C}^n$ *functions of bounded L-index,* Mat. Stud., **4** (1995), 53–58. (in Ukrainian)) is generalised for $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, $z \in \mathbb{C}^n$. We proved criteria of boundedness of **L**-index in joint variables and established a connection between the classes of entire functions of bounded l_j-index in each direction $\mathbf{e}_j$ and functions of bounded **L**-index in joint variables. We deduce new sufficient conditions of boundedness of **L**-index in joint variables. The obtained restrictions describe the behaviour of logarithmic derivative in each variable and the distribution of zeros.

1. Introduction. The papers of M. T. Bordulyak and M. M. Sheremeta ([1, 2]) are devoted to the investigation of the concept of an entire function of bounded **L**-index in $\mathbb{C}^n$ $(n \geq 2)$ (henceforth, an entire function of bounded **L**-index in joint variables). It is a multidimensional generalization of the concept of an entire function of the bounded l-index ([4, 5]) in $\mathbb{C}$. For $\mathbf{L} \equiv (\underbrace{1, \ldots, 1}_{n \text{ times}})$ entire functions of bounded index in joint variables have been studied in the papers of M. Salmassi, F. Nuray, R. F. Patterson ([6, 7, 8]). They found applications of bounded index in joint variables to value distribution theory. Namely, F. Nuray and R. Patterson investigated the relationship between the concept of bounded index and the radius of p-valence (univalence at $p = 1$) of entire bivariate functions and their partial derivatives at arbitrary points of $\mathbb{C}^2$. J. Gopala Krishna and S. M. Shah ([9]) introduced the concept of an analytic in a domain (a nonempty connected open set) $\Omega \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ function of bounded index and investigated the index boundedness of analytic solutions of partial differential equations.

In the general case it is very difficult to prove multivariable analogues of characterizations of entire function of bounded l-index from [5]. On the one hand, in ([1]) there are formulated counterparts of the well-known theorems from [5] on properties of entire function of bounded **L**-index in $\mathbb{C}^n$ (bounded **L**-index in joint variables) without proofs, except [3]. On the other hand, nowadays we do not know a complete analogue of an important criterion of l-index boundedness which has been established for the case of functions of one variable. This assertion contains necessary and sufficient conditions that an entire function has bounded l-index in the terms of the boundedness of its logarithmic derivative and zero

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counting function. Therefore, in the present article the following problem is considered: *is there a certain counterpart of the mentioned criterion for entire functions of bounded* **L**-*index in joint variables?*

We remark that the concept of an entire function of bounded L-index in direction is more flexible in some ways ([10]–[14]). In particular, Theorem 1 (for a complete proof see [10]) is such a criterion in this case.

In this paper, we generalize a concept of bounded **L**-index in joint variables from [1]. Instead of $\mathbf{L}(z) = (l_1(|z_1|), l_2(|z_2|), \dots, l_n(|z_n|))$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ as in [1, 2, 3] we consider $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)$ are positive continuous function of $z \in \mathbb{C}^n$. For this notion there are presented complete proofs of generalizations of some theorems from [1]. They are Theorem 3, Theorem 4 and Corollary 1 in the present article. Using these assertions we deduce that if an entire in $\mathbb{C}^n$ function F is of bounded l_j-index in every direction $\mathbf{e}_j = (0, \dots, 0, \underbrace{1}_{j-\text{th place}}, 0, \dots, 0)$ then F is of bounded **L**-index in joint variables (Theorem 6).

Employing Theorems 1 and 6 we deduct sufficient conditions of boundedness of **L**-index in joint variables containing restrictions by the logarithmic derivatives in each variable and by the distribution of zeros (Theorems 7 and 8).

2. Main definitions and notation. We need some standard notation. Let $\mathbb{R}_+ = [0, +\infty)$. Denote

$$\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n_+, \quad \mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n_+, \quad \mathbf{e}_j = (0, \dots, 0, \underbrace{1}_{j-\text{th place}}, 0, \dots, 0) \in \mathbb{R}^n_+.$$

For $R = (r_1, \dots, r_n) \in \mathbb{R}^n_+$ and $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ denote $\|R\| = r_1 + \dots + r_n$, $K! = k_1! \cdot \ldots \cdot k_n!$. For $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, $b = (b_1, \dots, b_n) \in \mathbb{C}^n$, we put

$$ab = (a_1 b_1, \cdots, a_n b_n), \quad a/b = (a_1/b_1, \dots, a_n/b_n), \ b \neq \mathbf{0}, \quad a^b = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}, \ b \in \mathbb{Z}^n_+,$$

and the notation $a < b$ means that $a_j < b_j$ $(j = 1, \dots, n)$; the relation $a \leq b$ is defined similarly.

The polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j, \ j = 1, \dots, n\}$ is denoted by $D^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j = 1, \dots, n\}$ is denoted by $T^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| \leq r_j, \ j = 1, \dots, n\}$ is denoted by $D^n[z^0, R]$. For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of an entire function $F(z) = F(z_1, \dots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \dots + k_n} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z)$ are positive continuous functions of $z \in \mathbb{C}^n$, $j \in \{1, 2, \dots, n\}$.

An entire function $F(z)$, $z \in \mathbb{C}^n$, is called *a function of bounded* **L**-*index in joint variables,* if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J! \mathbf{L}^J(z)} \leq \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \ K \in \mathbb{Z}^n_+, \ \|K\| \leq m \right\}. \tag{1}$$

If $l_j = l_j(|z_j|)$ then we obtain the concept of entire functions of bounded **L**-index in the sense of definition in the papers [1, 2]. If $l_j(z_j) \equiv 1, \ j \in \{1, 2, \dots, n\}$, then the entire function is called a *function of bounded index in joint variables.*

The least integer m for which (1) holds is called the **L**-*index in joint variables of the function* F and is denoted by $N(F,\mathbf{L})$.

If inequality (1) does not hold for any m then we set $N(F,\mathbf{L})=\infty$ and F is called *a function of unbounded* **L**-*index in joint variables.*

Besides, by $N(F,z^0,\mathbf{L})$ we denote the **L**-index in joint variables of the function F at the point z^0, i.e. it is the least integer m for which inequality (1) holds with z^0 instead of z. Clearly that $N(F,\mathbf{L})=\sup\{N(F,z^0,\mathbf{L})\colon z^0\in\mathbb{C}^n\}$.

Remark 1. It is obvious that if F is a polynomial in each variable then for any vector-function **L** its **L**-index satisfies $N(F,\mathbf{L})<\infty$. The function $F(z_1,z_2)=e^{z_1z_2}$ has bounded **L**-index in joint variables with $\mathbf{L}(z_1,z_2)=(|z_2|+1,|z_1|+1)$ and $N(F,\mathbf{L})=0$. But it is impossible to find the **L**-index for this function in the case $\mathbf{L}(z_1,z_2)=(l_1(|z_1|),l_2(|z_2|))$ (i.e. in the sense of the definition from [1]).

Let $L\colon\mathbb{C}^n\to\mathbb{R}_+$ be a continuous function. We need the following definition (see [10]–[14]). An entire function $F(z)$, $z\in\mathbb{C}^n$, is called *a function of bounded L-index in a direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, if there exists $m_0\in\mathbb{Z}_+$ such that for every $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right|\le\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right|\colon 0\le k\le m_0\right\},\tag{2}$$

where

$$\frac{\partial^0F(z)}{\partial\mathbf{b}^0}:=F(z),\ \frac{\partial F(z)}{\partial\mathbf{b}}:=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j=\langle\mathbf{grad}F,\overline{\mathbf{b}}\rangle,\ \frac{\partial^kF(z)}{\partial\mathbf{b}^k}:=\frac{\partial}{\partial\mathbf{b}}\Big(\frac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\Big),\ k\ge 2.$$

The least such an integer $m_0=m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *of the entire function $F(z)$ and is denoted by* $N_{\mathbf{b}}(F,L)=m_0$. If such m_0 does not exist then F is called *a function of unbounded L-index in the direction* **b** and we write $N_{\mathbf{b}}(F,L)=\infty$.

If $L(z)\equiv 1$ then $F(z)$ is called *a function of bounded index in the direction* **b** and $N_{\mathbf{b}}(F)=N_{\mathbf{b}}(F,1)$.

In the case of $n=1$ we obtain the definition of an entire function of one variable of bounded l-index (see [4, 5]); in the case of $n=1$ and $L(z)\equiv 1$ it is reduced to the definition of a bounded index, proposed by B. Lepson ([15]).

If $\mathbf{b}=\mathbf{e}_j$ then we obtain the definition of an entire function F of *uniformly bounded L-index* in variable z_j.

For $\eta>0$, $z\in\mathbb{C}^n$, $t\in\mathbb{C}$, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and a positive continuous function $L\colon\ \mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(z,t_0,\eta)=\inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}\colon|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\right\},\tag{3}$$

$$\lambda_1^{\mathbf{b}}(z,\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,t_0,\eta)\colon t_0\in\mathbb{C}\},\quad\lambda_1^{\mathbf{b}}(\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,\eta)\colon z\in\mathbb{C}^n\},\tag{4}$$

$$\lambda_2^{\mathbf{b}}(z,t_0,\eta)=\sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}\colon|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\right\},\tag{5}$$

$$\lambda_2^{\mathbf{b}}(z,\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta)\colon t_0\in\mathbb{C}\},\quad\lambda_2^{\mathbf{b}}(\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,\eta)\colon z\in\mathbb{C}^n\}.\tag{6}$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L which satisfy the condition

$$(\forall\eta\ge 0)\colon\ 0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty.\tag{7}$$

For $R \in \mathbb{R}^n_+$, $j \in \{1, \dots, n\}$ and $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ we define

$$\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{C}^n} \inf \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in D^n \left[z^0, \frac{R}{\mathbf{L}(z^0)} \right] \right\},$$
$$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{C}^n} \sup \left\{ \frac{l_j(z)}{l_j(z^0)} : z \in D^n \left[z^0, \frac{R}{\mathbf{L}(z^0)} \right] \right\},$$
$$\Lambda_1(R) = (\lambda_{1,j}(R), \dots, \lambda_{1,n}(R)), \quad \Lambda_2(R) = (\lambda_{2,1}(R), \dots, \lambda_{2,n}(R)).$$

By Q^n we denote the class of functions $\mathbf{L}(z)$ which for every $R \in \mathbb{R}^n_+$ and $j \in \{1, \dots, n\}$ satisfy $0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < +\infty$.

3. Auxiliary propositions. For a given $z^0 \in \mathbb{C}^n$ we denote $g_{z^0}(t) := F(z^0 + t\mathbf{b})$. If one has $g_{z^0}(t) \neq 0$ for all $t \in \mathbb{C}$, then $G_r^{\mathbf{b}}(F, z^0) := \varnothing$; if $g_{z^0}(t) \equiv 0$, then $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} \colon t \in \mathbb{C}\}$. And if $g_{z^0}(t) \not\equiv 0$ and a_k^0 are zeros of the function $g_{z^0}(t)$, then $G_r^{\mathbf{b}}(F, z^0) := \cup_k \{z^0 + t\mathbf{b} \colon |t - a_k^0| \le \frac{r}{L(z^0 + a_k^0 \mathbf{b})}\}$, $r > 0$. Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{C}^n} G_r^{\mathbf{b}}(F, z^0). \tag{8}$$

We remark that if $L(z) \equiv 1$, then $G_r^{\mathbf{b}}(F) \subset \{z \in \mathbb{C}^n \colon \operatorname{dist}(z, Z_F) < r|\mathbf{b}|\}$, where Z_F is the zero set of the function F. By $n\big(r, z^0, t_0, 1/F\big) = \sum_{|a_k^0 - t_0| \le r} 1$ we denote the counting function of the zero sequence (a_k^0).

Theorem 1 ([10])**.** *Let F be an entire in $\mathbb{C}^n$ function, $L \in Q^n_{\mathbf{b}}$. $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ iff the following conditions hold*

1) *for every $r > 0$ there exists $P = P(r) > 0$ such that for each $z \in \mathbb{C}^n \setminus G_r^{\mathbf{b}}(F)$*

$$\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}} \right| \le PL(z); \tag{9}$$

2) *for every $r > 0$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for every $z^0 \in \mathbb{C}^n$, satisfying $F(z^0 + t\mathbf{b}) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$*

$$n\left(\frac{r}{|\mathbf{b}| L(z^0 + t^0 \mathbf{b})}, z^0, t_0, \frac{1}{F} \right) \le \widetilde{n}(r). \tag{10}$$

The following characterization of a function of bounded L-index in direction gives an estimate of the maximum modulus on a greater circle by the maximum modulus on a lesser circle.

Theorem 2 ([10])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire in $\mathbb{C}^n$ function $F(z)$ is of bounded L-index in direction $\mathbf{b}$ iff for every r_1 and r_2 such that $0 < r_1 < r_2 < +\infty$, there exists a number $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{C}^n$ and $t_0 \in \mathbb{C}$*

$$\max \left\{ |F(z^0 + t\mathbf{b})| \colon |t - t_0| = \frac{r_2}{L(z^0 + t_0 \mathbf{b})} \right\} \le$$
$$\le P_1 \max \left\{ |F(z^0 + t\mathbf{b})| \colon |t - t_0| = \frac{r_1}{L(z_0 + t_0 \mathbf{b})} \right\}. \tag{11}$$

4. Behaviour of derivatives of function of bounded L-index in joint variables.

Theorem 3. *Let $\mathbf{L} \in Q^n$. In order that an entire function F be of bounded $\mathbf{L}$-index in joint varibles it is necessary and sufficient that for every $R \in \mathbb{R}^n_+$ there exist numbers $n_0 = n_0(R) \in \mathbb{Z}_+$ and $p_0 = p_0(R) \geq 1$ such that for each $z^0 \in \mathbb{C}^n$ and for some $K^0 = K^0(z^0) \in \mathbb{Z}^n_+$, $\|K^0\| \leq n_0$,*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\ \|K\| \leq n_0, z \in E\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \leq p_0 \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{12}$$

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}) < \infty$. For any $R = (r_1, \dots, r_n) \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ we put $p_1 = p_1(R) = \min\{\lambda_{1,j}(R)\colon\ j \in \{1, \dots, n\}\}$, $p_2 = p_2(R) = \max\{\lambda_{2,j}(R)\colon\ j \in \{1, \dots, n\}\}$, $q = q(R) = [2(N+1)p_1^N p_2^{N+1}\|R\|] + 1$, and for $m = 0, \dots, q$ and $z^0 \in \mathbb{C}^n$ we denote

$$S_m(z^0, R) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\| \leq N, z \in D^n\left[z^0, \frac{mR}{q\mathbf{L}(z^0)}\right]\right\},$$
$$S^*_m(z^0, R) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)}\colon \|K\| \leq N, z \in D^n\left[z^0, \frac{mR}{q\mathbf{L}(z^0)}\right]\right\}.$$

Since $D^n[z^0, \frac{mR}{q\mathbf{L}(z^0)}] \subset D^n[z^0, \frac{R}{\mathbf{L}(z^0)}]$, we have

$$\begin{aligned} S_m(z^0, R) &= \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z_0)}\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z)}\colon\ \|K\| \leq N, z \in D^n\left[z^0, \frac{mR}{q\mathbf{L}(z^0)}\right]\right\} \leq \\ &\leq \max\left\{\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z)}\colon\ \|K\| \leq N, z \in D^n\left[z^0, \frac{mR}{q\mathbf{L}(z^0)}\right]\right\} S^*_m(z^0, R) \leq \\ &\leq S^*_m(z^0, R)(\min\{\Lambda_1^K(R)\colon\ \|K\| \leq N\})^{-1} \leq p_1^{-N} S^*_m(z^0, R) \end{aligned} \tag{13}$$

and similarly

$$\begin{aligned} S_m(z^0, R) &\geq \min\left\{\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z)}\colon\ \|K\| \leq N, z \in D^n\left[z^0, \frac{mR}{q\mathbf{L}(z^0)}\right]\right\} S^*_m(z^0, R) \geq \\ &\geq S^*_m(z^0, R)(\max\{\Lambda_2^K(R)\colon\ \|K\| \leq N\})^{-1} \geq p_2^{-N} S^*_m(z^0, R). \end{aligned} \tag{14}$$

Let $K^{(m)}$, $\|K^{(m)}\| \leq N$, and $z^{(m)} \in D^n[z^0, \frac{mR}{q\mathbf{L}(z^0)}]$ be such that

$$S^*_m(z^0, R) = \frac{|F^{(K^{(m)})}(z^{(m)})|}{K^{(m)}!\mathbf{L}^{K^{(m)}}(z^0)}. \tag{15}$$

Since by the maximum principle $z^{(m)} \in T^n(z^0, \frac{mR}{q\mathbf{L}(z^0)})$, we have $z^{(m)} \neq z^0$. We choose

$$z^{(m)}_* = z^0 + \frac{m-1}{m}(z^{(m)} - z^0).$$

Then for all $j = 1, \dots, n$ we obtain

$$|z^{(m)}_{*j} - z^0_j| = \frac{m-1}{m}|z^{(m)}_j - z^0_j| = \frac{(m-1)r_j}{ql_j(z^0)}, \tag{16}$$

$$|z^{(m)}_{*j} - z^{(m)}_j| = \frac{1}{m}|z^{(m)}_j - z^0_j| = \frac{r_j}{ql_j(z^0)}. \tag{17}$$

By (16), $z_*^{(m)} \in T^n(z^0, \frac{(m-1)R}{q\mathbf{L}(z^0)})$ and, thus,

$$S^*_{m-1}(z^0, R) \geq \frac{|F^{(K^{(m)})}(z_*^{(m)})|}{K^{(m)}!\mathbf{L}^{K^{(m)}}(z^0)}.$$

From (15) by the definition of $S^*_m(z^0, R)$ we have

$$0 \leq S^*_m(z^0, R) - S^*_{m-1}(z^0, R) \leq \frac{|F^{K^{(m)}}(z^{(m)})| - |F^{K^{(m)}}(z_*^{(m)})|}{K^{(m)}!L^{K^{(m)}}(z^0)} =$$
$$= \frac{1}{K^{(m)}!\mathbf{L}^{K^{(m)}}(z^0)} \int_0^1 \frac{d}{dt}|F^{(K^{(m)})}(z_*^{(m)} + t(z^{(m)} - z_*^{(m)}))|dt.$$

But for every complex-valued function $\varphi(t)$, $t \in \mathbb{R}$, the inequality $\frac{d}{dt}|\varphi(t)| \leq |\frac{d}{dt}\varphi(t)|$ holds. Then

$$S^*_m(z^0, R) - S^*_{m-1}(z^0, R) \leq \frac{1}{K^{(m)}!\mathbf{L}^{K^{(m)}}(z^0)} \int_0^1 \sum_{j=1}^n |z_j^{(m)} - z_{*j}^{(m)}| \times$$
$$\times \left| \frac{\partial^{\|K^{(m)}\|+1} F}{\partial z_1^{k_1^{(m)}} \dots \partial z_j^{k_j^{(m)}+1} \dots \partial z_n^{k_n^{(m)}}}(z_*^{(m)} + t(z^{(m)} - z_*^{(m)})) \right| dt = \frac{1}{K^{(m)}!\mathbf{L}^{K^{(m)}}(z^0)} \times$$
$$\times \sum_{j=1}^n |z_j^{(m)} - z_{*j}^{(m)}| \left| \frac{\partial^{\|K^{(m)}\|+1} F}{\partial z_1^{k_1^{(m)}} \dots \partial z_j^{k_j^{(m)}+1} \dots \partial z_n^{k_n^{(m)}}}(z_*^{(m)} + t^*(z^{(m)} - z_*^{(m)})) \right|, \tag{18}$$

where $0 \leq t^* \leq 1$. Hence, $z_*^{(m)} + t^*(z^m - z_*^{(m)}) \in D^n[z^0, \frac{mR}{q\mathbf{L}(z^0)}]$. But as in the proof of inequalities (13) and (14), in view of (1) for $z \in D^n[z^0, \frac{mR}{q\mathbf{L}(z^0)}]$ and $\|J\| \leq N+1$ we have

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z^0)} \leq p_2^{N+1} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \leq p_2^{N+1} \max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leq N \right\} \leq$$
$$\leq p_2^{N+1} p_1^N \max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} : \|K\| \leq N \right\} \leq p_1^N p_2^{N+1} S^*_m(z^0, R).$$

Then

$$\frac{\left| \frac{\partial^{\|K\|+1} F}{\partial z_1^{k_1^{(m)}} \dots \partial z_j^{k_j^{(m)}+1} \dots \partial z_n^{k_n^{(m)}}}(z_*^{(m)} + t^*(z^{(m)} - z_*^{(m)})) \right|}{k_1^{(m)}! \dots (k_j^{(m)}+1)! \dots k_n^{(m)}! l_1^{k_1^{(m)}}(z^0) \dots l_j^{k_j^{(m)}+1}(z^0) \dots l_n^{k_n^{(m)}}(z^0)} \leq p_1^N p_2^{N+1} S^*_m(z^0, R),$$

and from (18) we obtain

$$S^*_m(z^0, R) - S^*_{m-1}(z^0, R) \leq p_1^N p_2^{N+1} S^*_m(z^0, R) \sum_{j=1}^n (k_j^{(m)}+1) l_j(z^0) |z_j^{(m)} - z_{*j}^{(m)}| \leq$$
$$\leq \frac{p_1^N p_2^{N+1} S^*_m(z^0, R)}{q} \sum_{j=1}^n (k_j^{(m)}+1) r_j \leq \frac{p_1^N p_2^{N+1} (N+1)\|R\|}{q} S^*_m(z^0, R) \leq \frac{S^*_m(z^0, R)}{2}.$$

It follows that $S^*_m(z^0,R)\le 2S^*_{m-1}(z^0,R)$, and in view of (13) and (14) we have $S_m(z^0,R)\le 2p_1^N S^*_{m-1}(z^0,R)\le 2p_1^N p_2^N S_{m-1}(z^0,R)$. Then

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\ \|K\|\le N, z\in D^n\left[z^0,\frac{R}{\mathbf{L}(z^0)}\right]\right\}=S_q(z^0,R)\le$$

$$\le 2p_1^Np_2^NS_{q-1}(z^0,R)\le\cdots\le(2p_1^Np_2^N)^qS_0(z^0,R)=(2p_1^Np_2^N)^q\max\left\{\frac{|F^{(K)}(z^0)|}{K!L^K(z^0)}\colon\|K\|\le N\right\}.$$

This inequality implies (12) with $p_0=(2p_1^Np_2^N)^q$ and some K^0, $\|K^0\|\le N=n_0$.

The necessity of condition (12) is proved.

Now we prove the sufficiency. We choose $R=\mathbf{2}=(2,\dots,2)$. Then there exists $n_0\in\mathbb{Z}_+$ and $p_0>1$ such that for every $z^0\in\mathbb{C}^n$ and for some $K^0\in\mathbb{Z}^n_+$, $\|K\|\le n_0$ inequality (12) holds. We put

$$s_0=\left[\frac{n_0\ln\max\{\lambda_{2,j}(\mathbf{2})\colon\ j\in\{1,\dots,n\}\}+\ln p_0}{\ln 2}\right]+1$$

and for every $z^0\in\mathbb{C}^n$ and some $K^0=K^0(z^0)\in\mathbb{Z}^n_+$ we write Cauchy's formula

$$\frac{F^{(K+S)}(z^0)}{S!}=\frac{1}{(2\pi i)^n}\int_{T^n\left(z^0,\frac{2}{\mathbf{L}(z^0)}\right)}\frac{F^{(K)}(z)}{(z-z^0)^{S+\mathbf{e}}}dz.$$

Hence in view of (12), we obtain that

$$\frac{|F^{(K+S)}(z^0)|}{S!}\le\left(\frac{\mathbf{L}(z^0)}{2}\right)^S\frac{p_0|F^{(K^0)}(z^0)|}{(K^0)!\mathbf{L}^{K^0}(z^0)}K!\mathbf{L}^K(z^0)(p_2(\mathbf{2}))^{n_0}$$

for every $K\in\mathbb{Z}^n_+$, $\|K\|\le n_0$, $S\in\mathbb{Z}^n_+$. It follows that

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{K+S}(z^0)}\le\frac{p_0(p_2(\mathbf{2}))^{n_0}K!S!}{(K+S)!\mathbf{2}^SK^0!\mathbf{L}^{K^0}(z^0)}|f^{(K^0)}(z^0)|\le$$

$$\le\frac{p_0(p_2(\mathbf{2}))^{n_0}}{2^{\|S\|}K^0!\mathbf{L}^{K^0}(z^0)}|F^{(K^0)}(z^0)|\le\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}$$

for all S, $\|S\|\ge s_0$, and for all K, $\|K\|\le n_0$.

Since $\|K^0\|\le n_0$, it follows from the previous inequality that for all $J\in\mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)}\le\max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)}\colon\ \|K\|\le s_0+n_0\right\},$$

where s_0 and n_0 do not depend on z^0, i.e. the function F has $\mathbf{L}$-index in joint variables $N(F,\mathbf{L})\le s_0+n_0$. □

Corollary 1. *Let $\mathbf{L}\in Q^n$ and an entire function f be of bounded $\mathbf{L}$-index in joint variables with $N(F,\mathbf{L})=N<\infty$. Then for each $R\in\mathbb{R}^n_+$ there exists $p=p(R)\ge 1$ such that for every $z^0\in\mathbb{C}^n$ for a some $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\le N$,*

$$\max\left\{|F^{(K^0)}(z)|\colon z\in D^n\left[z^0,\frac{R}{\mathbf{L}(z^0)}\right]\right\}\le p|F^{(K^0)}(z^0)|. \tag{19}$$

Proof. The arguments from the proof of Theorem 3 imply that inequality (12) is true for some K^0, $\|K^0\| \le N = n_0$. As in the proof of (14), we have

$$\frac{p_0|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)} \ge \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z)}\colon\ z \in D^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \ge$$

$$\ge \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z^0)}\colon\ z \in D^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \times$$

$$\times \min\left\{\frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z)}\colon z \in D^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]\right\} \ge \frac{\max\left\{|F^{(K^0)}(z)|\colon z \in D^n[z^0, \frac{R}{\mathbf{L}(z^0)}]\right\}}{K^0!\mathbf{L}^{K^0}(z^0)(p_2(R))^N},$$

where $p_2(R) = \max\{\lambda_{2,j}(R)\colon\ j \in \{1, \dots, n\}\}$. The obtained inequality implies (19) with $p = p_0(p_2(R))^N$. □

5. Local behaviour of function of bounded L-index in joint variables. For an entire function $F(z)$ we put

$$M(R, z^0, F) = \max\{|F(z)|\colon z \in T^n(z^0, R)\},$$

where $z^0 \in \mathbb{C}^n$, $R = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, $T^n(z^0, R) = \{z \in \mathbb{C}^n\colon |z_j - z_j^0| = r_j,\ j = 1, \dots, n\}$ be the skeleton of the polydisc $D^n(z^0, R)$ or of the closed polydisc $D^n[z^0, R]$. Then $M(R, z^0, F) = \max\{|F(z)|\colon z \in D^n[z^0, R]\}$, because the maximum modulus for an entire function in a closed polydisc is attained on its skeleton.

Theorem 4. *Let $\mathbf{L} \in Q^n$. An entire function F has bounded $\mathbf{L}$-index in joint variables if and only if for any R', R'', $\mathbf{0} < R' < R''$, there exists a number $p_1 = p_1(R', R'') \ge 1$ such that for every $z^0 \in \mathbb{C}^n$*

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \le p_1 M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right). \tag{20}$$

Proof. Let $N(F, \mathbf{L}) = N < +\infty$. Suppose that inequality (20) does not hold i.e. there exist R', R'', $\mathbf{0} < R' < R''$, such that for each $p_* \ge 1$ and for some $z^0 = z^0(p_*)$

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) > p_* M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right). \tag{21}$$

By Corollary 1, there exists a number $p_0 = p_0(R'') \ge 1$ such that for every $z^0 \in \mathbb{C}^n$ and some $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \le N$, one has

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F^{(K^0)}\right) \le p_0|F^{(K^0)}(z^0)|. \tag{22}$$

We put

$$b_1 = p_0\left(\prod_{j=2}^{n}\lambda_{2,j}^N(R'')\right)(N!)^{n-1}\left(\sum_{j=1}^{N}\frac{(N-j)!}{(r_1'')^j}\right)\left(\frac{r_1''r_2''\dots r_n''}{r_1'r_2'\dots r_n'}\right)^N,$$

$$b_2 = p_0\left(\prod_{j=3}^{n}\lambda_{2,j}^N(R'')\right)(N!)^{n-2}\left(\sum_{j=1}^{N}\frac{(N-j)!}{(r_2'')^j}\right)\left(\frac{r_2''\dots r_n''}{r_2'\dots r_n'}\right)^N \max\left\{1, \frac{1}{(r_1')^N}\right\},$$

$$\dots$$

$$b_{n-1} = p_0 \lambda_{2,n}^N(R') N! \left(\sum_{j=1}^N \frac{(N-j)!}{(r''_{n-1})^j} \right) \left(\frac{r''_{n-1} r''_n}{r'_{n-1} r'_n} \right)^N \max \left\{ 1, \frac{1}{(r'_1 \dots r'_{n-2})^N} \right\},$$

$$b_n = p_0 \left(\sum_{j=1}^N \frac{(N-j)!}{(r''_n)^j} \right) \left(\frac{r''_n}{r'_n} \right)^N \max \left\{ 1, \frac{1}{(r'_1 \dots r'_{n-1})^N} \right\}$$

and

$$p_* = (N!)^n p_0 \left(\frac{r''_1 r''_2 \dots r''_n}{r'_1 r'_2 \dots r'_n} \right)^N + \sum_{k=1}^n b_k + 1.$$

Let $z^0 = z^0(p_*)$ be a point for which inequality (21) holds and K^0 is such for which (22) holds and

$$M\left(\frac{R'}{L(z^0)}, z^0, F \right) = |F(z^*)|, \ M\left(\frac{R''}{L(z^0)}, z^0, F^{(J)} \right) = |F^{(J)}(z^*_J)|$$

for every $J \in \mathbb{Z}^n_+$, $\|J\| \leq N$. We apply Cauchy's inequality

$$|F^{(J)}(z^0)| \leq J! \left(\frac{L(z^0)}{R'} \right)^J |F(z^*)| \tag{23}$$

for estimate the difference

$$\begin{aligned} &|F^{(J)}(z^*_{J,1}, z^*_{J,2}, \dots, z^*_{J,n}) - F^{(J)}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})| = \\ &= \left| \int_{z^0_1}^{z^*_{J,1}} \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}} (\xi, z^*_{J,2}, \dots, z^*_{J,n}) d\xi \right| \leq \\ &\leq \left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}} (z^*_{(j_1+1, j_2, \dots, j_n)}) \right| \frac{r''_1}{l_1(z^0)}. \end{aligned} \tag{24}$$

Since $(z^0_1, z^*_{J,2}, \dots, z^*_{J,n}) \in D^n[z^0, \frac{R''}{\mathbf{L}(z^0)}]$, for all $k \in \{1, \dots, n\}$ $|z^*_{J,k} - z^0_k| = \frac{r''_k}{l_k(z^0)}$ and $l_k(z^*_J) \leq \lambda_{2,k}(R'') l_k(z^0)$ by Theorem 3 and in view of (23) with $J = K^0$ we have

$$\begin{aligned} &|F^{(J)}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})| \leq \frac{J! l_1^{j_1}(z^0) \prod_{k=2}^n l_k^{j_k}(z^*_J)}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 |F^{(K^0)}(z^0)| \leq \\ &\leq \frac{J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 K^0! \left(\frac{\mathbf{L}(z^0)}{R'} \right)^{K^0} |F(z^*)| = \frac{p_0 J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{(R')^{K^0}} |F(z^*)|. \end{aligned} \tag{25}$$

From inequalities (24) and (25) it follows that

$$\begin{aligned} &\left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}} (z^*_{(j_1+1, j_2, \dots, j_n)}) \right| \geq \frac{l_1(z^0)}{r''_1} \left\{ |F^{(J)}(z^*_j)| - |F^{(J)}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})| \right\} \geq \\ &\geq \frac{l_1(z^0_1)}{r''_1} |F^{(J)}(z^*_j)| - \frac{p_0 J! \mathbf{L}^{(j_1+1, j_2, \dots, j_n)}(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{r''_1 (R')^{K^0}} |F(z^*)|. \end{aligned}$$

Then

$$|F^{(K^0)}(z^*_{K^0})| \geq \frac{l_1(z^0)}{r''_1}\left|\frac{\partial^{\|K^0\|-1}f}{\partial z_1^{k_1^0-1}\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(k_1^0-1,k_2^0,\dots,k_n^0)})\right|-$$
$$-\frac{p_0(k_1^0-1)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')}{r''_1(R')^{K^0}}|F(z^*)|\geq$$
$$\geq\frac{l_1^2(z^0)}{(r''_1)^2}\left|\frac{\partial^{\|K^0\|-2}f}{\partial z_1^{k_1^0-2}\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(k_1^0-2,k_2^0,\dots,k_n^0)})\right|-$$
$$-\frac{p_0(k_1^0-2)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')}{(r''_1)^2(R')^{K^0}}|F(z^*)|-$$
$$-\frac{p_0(k_1^0-1)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(r''_i)}{r''_1(R')^{K^0}}|F(z^*)|\geq$$
$$\dots$$
$$\geq\frac{l_1^{k_1^0}(z^0)}{(r''_1)^{k_1^0}}\left|\frac{\partial^{\|K^0\|-k_1^0}f}{\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(0,k_2^0,\dots,k_n^0)})\right|-$$
$$-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r''_1)^{j_1}}|F(z^*)|\geq$$
$$\dots$$
$$\geq\frac{l_1^{k_1^0}(z^0)}{(r''_1)^{k_1^0}}\frac{l_2^{k_2^0}(z^0)}{(r''_2)^{k_2^0}}\left|\frac{\partial^{\|K^0\|-k_1^0-k_2^0}f}{\partial z_3^{k_3^0}\dots\partial z_n^{k_n^0}}(z^*_{(0,0,k_3^0,\dots,k_n^0)})\right|-$$
$$-\frac{l_1^{k_1^0}(z^0)p_0L^{(0,k_2^0,\dots,k_n^0)}(z^0)}{(r''_1)^{k_1^0}(R')^{K^0}}\left(\prod_{i=3}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_3^0!\dots k_n^0!\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r''_2)^{j_2}}|F(z^*)|-$$
$$-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r''_1)^{j_1}}|F(z^*)|\geq$$
$$\dots$$
$$\geq\left(\frac{L(z^0)}{R''}\right)|F(z^*_{\mathbf{0}})|-|F(z^*)|\sum_{i=1}^b\tilde{b}_i, \tag{26}$$

where in view of the inequalities $\lambda_{2,i}(R'')\geq 1$ and $R''\geq R'$ we have

$$\tilde{b}_1=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r''_1)^{j_1}}=$$
$$=\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left(\frac{R''}{R'}\right)^{K^0}p_0\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r''_1)^{j_1}}\leq\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_1,$$

$$\tilde{b}_2=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=3}^n\lambda_{2,i}^{k_i^0}(R'')\right)\frac{k_3^0!\dots k_n^0!}{(r''_1)^{k_1^0}}\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r''_2)^{j_2}}\leq\left(\frac{L(z^0)}{R''}\right)^{K^0}b_2,$$

$$\dots$$

$$\tilde{b}_{n-1}=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\lambda_{2,n}^{k_n^0}(R'')\frac{k_n^0!}{(r''_1)^{k_1^0}\dots(r''_{n-2})^{k_{n-2}^0}}\times$$

$$\times\sum_{j_{n-1}=1}^{k_{n-1}^0}\frac{(k_{n-1}^0-j_{n-1})!}{(r''_{n-1})^{j_{n-1}}}\le\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_{n-1},$$

$$\tilde{b}_n=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\frac{1}{(r''_1)^{k_1^0}\dots(r''_{n-1})^{k_{n-1}^0}}\sum_{j_n=1}^{k_n^0}\frac{(k_n^0-j_n)!}{(r''_n)^{j_n}}\le\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_n.$$

Thus, (26) implies that

$$|F^{(K^0)}(z^*_{K^0})|\ge\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{\frac{|F(z^*_{\mathbf{0}})|}{|F(z^*)|}-\sum_{j=1}^n b_j\right\}.$$

But in view of (21) and a choice of p_* we have $\frac{|F(z^*_{\mathbf{0}})|}{|F(z^*)|}\ge p_*>\sum_{j=1}^n b_j$. Thus, in view of (22) and (23) we obtain

$$|F^{(K^0)}(z^*_{K^0})|\ge\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{p_*-\sum_{j=1}^n b_j\right\}\ge$$

$$\ge\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left\{p_*-\sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^0)|(R')^{K^0}}{K^0!\mathbf{L}^{K^0}(z^0)}\ge\left(\frac{r'_1\dots r'_n}{r''_1\dots r''_n}\right)^N\left\{p_*-\sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^*_{K^0})|}{p_0(n!)^n}.$$

Hence, we have $p_*\le p_0\big(\frac{r'_1\dots r'_n}{r''_1\dots r''_n}\big)^N(N!)^n+\sum_{j=1}^n b_j$, but this contradicts the choice of p_*. The necessity of (26) is proved.

Now we prove the sufficiency. Let $z^0\in\mathbb{C}^n$ be an arbitrary point. We expand a function F in power series

$$F(z)=\sum_{K\ge\mathbf{0}}b_K(z-z^0)^K=\sum_{k_1,\dots,k_n\ge0}b_{k_1,\dots,k_n}(z_1-z_1^0)^{k_1}\dots(z_n-z_n^0)^{k_n},\tag{27}$$

where $b_K=b_{k_1,\dots,k_n}=\frac{F^{(K)}(z^0)}{K!}$.

Let $\mu(R,z^0,F)=\max\{|b_K|R^K\colon\ K\ge\mathbf{0}\}$ be the maximal term of series (27) and $\mathcal{N}(R)=(\nu_1^0(R),\dots,\nu_n^0(R))$ a set of indices such that

$$\mu(R,z^0,F)=|b_{\mathcal{N}(R)}|R^{\mathcal{N}(R)},\ \|\mathcal{N}(R)\|=\max\{\|K\|\colon K\ge\mathbf{0},\ |b_K|R^K=\mu(R,z^0,F)\}.$$

Then in view of inequality (23) we obtain

$$\mu(R,z^0,F)\le M(R,z^0,F)\le\sum_{K\ge\mathbf{0}}|b_K|(2R)^{K}\mathbf{2}^{-K}\le2^n\mu(2R,z^0,F),$$

$$\ln\mu(R,z^0,F)=\ln\{|b_{\mathcal{N}(R)}|(2R)^{\mathcal{N}(R)}\mathbf{2}^{-\mathcal{N}(R)}\}\le\ln\mu(2R,z^0,F)-\|\mathcal{N}(R)\|\ln2,$$

$$\|\mathcal{N}(R)\|\le\frac{1}{\ln2}\{\ln\mu(2r,z^0,F)-\ln(R,z^0,F)\}\le$$

$$\leq \frac{1}{\ln 2}\left\{\ln M(2R, z^0, F) - \ln M\left(\frac{R}{2}, z^0, F\right)\right\} + n. \tag{28}$$

Now let $N(F, z^0, \mathbf{L})$ be the **L**-index of function F in joint variables at a point z^0. It is easy show that

$$N(F, z^0, \mathbf{L}) \leq \left\| \mathcal{N}\left(\frac{\mathbf{e}}{\mathbf{L}(z^0)}, z^0, F\right)\right\|. \tag{29}$$

We put in (20) $R'' = \mathbf{2}$ and $R' = \frac{\mathbf{e}}{2}$. Then $M(\frac{\mathbf{2}}{\mathbf{L}(z^0)}, z^0, F) \leq p_1 M(\frac{\mathbf{e}}{\mathbf{2L}(z^0)}, z^0, F)$. Using (28), (29), we obtain that $N(F, z^0, \mathbf{L}) \leq n + \frac{\ln p_1(\frac{\mathbf{e}}{\mathbf{2}}, \mathbf{2})}{\ln 2}$ for every $z^0 \in \mathbb{C}^n$, which implies that F has bounded **L**-index in joint variables. □

Using the arguments of the proof of Theorem 4, one can prove the following theorem.

Theorem 5. *Let $\mathbf{L} \in Q^n$. An entire function F is of bounded **L**-index in joint variables if and only if there exist numbers R', R'', $\mathbf{0} < R' < \mathbf{e} < R''$, and $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{C}^n$ inequality (20) holds.*

6. Boundedness of L-index in every direction $\mathbf{e}_j$. The boundedness of l_j-index of a function $F(z)$ in every variable z_j, generally speaking, does not imply the boundedness of **L**-index in joint variables (see example in [1]). But, if a function F has bounded l_j-index in every direction $\mathbf{e}_j$, $j \in \{1, \ldots, n\}$, then F is a function of bounded **L**-index in joint variables.

Theorem 6. *Let $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $l_j \in Q^n_{\mathbf{e}_j}$ ($j \in \{1, \ldots, n\}$). If an entire in $\mathbb{C}^n$ function F has bounded l_j-index in a direction $\mathbf{e}_j$ for every $j \in \{1, \ldots, n\}$, then F is of bounded **L**-index in joint variables.*

Proof. Let an entire in $\mathbb{C}^n$ function F be of bounded l_j-index in every direction $\mathbf{e}_j$. Then by Theorem 2 for every $j \in \{1, \ldots, n\}$ and arbitrary $0 < r' < r'' < \infty$ there exists a number $p_j = p_j(r', r'')$ such that for every $z_j^0 \in \mathbb{C}$ and for all $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbb{C}^{n-1}$ inequality

$$\max\left\{|F(z)|\colon |z_i - z_i^0| = \frac{r_i''}{l_i(z_1, \ldots, z_{i-1}, z_i^0, z_{i+1}, \ldots, z_n)}\right\} \leq p_i(r_i', r_i'')\times$$
$$\times \max\left\{|F(z)|\colon |z_i - z_i^0| = \frac{r_i'}{l_i(z_1, \ldots, z_{i-1}, z_i^0, z_{i+1}, \ldots, z_n)}\right\} \tag{30}$$

holds.

Obviously, if for every $j \in \{1, \ldots, n\}$ $l_j \in Q^n_{\mathbf{e}_j}$ then $\mathbf{L} \in Q^n$. Let z^0 be an arbitrary point of $\mathbb{C}^n$, and a point $z^* \in T^n(z^0, \frac{R''}{\mathbf{L}(z^0)})$ is such that $M(\frac{R''}{\mathbf{L}(z^0)}, z^0, F) = |F(z^*)|$. We choose $R'' > \mathbf{e}$ and $R' < \Lambda_1(R'')$. Then inequality (30) implies that

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \leq \max\left\{|F(z_1, z_2^*, z_3^*, \ldots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r_1''}{l_1(z_1^0, z_2^*, z_3^*, \ldots, z_n^*)}\right\} \leq$$
$$\leq \max\left\{|F(z_1, z_2^*, \ldots, z_n^*)|\colon |z_1 - z_1^0| = \frac{r_1''}{l_1(z_1^0, z_2^*, \ldots, z_n^*)}\frac{l_1(z_1^0, z_2^*, \ldots, z_n^*)}{l_1(z^0)}\right\} \leq$$
$$\leq \max\left\{|F(z_1, z_2^*, \ldots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r_1''\lambda_{2,1}(R'')}{l_1(z_1^0, z_2^*, \ldots, z_n^*)}\right\} \leq$$
$$\leq p_1(r_1', r_1''\lambda_{2,1}(R''))\max\left\{|F(z_1, z_2^*, \ldots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r_1'}{l_1(z_1^0, z_2^*, \ldots, z_n^*)}\right\} =$$

$$= p_1(r_1', r_1''\lambda_{2,1}(R'')) \max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon |z_1 - z_1^0| = \frac{r_1'}{l_1(z^0)} \frac{l_1(z^0)}{l_1(z_1^0, z_2^*, \dots, z_n^*)}\right\} \leq$$

$$\leq p_1(r_1', r_1''\lambda_{2,1}(R'')) \max\left\{|F(z_1, z_2^*, \dots, z_n^*)|\colon\ |z_1 - z_1^0| = \frac{r_1'}{\lambda_{1,1}(R'')l_1(z^0)}\right\} =$$

$$= p_1(r_1', r_1''\lambda_{2,1}(R''))|F(z_1^{**}, z_2^*, \dots, z_n^*)| \leq p_1(r_1', r_1''\lambda_{2,1}(R''))\times$$

$$\times \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon\ |z_2 - z_2^0| = \frac{r_2''}{l_2(z^0)}\right\} = p_1(r_1', r_1''\lambda_{2,1}(R''))\times$$

$$\times \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon\ |z_2 - z_2^0| = \frac{r_2''}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)} \frac{l_2(z_1^{**}, z_2^0, \dots, z_n^*)}{l_2(z^0)}\right\} \leq$$

$$\leq p_1(r_1', r_1''\lambda_{2,1}(R'')) \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r_2''\lambda_{2,2}(R'')}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)}\right\} \leq$$

$$\leq \prod_{j=1}^{2} p_j(r_j', r_j''\lambda_{2,j}(R'')) \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r_2'}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)}\right\} \leq$$

$$\leq \prod_{j=1}^{2} p_j(r_j', r_j''\lambda_{2,j}(R'')) \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r_2'}{\lambda_{1,2}(R'')l_2(z^0)}\right\} =$$

$$= \prod_{j=1}^{2} p_j(r_j', r_j''\lambda_{2,j}(R''))|F(z_1^{**}, z_2^{**}, z_3^*, \dots, z_n^*)| \leq \ldots \leq \prod_{j=1}^{n} p_j(r_j', r_j''\lambda_{2,j}(R''))\times$$

$$\times \max\left\{|F(z_1, z_2, \dots, z_n)|\colon\ |z_j - z_j^0| = \frac{r_j'}{\lambda_{1,j}(R'')l_j(z^0)}, j \in \{1, \dots, n\}\right\} =$$

$$= \prod_{j=1}^{n} p_j(r_j', r_j''\lambda_{2,j}(R''))M\left(\frac{R'}{\Lambda_1(R'')\mathbf{L}(z^0)}, z^0, f\right).$$

Hence, by Theorem 5 f is of bounded $\mathbf{L}$-index in joint variables. □

Note that if an entire function F in $\mathbb{C}^n$ has bounded index in the direction $\mathbf{e}_j$ for every $j \in \{1, \dots, n\}$, then F is of bounded index in joint variables.

Using Theorem 6 and Theorem 1 it can be obtained sufficient conditions of boundedness of $\mathbf{L}$-index in joint variables.

Theorem 7. *Let $F(z)$ be an entire in $\mathbb{C}^n$ function and for every $j \in \{1, 2, \dots, n\}$ $l_j(z) \in Q_{\mathbf{e}_j}^n$ and $\mathbb{C}^n \setminus G_r^{\mathbf{e}_j}(F) \neq \varnothing$. If for each $j \in \{1, 2, \dots, n\}$ conditions*

1) for every $r > 0$ there exist $P_j = P_j(r) > 0$ such that for all $z \in \mathbb{C}^n \setminus G_r^{\mathbf{e}_j}(F)$

$$\left|\frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{e}_j}\right| \leq P_j l_j(z); \tag{31}$$

2) for every $r > 0$ there exists $\widetilde{n}_j(r) \in \mathbb{Z}_+$ that for all $z^0 \in \mathbb{C}^n$, such that $F(z^0 + t\mathbf{e}_j) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$

$$n_{\mathbf{e}_j}\left(\frac{r}{l_j(z^0 + t_0\mathbf{e}_j)}, z^0, t_0, \frac{1}{F}\right) \leq \widetilde{n}_j(r), \tag{32}$$

hold then $F(z)$ has bounded $\mathbf{L}$-index in joint variables.

Proof. We remark that $|\mathbf{b}| = |\mathbf{e}_j| = 1$, $L(z^0 + t_0\mathbf{b}) = L(z^0 + t_0\mathbf{e}_j) = L(z_1^0, \dots, z_{j-1}^0, z_j^0 + t, z_{j+1}^0, \dots, z_n^0) = l_j(z^0 + t\mathbf{e}_j)$, then by the assumptions of this theorem and Theorem 1, $F(z)$ is of bounded l_j-index in the direction $\mathbf{e}_j$. Hence, applying Theorem 5, we obtain that $F(z)$ is of bounded **L**-index in joint variables, where $\mathbf{L}(z) = (l_1(z_1), l_2(z_2), \dots, l_n(z_n))$. □

After introducing new notations this theorem can be paraphrased. For a given point $z^0 \in H_j \equiv \{(z_1^0, \dots, z_{j-1}^0, 0, z_{j+1}^0, \dots, z_n^0)\colon z_k \in \mathbb{C}, k = 1, \dots, j-1, j+1, \dots, n\}$ we denote $\widetilde{g}_{z^0}(\widetilde{t}) \equiv F(z_1^0, \dots, z_{j-1}^0, \widetilde{t}, z_{j+1}^0, \dots, z_n^0)$. If $\widetilde{g}_{z^0}(\widetilde{t}) \not\equiv 0$ and $\widetilde{a}_k^0$ are zeros of the function $\widetilde{g}_{z^0}(\widetilde{t})$ then

$$G_r^{\mathbf{e}_j}(F, z^0) = \bigcup_k \left\{ z^0 + t\mathbf{e}_j\colon |t - a_k^0| \le \frac{r}{l_j(z^0 + a_k^0\mathbf{e}_j)} \right\} =$$

$$= \bigcup_k \left\{ (z_1^0, \dots, z_{j-1}^0, z_j^0 + t, z_{j+1}^0, \dots, z_n^0)\colon |(z_j^0 + t) - (z_j^0 + a_k^0)| \le \frac{r}{l_j(z^0 + a_k^0\mathbf{e}_j)} \right\} =$$

$$= \bigcup_k \left\{ (z_1^0, \dots, z_{j-1}^0, \widetilde{t}, z_{j+1}^0, \dots, z_n^0)\colon |\widetilde{t} - \widetilde{a}_k^0| \le \frac{r}{l_j(z^0 + a_k^0\mathbf{e}_j)} \right\},$$

where $\widetilde{t} = z_j^0 + t$, $\widetilde{a}_k^0 = z_j^0 + a_k^0$.

As above, if for all $\widetilde{t} \in \mathbb{C}$ one has $F(z_1^0, \dots, z_{j-1}^0, \widetilde{t}, z_{j+1}^0, \dots, z_n^0) \ne 0$, then we put $G_r^{\mathbf{e}_j}(F, z^0) = \varnothing$. If $\widetilde{g}_{z^0}(\widetilde{t}) \equiv 0$ for a given z^0, then we put

$$G_r^{\mathbf{e}_j}(F, z^0) = \{(z_1^0, \dots, z_{j-1}^0, \widetilde{t}, z_{j+1}^0, \dots, z_n^0)\colon \widetilde{t} \in \mathbb{C}\}.$$

Denote $G_r^{\mathbf{e}_j}(F) = \bigcup_{z^0 \in H_j} G_r^{\mathbf{e}_j}(F, z^0)$, and $n\big(r, z^0, \widetilde{t}_0, 1/F\big) = \sum_{|\widetilde{a}_k^0 - \widetilde{t}_0| \le r} 1$.

The following theorem is valid.

Theorem 8. *Let $F(z)$ be an entire in $\mathbb{C}^n$ function and for all $j \in \{1, 2, \dots, n\}$ $l_j \in Q_{\mathbf{e}_j}^n$ and $\mathbb{C}^n \setminus G_r^{\mathbf{e}_j}(F) \ne \varnothing$. If for every $j = 1, 2, \dots, n$ conditions*

1) for every $r > 0$ there exists $P_j = P_j(r) > 0$ such that for all $z \in \mathbb{C}^n \setminus G_r^{\mathbf{e}_j}(F)$

$$\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial z_j} \right| \le P_j l_j(z); \tag{33}$$

2) for every $r > 0$ there exists $\widetilde{n}_j(r) \in \mathbb{Z}_+$ such that for all $z^0 \in H_j$, $F(z_1^0, \dots, z_{j-1}^0, \widetilde{t}, z_{j+1}^0, \dots, z_n^0) \not\equiv 0$, and for all $\widetilde{t}_0 \in \mathbb{C}$

$$n\left(\frac{r}{l_j(z^0 + \widetilde{t}_0\mathbf{e}_j)}, z^0, \widetilde{t}_0, \frac{1}{F} \right) \le \widetilde{n}_j(r), \tag{34}$$

hold then $F(z)$ has bounded **L***-index in joint variables.*

Remark 2. We do not know whether the converse proposition is true, i.e. does the boundedness of **L**-index in joint variables imply (31) and (32). This problem is reduced to the question of does the boundedness of **L**-index in joint variables imply the boundedness of l_j-index in the direction $\mathbf{e}_j$, i.e. the uniform boundedness of l_j-index in each variable z_j. The **problem has been open** since the early 90's when these functions began to be studied in [1, 2].

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